Lecture notes on Homogenization

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Chapter 1

A brief Abstract

In chapter 2 we introduce the notion of $\Gamma-$convergence and it's main properties together with two basic examples. The first example is the $\Gamma-$limit of the Modica-Mortola functional which relates phase transition type problems with minimal surfaces. The second, is an example from homogenization.

In chapter 3, we introduce the notion of G-convergence for elliptic equations in divergence form together with the main properties. Then, we state the homogenization theorem of G-convergence and we also provide an example for linear elliptic operators.

Furthermore, in chapter 4, we illustrate the relationship between G-convergence with $\Gamma-$convergence and the second example stated in the beginning allow us to compare the G-limit with the weak* L^∞ limit.

Finally, in chapter 5, we introduce the notion of H-convergence that generalizes the notion of G-convergence in the non symmetric case. Moreover, we provide the basic properties of H-convergence and the main tool for the homogenization theorem based on compensated compactness, that is, the div-curl lemma. Additionally, we state and prove the homogenization theorem for H-convergence and define the corrector matrix that gives an approximation for the solutions of the homogenization problem, i.e. the corrector result.

These lecture notes provide some of the basic homogenization techniques and the notions of convergence for the case of linear operators, however these can be extended for nonlinear operators, the so called monotone operators. We refer the interested reader to Chapter 5,6 in [12] and in [24] for nonlinear elliptic and parabolic operators.

Chapter 2

An introduction to $\Gamma-$convergence

The notion of $\Gamma-$convergence was introduced by E. De Giorgi and T. Franzoni in [9] and in particular relates phase transition type problems with the theory of minimal surfaces. We can think of this notion as a generalization of the Direct Method in the Calculus of Variations as follows, if F_0 is lower semicontinuous and coercive we can take $F_\varepsilon = F_0$ and then the Γ limit of F_ε equals F_0.

One application of $\Gamma-$convergence is the proof of existence of minimizers of a limiting functional, say F_0 by utilizing an appropriate sequence of functionals F_ε that we know they admit a minimizer and the $\Gamma-$limit of F_ε is F_0. Also vise virsa (see [23]), we can obtain information for the F_ε energy functional from the properties of minimizers of the limiting functional F_0.

We now provide the definition of $\Gamma-$convergence in metric spaces.

Definition 2.1. Let X be a metric space, and for $\varepsilon > 0$ let $F_\varepsilon\,,\ F\ : X \to [0, +\infty]$. We say that F_ε $\Gamma-$converge to F on X as $\varepsilon \to 0$, if the following two condition hold

$$\begin{gathered}(LB)\ \forall\ \ u \in X\ \text{ and }\ \forall\ \ (u_\varepsilon) \subset X\ \text{ such that }\ u_\varepsilon \to u\ \text{ in }\ X,\ \text{there holds} \\ F(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon)\end{gathered} \tag{2.1}$$

$$\begin{gathered}(UB)\ \forall\ \ u \in X,\ \text{ there exist }\ (u_\varepsilon) \subset X\ \text{ such that }\ u_\varepsilon \to u\ \text{ in }\ X\ \text{ and} \\ \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq F(u)\end{gathered} \tag{2.2}$$

If F_ε $\Gamma-$converge to F we write $F_\varepsilon \xrightarrow{\Gamma} F$.

Note: Sometimes we replace the strong convergence $u_\varepsilon \to u$ in X with the weak convergence $u_\varepsilon \rightharpoonup u$. In homogenization and in the relationship between $\Gamma-$convergence and G-convergence the weak convergence is often more convenient. In fact, there is a more general definition of $\Gamma-$convergence in topological spaces and we refer the reader in [6] (or [2]) for further details.

Remark 2.2. Notice also that if (LB) inequality holds, we can replace the (UB) inequality in the definition 2.1, by

$$\begin{aligned} &\forall\ u \in X,\ \text{ there exist }\ (u_\varepsilon) \subset X\ \text{ such that }\ u_\varepsilon \to u\ \text{ in }\ X\ \text{ and} \\ &\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = F(u) \end{aligned} \tag{2.3}$$

The main properties of $\Gamma-$convergence are listed in the following statement:

Proposition 2.3. *We have the following:*
(i) The $\Gamma-$limit F is always lower semicontinuous on X.
(ii) (Stability under continuous perturbations) If $F_\varepsilon \xrightarrow{\Gamma} F$ and G is continuous, then $F_\varepsilon + G \xrightarrow{\Gamma} F + G$.
(iii) (Stability of minimizing sequences) If $F_\varepsilon \xrightarrow{\Gamma} F$ and u_ε minimizes F_ε over X, then every limit point of (u_ε) minimizes F over X.

Proof. (i) see Proposition 6.8 in [6] (or Proposition 1.28 in [2]).
(ii) If (2.1) holds, then for all $u \in X$ and $u_\varepsilon \to u$, we get

$$F(u) + G(u) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) + \lim_{\varepsilon \to 0} G(u_\varepsilon) = \liminf_{\varepsilon \to 0}(F_\varepsilon(u_\varepsilon) + G(u_\varepsilon)) \tag{2.4}$$

and if (2.2) inequality holds, in view of Remark 2.2 we have

$$F(u) + G(u) = \lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) + \lim_{\varepsilon \to 0} G(u_\varepsilon) = \lim_{\varepsilon \to 0}(F_\varepsilon(u_\varepsilon) + G(u_\varepsilon)) \tag{2.5}$$

and (u_ε) is a recovery sequence also for $F + G$.
(iii) Consider a subsequence (u_{ε_k}) such that $u_{\varepsilon_k} \to u$ in X. Without loss of generality we denote (u_{ε_k}) by (u_ε). Let $v \in X$, then by the (UB) inequality (2.2), $\exists\ v_\varepsilon$ such that

$$\limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \leq F(v) \tag{2.6}$$

In addition, by the (LB) inequality (2.1) and the minimality of u_ε,

$$F(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \leq \limsup_{\varepsilon\to 0} F_\varepsilon(v_\varepsilon) \leq F(v) \tag{2.7}$$

Thus, u is a minimizer of F. □

In particular, if F_ε satisfies the *Equicoercive Condition*:

$$(EC) \text{ If } F_\varepsilon(u_\varepsilon) \text{ is bounded, then } (u_\varepsilon) \text{ is precompact in } X \tag{2.8}$$

we have the following *fundamental theorem of* $\Gamma-$*convergence*:

Theorem 2.4. *If $F_\varepsilon \xrightarrow{\Gamma} F$ and F_ε satisfies (EC), then $\forall$ u_ε minimizing sequence of F_ε there exist a subsequence u_{ε_k} that converges to a minimizer of F.*

We will now provide two examples for the $\Gamma-$limit in a sequence of energy functionals. The first example arise from the theory of phase transition type problems and the Euler-Lagrange equations are the Allen-Cahn equations. The $\Gamma-$limit is the perimeter functional and thus a relationship between minimal surfaces and phase transition type problems occurs. The second example is from homogenization and shows that even if the pointwise limit of a sequence of functionals exist, it differs from the $\Gamma-$limit.

