Μάθημα 12ο   Ε8. Μερικές Διαφορικές Εξισώσεις Ι   Στρατής   21/5/2019

Χρονική Εξασθένηση

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^n, t>0 \\ u(x,0) = \varphi(x) & \\ u_t(x,0) = \psi(x) & \end{cases} \quad x \in \mathbb{R}^n$$

Υποθέτουμε επιπλέον ότι οι φ, ψ έχουν συμπαγή φορέα. Απλάδη μηδενίζονται έξω από την $B_R = \{x \in \mathbb{R}^n, |x| < R\}$

Θεωρώ τις $\|f\| = \max_{x \in B_R} |f(x)|$, $\|f\|_{C^1} = \max_{x \in B_R} [|f(x)| + |\nabla f(x)|]$

$n=1$ Τύπος d'Alembert

$$u(x,t) = \frac{1}{2}[\varphi(x+t) + \varphi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\, ds$$

$|\text{supp}\,\psi| < 2R$
- $|u(x,t)| \leq \|\varphi\| + R\|\psi\|$ Ανεξάρτητο φράγμα από το t
- $|u_1(x,t) - u_2(x,t)| \leq \|\varphi_1 - \varphi_2\| + R\|\psi_1 - \psi_2\|$

$n=2$ Τύπος Poisson

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\varphi(y) + t^2\psi(y) + t\,\nabla\varphi(y)(y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy$$

$|\text{supp}\,\varphi|, |\text{supp}\,\psi| \leq \pi R^2$

$t \geqslant \quad (|y-x| \leq t)$

- $|u(x,t)| \leq \frac{R^2}{t}[\|\varphi\|_{C^1} + \|\psi\|]$
- $|u_1(x,t) - u_2(x,t)| \leq \frac{R^2}{t}[\|\varphi_1 - \varphi_2\|_{C^1} + \|\psi_1 - \psi_2\|]$

$n=3$ Τύπος Kirchhoff

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} [\varphi(y) + t\psi(y) + \nabla\varphi(y)(y-x)]\, dS(y)$$

$\partial B(x,t) \longrightarrow |\partial B(x,t) \cap B_R| \leq 4\pi R^2$

$t \geqslant 1 \quad (|y-x| \leq t)$

- $|u(x,t)| \leq \frac{R^2}{t}[\|\varphi\|_{C^1} + \|\psi\|$
- $|u_1(x,t) - u_2(x,t)| \leq \frac{R^2}{t}[\|\varphi_1 - \varphi_2\|_{C^1} + \|\psi_1 - \psi_2\|]$

• Για $n =$ ζυγός $\exists\ C = C(R)$

$$|u(x,t)| \le C\cdot(1+t)^{\frac{1-n}{2}}(1+|t-|x||)^{\frac{1-n}{2}} \quad \forall x\in\mathbb{R}^n,\ \forall t>0$$

• Για $n =$ περιττός $\exists\ C = C(R)$

$$|u(x,t)| \le C(1+t)^{\frac{1-n}{2}}$$

$\mathbb{R}^n$, $n = 2k+1$, $(k = 1,2,3\dots)$

• $\varphi:\mathbb{R}\to\mathbb{R}$ C^{k+1}: $\frac{d^2}{dr^2}\left[\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}(r^{2k-1}\varphi(r))\right] = \left(\frac{1}{r}\frac{d}{dr}\right)^k\left(r^{2k}\frac{d\varphi(r)}{dr}\right)$

• $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\varphi(r)\right) = \sum_{j=0}^{k-1} b_j^k\, r^{j+1}\frac{d^j\varphi(r)}{dr^j}$

ανεξάρτητο της φ.

• $b_0^k = 1\cdot3\cdot5\cdot\ldots\cdot(2k-1)$

$$v(x;r,t) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\bar{u}(x;r,t)\right)$$

$$g(x;r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\bar{\varphi}(x;r)\right)$$

$$h(x,r) := \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\bar{\psi}(x;r)\right)$$

$\forall\, k\in\mathbb{N}$, $\forall\, x\in\mathbb{R}^n$ ΠΑΣ.Τ: 1-dim

$$\begin{cases} v_{tt} - v_{rr} = 0, & r>0,\ t>0 \\ v(x,r,0) = g(x,r) \\ v_t(x;r,0) = h(x,r) \\ v(x;0,t) = 0 \end{cases} \rightsquigarrow v(x;r,t) = \frac{1}{2}\left[g(x,t+r) - g(x,t-r)\right] + \frac{1}{2c}\int_{t-r}^{t+r} h(x,y)\,dy$$

• $u(x,t) = \lim_{r\to0}\bar{u}(x,r,t)$

$$\bar{u} = \frac{v}{b_0^k r} - \frac{b_1^k}{b_0^k}\, r\frac{\partial\bar{u}}{\partial r} - \dots - \frac{b_{k-1}^k}{b_0^k}\, r^{k-1}\frac{\partial^{k-1}\bar{u}}{\partial r^{k-1}} \longrightarrow$$

$$u(x,t) = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)}\varphi(y)\,dS(y)\right) + \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{\partial B(x,t)}\psi(y)\,dS(y)\right)$$