Example 2.5. *(Phase transition type problems and minimal surfaces)*
Let X be the space of the measurable functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ endowed with the L^1 norm and

$$F_\varepsilon(u,\Omega) := \begin{cases} \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)dx & , \ u \in W^{1,2}(\Omega;\mathbb{R}) \cap X \\ +\infty & , \ \textit{elsewhere in } X \end{cases}$$

$$F_0(u,\Omega) := \begin{cases} \sigma\mathcal{H}^{n-1}(Su) & , \ u \in BV(\Omega;\{-1,1\}) \cap X \\ +\infty & , \ \textit{elsewhere in } X \end{cases}$$

$$\textit{where } W : \mathbb{R} \to [0,+\infty) \ , \ \{W = 0\} = \{-1,1\} \ , \ \sigma = \int_{-1}^{1} \sqrt{2W(u)}du$$

and Su is the singular set of the BV function u.

Let now u_ε be a minimizer of F_ε subject to a mass constraint, that is, $\int_\Omega u = V \in (0, |\Omega|)$. The asymptotic behavior of u_ε was first studied by Modica and Mortola in [14] and by Modica in [15, 16].

To be more precise, they proved that $\Gamma-$limit $F_\varepsilon = F_0$. We briefly describe a possible physical meaning of this problem. Su is often considered as the interface between two fluids (i.e. the set of discontinuity points of u) and by Theorem 2.4 this interface minimizes the perimeter functional, which means that the surface that is formed between the two fluids is eventually a minimal surface subject to a volume constraint. In other words, the level sets of the minimizers u_ε of F_ε converge to minimal surfaces in some sense.

So, one of the most important outcomes of $\Gamma-$convergence in the scalar phase transition type problems is the relationship with minimal surfaces. This relationship is deeper as indicated in the De Giorgi conjecture (see [8]), which states that the level sets of global entire solutions of the scalar Allen-Cahn equations that are bounded and strictly monotone with respect to x_n, are hyperplanes if $n \leq 8$. The relationship with the Bernstein problem for minimal graphs is the reason why $n \leq 8$ appears in the conjecture. The Γ-limit of the $\varepsilon-$energy functional F_ε of the Allen-Cahn equation is a possible motivation behind the conjecture.

Example 2.6. *(An example from homogenization)*
Let X be the class of all $u \in H^1(0,1)$ such that $u(0) = u(1) = 0$, endowed with the strong topology of $L^2(0,1)$. Let A be the $1-$periodic function defined as

$$A(x) = \begin{cases} \alpha_1 & , \quad x \in [0, 1/2) \\ \alpha_2 & , \quad x \in [1/2, 1) \end{cases}$$

with $0 < \alpha_1 < \alpha_2 < +\infty$ and set

$$F_\varepsilon(u) := \int_0^1 A(\frac{x}{\varepsilon})|u'(x)|^2 dx \tag{2.9}$$

The functional F_ε $\Gamma-$converge on X to

$$F(u) := \alpha \int_0^1 |u'(x)|^2 dx \quad , \quad \alpha = \frac{2\alpha_1\alpha_2}{\alpha_1 + \alpha_2} \tag{2.10}$$

This is a simple example of homogenization. We just give a sketch for the proof and for further details and the proof of the $\Gamma-$limit see Chapter 24 and 25 in [6]).

1) Start with the constructive part of the proof, that is, with the upper bound inequality. Take u affine on $(0,1)$ and show that (2.2) can be fulfilled by suitable approximating functions u which are affine on every interval where $a(x/\varepsilon)$ is constant, that is, intervals of the type $[n\varepsilon, (n+1/2)\varepsilon)$ with $n = 0, 1, \ldots$.
2) Extend the previous construction to every u which is piecewise affine on $(0,1)$.
3) Since for every $u \in H^1(0,1)$ there exists an approximating sequence u_n by piecewise affine functions such that $u_n \to u$ and $F(u_n) \to F(u)$, a simple diagonal argument shows it is enough to verify condition (2.2) only for piecewise affine functions. So, use a proper density argument to conclude the proof of the upper bound inequality.
4) Try to understand why the approximation proposed in step 2 is optimal, and then prove the lower bound inequality, i.e. (2.1).

The choice of the L^2-topology is for the following reason:
Since

$$F_\varepsilon(u) \geq \alpha_1 \int_0^1 |u'(x)|^2 dx,$$

when $F_\varepsilon(u_\varepsilon)$ is bounded, the functions u_ε are weakly pre-compact in $H^1(0,1)$, but not strongly. Hence the compactness condition (EC) (i.e. (2.8)) is verified if we endow X with the L^2 topology (recall that the weak topology of H^1 is not metrizable, and anyhow conditions (LB) and (UB) in the Definition 2.1 remain unchanged if we replace L^2-topology with the weak H^1-topology.
Note also that the pointwise limit of F_ε as $\varepsilon \to 0$ is

$$\overline{F}(u) = \overline{\alpha} \int_0^1 |u'(x)|^2 dx \quad , \quad \text{where } \overline{\alpha} = \frac{\alpha_1 + \alpha_2}{2},$$

while α is the harmonic mean of α_1 , α_2, since $\alpha = \frac{2}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}}$ and so, $\alpha < \overline{\alpha}$.

Chapter 3

An introduction to G-convergence

The notion of G-convergence for second order linear elliptic operators was introduced by De Giorgi and S. Spagnolo in [10], [27], [28], [29] as the convergence, in a suitable topology, of the Green's operator associated to the Dirichlet boundary value problems, in the symmetric case. Moreover, there is a relation of G-convergence with $\Gamma-$convergence as we will see in the next chapter.

We note that throughout the rest of the chapters we use the notation $\langle \cdot, \cdot \rangle$ for both the inner product in a Hilbert space and for the inner product in $\mathbb{R}^n$.

Definition 3.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ and α , β constants such that $0 < \alpha \leq \beta < +\infty$. We denote by $\mathcal{M}_s(\Omega, \alpha, \beta)$ the set of all $n \times n$ symmetric matrices $A : \Omega \to \mathbb{M}^{n\times n}$, $(A_{ij} = A_{ji})$ satisfying the following properties:
(I) $A \in L^\infty(\Omega)^{n\times n}$ and $||A||_{L^\infty(\Omega)^{n\times n}} \leq \beta$. (i.e. $A_{ij} \in L^\infty(\Omega)$, $i, j \in \{1, ..., n\}$ and $|A(x)\xi| \leq \beta|\xi|$, for a.e. $x \in \Omega$ and $\forall\ \xi \in \mathbb{R}^n$)
(II) (Equicoercivity) $\exists\ \ \alpha > 0$ such that $A(x) \geq \alpha I$ a.e. in Ω, where I is the identity matrix. (i.e. $\langle A(x)\xi, \xi\rangle \geq \alpha|\xi|^2$, for a.e. $x \in \Omega$, $\forall \xi \in \mathbb{R}^n$)

Let A^ε be a sequence in $\mathcal{M}_s(\Omega, \alpha, \beta)$ and let $f \in H^{-1}(\Omega)$ (for the sake of simplicity and without loss of generality, we consider the right hand side term independent of ε). Let u^ε be the unique solution of

$$\begin{cases} -div(A^\varepsilon \nabla u^\varepsilon) = f \ \ , \text{ in } \ H^{-1}(\Omega) \\ u^\varepsilon \in H^1_0(\Omega) \end{cases} \tag{3.1}$$

The equation (3.1) can be written in the form

$$\int_\Omega A^\varepsilon \nabla u^\varepsilon \nabla v = \langle f, v\rangle \ \ , \ \forall\ v \in H^1_0(\Omega) \tag{3.2}$$

and for $v = u^\varepsilon$ this gives

$$\alpha||u^\varepsilon||_{H_0^1(\Omega)} \leq ||f||_{H^{-1}(\Omega)} \tag{3.3}$$

Hence, there exists a subsequence $u^{\varepsilon'}$ and $u \in H_0^1(\Omega)$ such that

$$u^{\varepsilon'} \rightharpoonup u \quad \text{weakly in } \; H_0^1(\Omega) \tag{3.4}$$

In addition, if we define $\sigma^\varepsilon = A^\varepsilon \nabla u^\varepsilon$ we have

$$||\sigma^\varepsilon||_{L^2(\Omega)^n} \leq ||A^\varepsilon||_{L^\infty(\Omega)^{n\times n}} ||\nabla u^\varepsilon||_{L^2(\Omega)^n} \leq \frac{\beta}{\alpha}||f||_{H^{-1}(\Omega)} \tag{3.5}$$

Thus, $\exists \; \sigma \in L^2(\Omega)^n$ such that

$$\begin{aligned} \sigma^{\varepsilon'} \rightharpoonup \sigma \quad &\text{weakly in } \; L^2(\Omega)^n \\ \text{and} \quad -div\sigma = f \quad &\text{in the sense of distributions} \end{aligned} \tag{3.6}$$

So the problem is the following: What can we say about u and σ? Does u satisfy an equation of the same type as u^ε?

Definition 3.2. (*G-convergence*) Let $A^\varepsilon \in \mathcal{M}_s(\Omega, \alpha, \beta)$ and let $A^0 \in \mathcal{M}_s(\Omega, \alpha, \beta)$. We say that A^ε *G-converges* to A^0 if $\forall \; f \in H^{-1}(\Omega)$ the solutions u^ε of the equations

$$\begin{cases} -div(A^\varepsilon(x)\nabla u^\varepsilon) = f \;\;, \text{ in } \; \Omega \\ u^\varepsilon \in H_0^1(\Omega) \end{cases} \tag{3.7}$$

satisfy $u^\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$, where u is the solution of

$$\begin{cases} -div(A^0(x)\nabla u) = f \;\;, \text{ in } \; \Omega \\ u \in H_0^1(\Omega) \end{cases} \tag{3.8}$$

We can replace the weak convergence in $H_0^1(\Omega)$ with the strong convergence in $L^2(\Omega)$ or some other convergence and the definition remains unchanged.

The main result (which motivates the definition) is the sequential compactness of the class of symmetric functions belonging to $\mathcal{M}_s(\Omega, \alpha, \beta)$ with respect to G-convergence.

Theorem 3.3. *(Homogenization) Let $(A^\varepsilon) \subset \mathcal{M}_s(\Omega, \alpha, \beta)$, then there is a subsequence $A^{\varepsilon'}$ of (A^ε) and $A^0 \in \mathcal{M}_s(\Omega, \alpha, \beta)$, such that $A^{\varepsilon'}$ G-converges to A^0.*

Proof. We will provide the proof in Chapter 5 for the non symmetric case in H-convergence which is more general. □

The original proof of Spagnolo is rather technical and uses results of semigroup theory for linear operators and the G-convergence of parabolic equations. Many different proofs have been given subsequently (see for example [26], [31]).

We note also that the G-convergence satisfy a localization property:

Theorem 3.4. *(localization) Assume that (A^ε) , (B^ε) , A^0 , B^0 belong in $\mathcal{M}_s(\Omega, \alpha, \beta)$. If (A^ε) G-converges to A^0 , (B^ε) G-converges to B^0 and $A^\varepsilon(x) = B^\varepsilon(x)$ for a.e. $x \in \Omega' \subset \Omega$ open, then $A^0(x) = B^0(x)$ for a.e. $x \in \Omega'$.*

Proof. See [29]. □

Additionally, the G-limit is unique:

Theorem 3.5. *(uniqueness) The G-limit of a G-converging sequence $(A^\varepsilon) \subset \mathcal{M}_s(\Omega, \alpha, \beta)$ is unique.*

Example 3.6. *(Homogenization of elliptic operators)*

Let $a_{ij} \in L^\infty(\mathbb{R}^n)$ and consider

$$f(x, \xi) = \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ ,$$

where $A = (a_{ij})$ is a $n \times n$ symmetric matrix satisfying

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \tag{3.9}$$

for a.e. $x \in \Omega$ *and for every* $\xi \in \mathbb{R}^n$.

Let $\xi \in \mathbb{R}^n$ *and let* $w_\xi \in H^1_{loc}(\mathbb{R}^n)$ *is such that* ∇w_ξ *is* $Y-$*periodic,* $\int_Y \nabla w_k dy = \xi$ *and*

$$\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j w_\xi) = 0 \tag{3.10}$$

It holds that w_ξ *is unique up to an additive constant and*

$$w_\xi = \sum_{k=1}^{n} \xi_k w_k(y) + c \tag{3.11}$$

where $e_1, ..., e_n$ *is the canonical basis of* $\mathbb{R}^n$, $c \in \mathbb{R}$ *and we write* w_k *instead of* w_{e_k}.

Consider now

$$f_0(\xi) = \int_Y \left(\sum_{i,j=1}^{n} a_{ij}\partial_i w_\xi \partial_j w_\xi \right) dy = \sum_{h,l=1}^{n} d_{hl}\xi_h\xi_l \tag{3.12}$$

where

$$d_{hl} = \int_Y \left(\sum_{i,j=1}^{n} a_{ij}\partial_i w_h \partial_j w_l \right) dy \tag{3.13}$$

for $h, l = 1, ..., n$. *The computation of* $f_0(\xi)$ *is therefore reduced to the solutions of the* n *boundary value problems* (3.10) *corresponding to* $\xi = e_1, ..., e_n$.