$\gamma_n = 1\cdot3\cdot5\cdot\ldots\cdot(n-2)$

ΕΒ. Μερικές Διαφορικές Εξισώσεις Ι Στράτης 21/5/2019

$\mathbb{R}^n$, $n = 2\cdot k$ $(k = 1, 2, \dots)$ μέθοδος της καθόδου

$u(x_1, x_2, \dots, x_n, t) \rightsquigarrow \tilde{u}(x_1, \dots, x_n, x_{n+1}, t) = u(x_1, \dots, x_n, t)$

$u_{tt} - \Delta u = 0$

$u(x_1, \dots, x_n, 0) = \varphi(x_1, \dots, x_n) \rightsquigarrow \tilde{\varphi}(x_1, \dots, x_n, x_{n+1}) = \varphi(x_1, \dots, x_n)$

$u_t(x_1, \dots, x_n, 0) = \psi(x_1, \dots, x_n) \rightsquigarrow \tilde{\psi}(x_1, \dots, x_n, x_{n+1}) = \psi(x_1, \dots, x_n)$

$\tilde{u}$: Η λύση της κυματικής εξίσωσης στον $\mathbb{R}^{n+1}$

$(\bar{x}, t) = (x_1, \dots, x_n, 0, t)$

$\bar{B}(\bar{x}, t)$:= μπάλα στον $\mathbb{R}^{n+1}$ κέντρου $\bar{x}$, ακτίνας t.

Ενδιάμεσος τύπος. $\displaystyle \fint_{\partial \bar{B}(\bar{x},t)} \varphi(y)\,ds(y) = \frac{1}{(n+1)\alpha(n+1)t^n} \int_{\partial \bar{B}(\bar{x},t)} \tilde{\varphi}(y)\,ds(y) = \frac{2}{(n+1)\alpha(n+1)} \int_{B(x,t)} \varphi(y)(1+|\nabla \gamma(y)|^2)^{1/2}\,dy$

$\gamma(y) = (t^2 - |y-x|^2)^{1/2}$

$$u(x,t) = \frac{1}{\sigma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{\varphi(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{\psi(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\right]$$

$\sigma_n = 2 \cdot 4 \cdot \ldots \cdot (n-2)\cdot n$

Για τη μη ομογενή κυματική εξίσωση στον $\mathbb{R}^n$

$$\begin{cases} u_{tt} - \Delta u = f(x,t) & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = \varphi(x) & \\ u_t(x,0) = \psi(x) & \Big\}\ x \in \mathbb{R}^n \end{cases}$$

Εισάγουμε $v = u_t$, γράφουμε το πρόβλημα :

$$\begin{cases} \begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} \\ \begin{pmatrix} u \\ v \end{pmatrix}(x,0) = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \end{cases}$$

Ορίζουμε: $U := \begin{pmatrix} u \\ v \end{pmatrix}$, $F := \begin{pmatrix} 0 \\ f \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$, $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$

$$\begin{cases} U_t - AU = F \\ U(x,0) = \Phi(x) \end{cases} \longrightarrow U(t) = \underbrace{S(t)\Phi}_{U_0(t)} + \int_0^t S(t-s)F(s)ds$$

$S(t)$: ο επιλύων τελεστής του $\begin{cases} U_t + AU = 0 \\ U(x,0) = \Phi(x) \end{cases}$

το οποίο έχει λύση $U_0(t) = S(t)\Phi$

• για $n=3$: $u_0(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \left[\varphi(y) + \nabla\varphi(y)(y-x) + t\,\psi(y)\right] ds(y)$

$$S(t)\Phi = S(t)\begin{pmatrix}\varphi \\ \psi\end{pmatrix} = \begin{pmatrix} u_0 \\ (u_0)_t \end{pmatrix}$$

$$S(t-s)\cdot F(s) = S(t-s)\begin{pmatrix} 0 \\ f(s)\end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi(t-s)}\int_{\partial B(x,t-s)} f(y,s)ds(y) \\ \frac{\partial}{\partial t}\left(\frac{1}{4\pi(t-s)}\int_{\partial B(x,t-s)} f(y,s)ds(y)\right)\end{pmatrix}$$

Τελικά

$$u(x,t) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \left[\varphi(y) + \nabla\varphi(y)(y-x) + t\,\psi(y)\right]ds(y) + \frac{1}{4}\int_0^t \frac{1}{t-s}\int_{\partial B(x,t-s)} f(y,s)ds(y)dy$$

Παρατήρηση: Έστω $\varphi = \psi \equiv 0$ τότε

$$\underbrace{u(x,t) = \frac{1}{4\pi}\int_{B(x,t-s)} \frac{f(y,t-|y-x|)}{|y-x|}dy}_{} \quad x\in\mathbb{R}^3,\ t>0$$

Υστερημένο δυναμικό (retarded potential)

• $n=2$

$$u(x,t) = \frac{1}{2\pi t}\int_{B(x,t)} \frac{t\varphi(y) + t^2\psi(y) + t\,\nabla\varphi(y)(y-x)}{(t^2-|y-x|^2)^{1/2}}dy + \frac{1}{2\pi}\int_0^t \frac{1}{(t-s)^2}\int_{B(x,t-s)} \frac{(t-s)^2 f(y,s)}{(t^2-|y-x|^2)^{1/2}}dy\,ds$$