For every $\varepsilon > 0$ *let* A^ε *be the elliptic operator defined by*

$$A^\varepsilon u = -\sum_{i,j=1}^{n} \partial_i(a^\varepsilon_{ij}\partial_j u) \tag{3.14}$$

where $a^\varepsilon_{ij}(x) = a_{ij}(\frac{x}{\varepsilon})$, *and let* A^0 *be the elliptic operator defined by*

$$A^0 u = -\sum_{i,j=1}^{n} \partial_i(d_{ij}\partial_j u) \tag{3.15}$$

By Theorem 22.4 in [6], the sequence $(A^{\varepsilon'})$ *G-converges to* A^0 *in the strong topology of* $L^2(\Omega)$ *for every sequence* (ε') *of real positive numbers such that* $\varepsilon' \to 0$ *and for every bounded open* $\Omega \subset \mathbb{R}^n$.

Chapter 4

The relation of G-convergence and $\Gamma-$convergence

In this chapter we will see the relationship between G-convergence and $\Gamma-$convergence. To be more precise, there is a connection between the $\Gamma-$convergence of lower semicontinuous quadratic forms that are coercive and the G-convergence of the corresponding self-adjoint operators. We present the theorem in the case of elliptic operators with corresponding coefficients that belong in the class $\mathcal{M}_s(\Omega, \alpha, \beta)$ and we refer the reader to Theorem 13.5 and 13.12 in [6] for further details in the more general setting.

Theorem 4.1. *Let A^ε , $A^0 \in \mathcal{M}_s(\Omega, \alpha, \beta)$ and consider*

$$\begin{aligned} F_\varepsilon(u) &= \int_\Omega A^\varepsilon \nabla u \nabla u dx \\ F_0(u) &= \int_\Omega A^0 \nabla u \nabla u dx \end{aligned} \tag{4.1}$$

Then the following conditions are equivalent:
(i) F_ε $\Gamma-$converges to F_0 in the weak topology of $H_0^1(\Omega)$.
(ii) For any linear map $H(u) = \langle f, u \rangle$, $f \in H_0^1(\Omega)$, it holds

$$\min_{u \in H_0^1(\Omega)} (F_0(u) + H(u)) = \lim_{\varepsilon \to 0} \min_{u \in H_0^1(\Omega)} (F_\varepsilon(u) + H(u))$$

(iii) A^ε G-converges to A^0 in the weak topology of $H_0^1(\Omega)$.
These equivalences also hold if we replace the weak topology of $H_0^1(\Omega)$ with the strong topology of $L^2(\Omega)$.

Proof. (i)⇒(ii). Let $f \in H_0^1(\Omega)$ and H be a linear map defined as $H(u) = \langle f, u\rangle$ for every $u \in H_0^1(\Omega)$, where $\langle \cdot, \cdot\rangle$ is the inner product in $H_0^1(\Omega)$. By the second property in Proposition 2.3 the sequence of functionals $F_\varepsilon + H$ $\Gamma-$converges to $F_0 + H$ in the weak topology of $H_0^1(\Omega)$. By the property (II) in definition 3.1 and Hölder inequality we have that

$$F_\varepsilon(u) + H(u) \geq \alpha C||u||^2_{H_0^1(\Omega)} - ||f||_{H_0^1(\Omega)}||u||_{H_0^1(\Omega)} \tag{4.2}$$

This means that $F_\varepsilon(u) + H(u)$ tends to $+\infty$ as $||u||_{H_0^1(\Omega)}$ tends to $+\infty$ and since $H_0^1(\Omega)$ is reflexible, it holds that $F_\varepsilon + H$ is equi-coercive. Then we apply Theorem 7.8 in [6] and we conclude.

(ii)⇒(iii). Assume (ii), then for every $f \in H_0^1(\Omega)$, the points $u^0 = (\mathcal{A}^0)^{-1}f$ and $u^\varepsilon = (\mathcal{A}^\varepsilon)^{-1}f$ are the minimum points of the functionals $F_0(v) - 2\langle f, v\rangle$ and $F_\varepsilon(v) - 2\langle f, v\rangle$ respectively, where

$$\mathcal{A}^\varepsilon u^\varepsilon = -div(A^\varepsilon \nabla u^\varepsilon) \quad \text{and} \quad \mathcal{A}^0 u^0 = -div(A^0 \nabla u^0) \tag{4.3}$$

It suffices to prove that $u^\varepsilon \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$. We have $\mathcal{A}^\varepsilon u^\varepsilon = f$ and $\mathcal{A}^0 u^0 = f$ and thus

$$\begin{gathered} F_\varepsilon(u^\varepsilon) = \langle A^\varepsilon \nabla u^\varepsilon, \nabla u^\varepsilon\rangle = \langle \mathcal{A}^\varepsilon u^\varepsilon, u^\varepsilon\rangle = \langle f, u^\varepsilon\rangle \\ \Rightarrow \langle f, u^\varepsilon\rangle = -F_\varepsilon(u^\varepsilon) + 2\langle f, u^\varepsilon\rangle = -\min_{v \in H_0^1(\Omega)}(F_\varepsilon(v) - 2\langle f, v\rangle) \\ \text{and similarly } \langle f, u^0\rangle = -\min_{v \in H_0^1(\Omega)}(F_0(v) - 2\langle f, v\rangle) \end{gathered} \tag{4.4}$$

so we get

$$\langle f, (\mathcal{A}^0)^{-1}f\rangle = \lim_{\varepsilon \to 0}\langle f, (\mathcal{A}^\varepsilon)^{-1}f\rangle \tag{4.5}$$

for every $f \in H_0^1(\Omega)$. Now we apply (4.5) to $f + g$ and $f - g$ and by the polarization identity this implies

$$\langle g, (\mathcal{A}^0)^{-1}f\rangle = \lim_{\varepsilon \to 0}\langle g, (\mathcal{A}^\varepsilon)^{-1}f\rangle \tag{4.6}$$

for every $f, g \in H_0^1(\Omega)$. Hence $(\mathcal{A}^\varepsilon)^{-1}f$ converges to $(\mathcal{A}^0)^{-1}f$ weakly in $H_0^1(\Omega)$ for every $f \in H_0^1(\Omega)$. That is, u^ε converges weakly in $H_0^1(\Omega)$ to u^0 and we conclude.

The last assertion, (iii)⇒(i), is more technical and we refer the reader to the proof of Theorem 13.5 in [6] (and to Theorem 13.12 for the case of strong convergence). □

Remark 4.2. If $(A^\varepsilon) \subset \mathcal{M}_s(\Omega,\alpha,\beta)$ and $A^\varepsilon \to A^0$ strongly in $L^\infty(\Omega)^{n\times n}$, we can pass to the limit in $A^\varepsilon \nabla u^\varepsilon$ and we have:

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u \quad , \text{ weakly in } L^2(\Omega)^n \tag{4.7}$$

and hence u is the solution (unique since $A^0 \in \mathcal{M}_s(\Omega,\alpha,\beta)$) to

$$\begin{cases} -div(A^0\nabla u) = f \ , \text{ in } \ \Omega \\ u \in H_0^1(\Omega) \end{cases} \tag{4.8}$$

We note that the previous result does not hold if we do not have the strong convergence of the sequence (A^ε). That is, in view of Theorem 3.3, the G-limit of A^ε (i.e. A^0) is not equal with the weak* L^∞ limit of A^ε. In the example 2.6 in chapter 2, we illustrated a case where

$$F_\varepsilon(u) = \int_0^1 A(\frac{x}{\varepsilon})|u'(x)|^2 dx \quad , \quad F_0(u) = \alpha \int_0^1 |u'(x)|^2 dx \tag{4.9}$$

where $\alpha = \dfrac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}$ is the harmonic mean of α_1 , α_2 and it holds that $F_\varepsilon \xrightarrow{\Gamma} F_0$. By theorem 4.1 it holds that $A^\varepsilon = A(\frac{x}{\varepsilon})$ G-converges to $A^0 = \alpha I$, where I is the identity matrix (the equivalence of the two notions of convergence also holds with respect to the strong convergence).

On the other hand, the pointwise limit of F_ε as $\varepsilon \to 0$ is

$$\overline{F}(u) = \overline{\alpha} \int_0^1 |u'(x)|^2 dx$$

where $\overline{\alpha} = \dfrac{\alpha_1+\alpha_2}{2}$.

In other words, A^ε converges weakly* in L^∞ to $\overline{A} = \overline{\alpha} I$, and $\overline{\alpha} > \alpha = \dfrac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}$, since $\alpha_1 < \alpha_2$. Therefore, in this case we have a strict inequality $\overline{A} > A^0$ (that is, $\langle \overline{A}\xi,\xi\rangle > \langle A^0\xi,\xi\rangle \ , \ \forall \ \xi \in \mathbb{R}^n$).

However, if A^0 is the G-limit of A^ε, in general it holds that

$$A^0 \le \overline{A} \quad , \text{ where } \overline{A} \text{ is the weak* } L^\infty \text{ limit of } A^\varepsilon. \tag{4.10}$$

One way to see this inequality is by Proposition 5.1 in [6], where there is a similar inequality that compare the $\Gamma-$limit of a sequence of functionals with the pointwise limit, and then utilize theorem 4.1.

A simpler way to prove (4.10) is the following:
Let A^0 be the G-limit of A^ε , $\overline{A}$ be the weak* L^∞ limit of A^ε and $\underline{A}$ be the the weak* L^∞ limit of $(A^\varepsilon)^{-1}$. Then it holds

$$\langle (A^\varepsilon)^{-1}(A^\varepsilon\lambda - \mu), (A^\varepsilon\lambda - \mu)\rangle \geq 0 \ \ , \ \forall\lambda \ , \ \mu \in \mathbb{R}^n.$$

$$\Rightarrow \langle \lambda, A^\varepsilon\lambda\rangle - \langle \lambda, \mu\rangle - \langle (A^\varepsilon)^{-1}\mu, A^\varepsilon\lambda\rangle + \langle (A^\varepsilon)^{-1}\mu, \mu\rangle \geq 0$$

$$\Rightarrow \langle A^\varepsilon\lambda, \lambda\rangle - 2\langle \lambda, \mu\rangle + \langle (A^\varepsilon)^{-1}\mu, \mu\rangle \geq 0$$

since A^ε is self-adjoint. Now by taking the weak* L^∞ limit we have

$$\langle \overline{A}\lambda, \lambda\rangle - 2\langle \lambda, \mu\rangle + \langle \underline{A}\mu, \mu\rangle \geq 0$$

so we set $\mu = (\underline{A})^{-1}\lambda$ and thus

$$\langle \overline{A}\lambda, \lambda\rangle \geq \langle (\underline{A})^{-1}\lambda, \lambda\rangle$$

and we conclude since $(\underline{A})^{-1} = A^0$ (see also Definition 13.3 in [6]).

Chapter 5

H-convergence and the Homogenization theorem

The notion of G-convergence has been extended to the non-symmetric case by Murat and Tartar under the name of H-convergence (see [17], [34] and [35]). This notion of convergence has been extended to sequences of matrices which are not necessarily symmetric and it allow us to deal with problems that do not have a variational structure in general.

One could define similarly to the case of G-convergence the space of matrices in Definition 3.1 that are not necessarily symmetric. However if we define it this way, the H-limit provided by compactness Theorem 5.3 is in a larger class of matrices as we can see in [22] (or [3],[11]). So we follow the definition as in the recent lecture of Professor F. Murat.

Definition 5.1. Let Ω be a bounded open subset of $\mathbb{R}^n$ and α , β constants such that $0 < \alpha \leq \beta < +\infty$. We denote by $\mathcal{M}(\Omega, \alpha, \beta)$ the set of all $n \times n$ matrices $A : \Omega \to \mathbb{M}^{n\times n}$ defined as:

$$\mathcal{M}(\Omega, \alpha, \beta) := \left\{ A \in L^\infty(\Omega)^{n\times n} \ : \ A \geq \alpha I \ \text{ and } \ A^{-1} \geq \frac{1}{\beta} I \ \text{ a.e. } x \in \Omega \right\} \tag{5.1}$$

where by $A \geq \alpha I$ we denote $\langle A(x)\xi, \xi\rangle \geq \alpha|\xi|^2$, $\forall \xi \in \mathbb{R}^n$ and I is the identity matrix.

Notice that, the $\frac{1}{\beta}-$coercivity of the inverse matrix A^{-1} gives in fact that $|A\xi| \leq \beta|\xi|$ for a.e. $x \in \Omega$ and $\forall \ \xi \in \mathbb{R}^n$, which is the property (I) in the Definition 3.1.

Let A^ε be a sequence in $\mathcal{M}(\Omega, \alpha, \beta)$ and let $f \in H^{-1}(\Omega)$ (for simplicity and without loss of generality we consider the right hand side term independent of ε).

Definition 5.2. (*H-convergence*) Let $A^\varepsilon \in \mathcal{M}(\Omega, \alpha, \beta)$ and let $A^0 \in \mathcal{M}(\Omega, \alpha', \beta')$. We say that A^ε *H-converges* to A^0 if $\forall\ f \in H^{-1}(\Omega)$ the solutions u^ε of the equations

$$\begin{cases} -div(A^\varepsilon(x)\nabla u^\varepsilon) = f \ \ , \text{ in } \ \Omega \\ u^\varepsilon \in H_0^1(\Omega) \end{cases} \tag{5.2}$$

satisfy

$$\begin{cases} (i) \ \ u^\varepsilon \rightharpoonup u^0 \ \text{ weakly in } \ H_0^1(\Omega) \\ (ii) \ \ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 u^0 \ \text{ weakly in } \ L^2(\Omega)^n \end{cases} \tag{5.3}$$

where u^0 is the solution of

$$\begin{cases} -div(A^0(x)\nabla u^0) = f \ \ , \text{ in } \ \Omega \\ u^0 \in H_0^1(\Omega) \end{cases} \tag{5.4}$$

Let us point out the main difference between these two notions of convergence. G-convergence deals with symmetric matrices and supposes the convergence of the solutions u^ε only. H-convergence is defined for general sequences (not necessarily symmetric) and supposes not only the convergence of solutions u^ε but also the convergence of $A^\varepsilon \nabla u^\varepsilon$. The main feature of H-convergence is that the additional condition on the convergence of $A^\varepsilon \nabla u^\varepsilon$ is essential in order to keep the main three properties stated in Chapter 3 for the G-convergence, i.e. (i) uniqueness (Theorem 3.5), (ii) locality (Theorem 3.4) and (iii) compactness (Theorem 3.3).

The main result, which motivates the definition of H-convergence, as in the case of G-convergence, is the sequential compactness with respect to the H-convergence.

Theorem 5.3. *(Homogenization) Let* $(A^\varepsilon) \subset \mathcal{M}(\Omega, \alpha, \beta)$*, then there is a subsequence* $A^{\varepsilon'}$ *of* (A^ε) *and* $A^0 \in \mathcal{M}(\Omega, \alpha, \beta)$*, such that* $A^{\varepsilon'}$ *H-converges to* A^0*.*

We note here again that, if we give an alternative definition of the space of matrices similar to Definition 3.1 (without symmetry), the H-limit provided by compactness Theorem 5.3 is in a larger class of matrices as we can see in [22] (or [3],[11]). In any case, the Proposition 5.6 that relates the two notions of convergence remains unchanged.

Before the proof of Theorem 5.3, we state some additional properties of H-convergence. As mentioned previously, the two of the main properties are:

Theorem 5.4. *(localization) Assume that* (A^ε) , (B^ε) *belong in* $\mathcal{M}(\Omega, \alpha, \beta)$ *which H-converge respectively to* A^0 *and* B^0. *If* $A^\varepsilon(x) = B^\varepsilon(x)$ *for a.e.* $x \in \Omega' \subset \Omega$ *open, then* $A^0(x) = B^0(x)$ *for a.e.* $x \in \Omega'$.

Theorem 5.5. *(uniqueness) The H-limit of a H-converging sequence* $(A^\varepsilon) \subset \mathcal{M}(\Omega, \alpha, \beta)$ *is unique.*

For a proof of Theorems 5.4 and 5.5 we refer to Proposition 1 in [22] or [11].

A natural question is what is the relation between the two converges for a sequence of symmetric matrices?

The answer is given by the next proposition.

Proposition 5.6. *Let* A^ε *be a sequence of symmetric matrices in* $\mathcal{M}_s(\Omega, \alpha, \beta)$, *then G-convergence is equivalent to H-convergence.*

Proof. The proof of this result makes use of a comparison theorem and we refer for it to De Giorgi and Spagnolo [10] and to Tartar [36]. □

Corollary 5.7. *Let* $(A^\varepsilon) \subset \mathcal{M}_s(\Omega, \alpha, \beta)$ *which G-converges to* A^0. *Then* $A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^0 \nabla u^0$ *weakly in* $L^2(\Omega)^n$.

One of the main tools for proving Theorem 5.3 is the compensated compactness due to F. Murat and L. Tartar (see for instance [17] and [32]).

As it is well known, the product of two weakly convergent sequences does not converge in general to the product of limits and this is the principal difficulty when trying to characterize the weak limit of $A^\varepsilon \nabla u^\varepsilon$ is terms of u^0. The compensated compactness show that under some additional assumptions, the product of two weak convergent sequences in $L^2(\Omega)^n$ converges in the sense of distributions to the product of the limits.

This result is interesting itself and is widely used in many applications

Theorem 5.8. *(div-curl lemma/ compensated compactness) Let $\Omega \subset \mathbb{R}^n$ open and (X^ε) , $(Y^\varepsilon) \subset L^2(\Omega)^n$ such that*

$$\begin{cases} X^\varepsilon \rightharpoonup X^0 \;\; \text{weakly in} \;\; L^2(\Omega)^n \\ Y^\varepsilon \rightharpoonup Y^0 \;\; \text{weakly in} \;\; L^2(\Omega)^n \end{cases} \tag{5.5}$$

Suppose that $(divX^\varepsilon)$ is compact in $H^{-1}(\Omega)$ and $(curlY^\varepsilon)$ is compact in $L^2(\Omega)^{n\times n}$, where the matrix $curlY^\varepsilon = ((curlY^\varepsilon)_{ij})_{1\leq i,j\leq n}$ is defined by

$$(curlY^\varepsilon)_{ij} = \frac{\partial Y_i^\varepsilon}{\partial x_j} - \frac{\partial Y_j^\varepsilon}{\partial x_i} \;\; , \; i,j = 1,...,n \tag{5.6}$$

Then

$$X^\varepsilon \cdot Y^\varepsilon = \sum_{i=1}^n X_i^\varepsilon Y_i^\varepsilon \rightharpoonup X^0 \cdot Y^0 = \sum_{i=1}^n X_i^0 Y_i^0 \;\; , \; in \; \mathcal{D}' \tag{5.7}$$

In the framework of H-convergence, the above theorem can be applied to the case:

$$X^\varepsilon = A^\varepsilon \nabla u^\varepsilon \quad \text{and} \quad Y^\varepsilon = \nabla v^\varepsilon$$

Proof of Theorem 5.8. So, we will prove the above theorem in the special case where $Y^\varepsilon = \nabla v^\varepsilon$, where v^ε bounded in $H^1(\Omega)$ and thus $v^\varepsilon \rightharpoonup v^0$ weakly in $H^1(\Omega)$ $(Y_k^\varepsilon = \frac{\partial v_i^\varepsilon}{\partial x_k}$ and so $curlY^\varepsilon \equiv 0)$.

We want to prove that $\forall\, \phi \in C_c^\infty(\Omega)$,

$$\int_\Omega \phi X^\varepsilon Y^\varepsilon \to \int_\Omega \phi X^0 Y^0 \tag{5.8}$$

where $Y^\varepsilon = \nabla v^\varepsilon$ and $Y^0 = \nabla v^0$.

We have

$$\begin{aligned} \int_\Omega \phi X^\varepsilon Y^\varepsilon = \int_\Omega \sum_{i=1}^n X_i^\varepsilon \frac{\partial v^\varepsilon}{\partial x_i} = -\int_\Omega \sum_{i=1}^n \frac{\partial \phi}{\partial x_i} X_i^\varepsilon v^\varepsilon - \int_\Omega \sum_{i=1}^n \phi \frac{\partial X_i^\varepsilon}{\partial x_i} v^\varepsilon \\ \Rightarrow \int_\Omega \phi X^\varepsilon \nabla v^\varepsilon = -\int_\Omega \nabla \phi X^\varepsilon v^\varepsilon - \int_\Omega divX^\varepsilon \phi v^\varepsilon \end{aligned} \tag{5.9}$$

For the first term of the last equation we have that $X^\varepsilon \rightharpoonup X^0$ weakly in $L^2(\Omega)^n$ and $v^\varepsilon \rightharpoonup v^0$ weakly in $H^1(\Omega)$ and by the Rellich–Kondrachov theorem, $v^\varepsilon \to v^0$ strongly in $L^2(\Omega)$.

Therefore

$$\int_\Omega \nabla\phi X^\varepsilon v^\varepsilon \to \int_\Omega \nabla\phi X^0 v^0 \tag{5.10}$$

For the second term, since $(divX^\varepsilon)$ is compact in $H^{-1}(\Omega)$ and $\phi v^\varepsilon \rightharpoonup \phi v^0$ weakly in $H^1_0(\Omega)$ it holds

$$\int_\Omega divX^\varepsilon \phi v^\varepsilon \to \int_\Omega divX^0 \phi v^0 \tag{5.11}$$

Thus,

$$\int_\Omega \phi X^\varepsilon \nabla v^\varepsilon \to -\int_\Omega \nabla\phi X^0 v^0 - \int_\Omega divX^0 \phi v^0 = \int_\Omega \phi X^0 \nabla v^0 \ , \ \forall \phi \in C_c^\infty(\Omega) \tag{5.12}$$

and we conclude. □

A proof of Theorem 5.8 can also be found in [12] (see Theorem 5.2.1).

In the proof of theorem 5.3 we also utilize the following:

Theorem 5.9. *Under the assumptions of Theorem 5.3, $\forall\ i = 1, ..., n$ there exist $w^{\varepsilon''} \in H^1(\Omega)$, where $(\varepsilon'') \subset (\varepsilon')$ and there exist h_i such that*

$$\begin{cases} w_i^{\varepsilon''} \rightharpoonup x_i \text{ weakly in } H^1(\Omega) \\ -div(A^{\varepsilon''}\nabla w_i^{\varepsilon''}) = h_i \in H^{-1}(\Omega) \end{cases} \tag{5.13}$$

and

$$A^{\varepsilon''}\nabla w_i^{\varepsilon''} \rightharpoonup \xi_i \text{ weakly in } L^2(\Omega)^n. \tag{5.14}$$

In addition, if we define a matrix $A^0 \in L^2(\Omega)^{n\times n}$ by

$$A^0 e_i = \xi_i \quad , \quad \text{where } \{e_i\}_{i=1,\dots,n} \text{ is the canonical basis of } \mathbb{R}^n \tag{5.15}$$

then it holds that $A^0 \in L^\infty(\Omega)^{n\times n}$ and $A^0 \geq \alpha I$.

Remark 5.10. The intuition for this theorem is given by the "Corrector result" in Theorem 5.14 which tell us that if u^ε is the solution of (5.2), then

$$\nabla u^\varepsilon \approx \sum_i \nabla w_i^\varepsilon \frac{\partial u^0}{\partial x_i}$$
$$\text{and } \nabla w_i^\varepsilon \rightharpoonup e_i \text{ weakly in } L^2$$

plus a τ^ε term that converges to zero strongly in $L^1(\Omega)^n$.

Now we proceed to the proof of the Homogenization theorem and we will prove theorem 5.9 afterwards.

Proof of Theorem 5.3. Let $\lambda \in \mathbb{R}^n$, by Theorem 5.9 we have

$$\sum_{i=1}^{n} \lambda_i w_i^{\varepsilon''} \rightharpoonup \sum_{i=1}^{n} \lambda_i x_i$$

weakly in $H^1(\Omega)$ and $\sum_{i=1}^{n} \nabla(\lambda_i x_i) = \sum_{i=1}^{n} \lambda_i e_i = \lambda$, where $e_1, ..., e_n$ is the canonical basis of $\mathbb{R}^n$.

Set

$$\begin{aligned} \mathscr{E}^{\varepsilon} &= A^{\varepsilon''}(\nabla u^{\varepsilon''} - \nabla w^{\varepsilon''})(\nabla u^{\varepsilon''} - \nabla w^{\varepsilon''}) \\ &= (\sigma^{\varepsilon''} - A^{\varepsilon''}\nabla w^{\varepsilon''})(\nabla u^{\varepsilon''} - \nabla w^{\varepsilon''}) \end{aligned} \tag{5.16}$$

For the first term of the last equation equals

$$\begin{aligned} &div\left(\sigma^{\varepsilon''} - A^{\varepsilon''}\nabla w^{\varepsilon''}\right) = -f + \sum_{i=1}^{n} \lambda_i h_i \\ \text{and } \ &\sigma^{\varepsilon''} - A^{\varepsilon''}\nabla w^{\varepsilon''} \rightharpoonup \sigma_0 - \sum_{i=1}^{n} \lambda_i \xi_i =: \sigma^0 - A^0 \lambda \end{aligned} \tag{5.17}$$

by Theorem 5.9.

The second term of the last equality in (5.16) has zero curl and

$$\nabla u^{\varepsilon''} - \nabla w^{\varepsilon''} \rightharpoonup \nabla u^0 - \lambda \quad , \quad \text{weakly in } L^2(\Omega)^n \tag{5.18}$$

Therefore, we utilize the div-curl lemma (that is, Theorem 5.8) by setting $X^{\varepsilon''} = \sigma^{\varepsilon''} - A^{\varepsilon''}\nabla w^{\varepsilon''}$ and $Y^{\varepsilon''} = \nabla u^{\varepsilon''} - \nabla w^{\varepsilon''}$ and obtain

$$\int_{\Omega} \phi \mathscr{E}^{\varepsilon} \to \int_{\Omega} \phi(\sigma^0 - A^0\lambda)(\nabla u^0 - \lambda) \ \ , \text{ in } \mathcal{D}'(\Omega) \ \ \forall \phi \in C_c^{\infty}(\Omega) \tag{5.19}$$

and $0 \leq \mathscr{E}^{\varepsilon}$, thus

$$\begin{aligned} &(\sigma^0 - A^0\lambda)(\nabla u^0(x) - \lambda) \geq 0 \quad , \quad \text{for a.e. } x \in \Omega \\ &\Rightarrow \sigma^0(x) = A^0(x)\nabla u^0(x) \quad , \quad \text{for a.e. } x \in \Omega \end{aligned} \tag{5.20}$$

Indeed, take the lebesgue points of σ^0 , ∇u^0 , A^0 and denote this set as Z. The first equation of (5.20) holds for all $x \in \Omega \setminus Z$, $\forall \lambda$, so for fixed x_0 and $\varepsilon > 0$ small, consider $\lambda = \nabla u^0(x_0) - \varepsilon\mu$, $\mu \in \mathbb{R}^n$, then

$$(\sigma^0(x_0) - A^0(x_0)(\nabla u^0(x_0) - \varepsilon\mu))\mu \geq 0 \tag{5.21}$$

so taking the limit as $\varepsilon \to 0$,

$$
\begin{aligned}
(\sigma^0(x_0) - A^0(x_0)\nabla u^0(x_0))\mu \geq 0 \ \ , \ \forall \mu \in \mathbb{R}^n \\
\Rightarrow \sigma^0(x_0) = A^0(x_0)\nabla u^0(x_0)
\end{aligned}
\tag{5.22}
$$

as claimed. □

Abstract Setting

For the proof of Theorem 5.9 we will utilize an abstract result for linear operators, in the spirit of Lax Milgram formulation.

First, we provide the following lemma.

Lemma 5.11. *Let V , W be two Banach spaces, where V is separable and W is reflexible. Let $\mathcal{A}^\varepsilon \in \mathcal{L}(V, W)$ such that $||\mathcal{A}^\varepsilon||_{\mathcal{L}(V,W)} \leq C$.*

Then there exists a subsequence $(\mathcal{A}^{\varepsilon'})$ and $\mathcal{A}^0 \in \mathcal{L}(V, W)$ such that $\forall f \in V$,

$$
\mathcal{A}^{\varepsilon'} f \rightharpoonup \mathcal{A}^0 f \ \ , \ \textit{weakly in } W. \tag{5.23}
$$

Proof. A diagonal process ensures the existence of a subsequence such that $\mathcal{A}^{\varepsilon'} x$ has a weak limit in W denoted by $\mathcal{A}^0 x$, $\forall x \in X$, where X is the countable dense subset of V. For $f \in V$ and $g \in W'$ we can prove that $\langle \mathcal{A}^{\varepsilon'} f, g\rangle_{W,W'}$ is Caushy sequence (by approximating f by elements $x \in X$). Denote by $\langle \mathcal{A}^0 f, g\rangle_{W,W'}$ the limit, then $\mathcal{A}^0$ is linear and bounded, thus

$$
||\mathcal{A}^0 f||_W \leq \liminf ||\mathcal{A}^{\varepsilon'} f||_W \leq C||f||_V \tag{5.24}
$$

□

We denote by V reflexible and separable Banach space with dual V' and by $\langle \cdot, \cdot \rangle_{V,V'}$ the dual pairing between V, V'.

Proposition 5.12. *Let* $\mathcal{A}^\varepsilon \in \mathcal{L}(V, V')$ *a sequence of linear operators such that*
(i) $\mathcal{A}^\varepsilon$ *is* $\alpha-$*coercive, that is,* $\langle \mathcal{A}^\varepsilon v, v\rangle_{V',V} \geq \alpha ||v||_V^2$, $\forall v \in V$,
(ii) $(\mathcal{A}^\varepsilon)^{-1}$ *exist and it is* $\frac{1}{\beta}-$*coercive, that is,* $\langle (\mathcal{A}^\varepsilon)^{-1} g, g\rangle_{V,V'} \geq \frac{1}{\beta}||g||_{V'}^2$, $\forall g \in V'$.

Then there exists a subsequence $(\mathcal{A}^{\varepsilon'})^{-1}$ *and* $\mathcal{A}^0 \in \mathcal{L}(V, V')$ *with* $\mathcal{A}^0$ $\alpha-$*coercive and* $(\mathcal{A}^0)^{-1}$ $\frac{1}{\beta}-$*coercive, such that* $\forall$ $f \in V'$

$$\begin{gathered} \mathcal{A}^{\varepsilon'} u^{\varepsilon'} = f \\ \textit{and } u^{\varepsilon'} \rightharpoonup u^0 \textit{ weakly in } V \textit{ , with } \mathcal{A}^0 u^0 = f \end{gathered} \tag{5.25}$$

Note that $u^{\varepsilon'} \rightharpoonup u^0$ *can be also written as* $(\mathcal{A}^{\varepsilon'})^{-1} f \rightharpoonup (\mathcal{A}^0)^{-1} f$.

Proof. Since $\mathcal{A}^\varepsilon$ is $\alpha-$coercive, we know that $(\mathcal{A}^\varepsilon)^{-1}$ exists and

$$||(\mathcal{A}^\varepsilon)^{-1}||_{\mathcal{L}(V',V)} \leq \frac{1}{\alpha}.$$

From Lemma 5.11, there exists a subsequence $(\mathcal{A}^{\varepsilon'})$ and $B^0 \in \mathcal{L}$(V,V') such that $||B^0||_{\mathcal{L}(\text{V,V'})} \leq \frac{1}{\alpha}$ and $\forall f \in V'$,

$$(\mathcal{A}^{\varepsilon'})^{-1} f \rightharpoonup B^0 f \text{ , weakly in } V. \tag{5.26}$$

Without loss of generality we still denote the subsequence by $\mathcal{A}^\varepsilon$.

We know that $(\mathcal{A}^\varepsilon)^{-1}$ is $\frac{1}{\beta}-$coercive, so

$$\begin{gathered} \langle (\mathcal{A}^\varepsilon)^{-1} f, f\rangle \geq \frac{1}{\beta}||f||^2 \\ \Rightarrow \langle B^0 f, f\rangle \geq \frac{1}{\beta}||f||^2 \end{gathered} \tag{5.27}$$

and thus B^0 is $\frac{1}{\beta}-$coercive and in particular B^0 is invertible.

We denote as $\mathcal{A}^0 := (B^0)^{-1}$. It remains to show that $\mathcal{A}^0$ is $\alpha-$coercive. For $f \in V'$, take $u^\varepsilon = (\mathcal{A}^\varepsilon)^{-1} f$ and we have

$$\begin{gathered} \alpha ||u^\varepsilon|| \leq \langle \mathcal{A}^\varepsilon u^\varepsilon, u^\varepsilon\rangle = \langle f, u^\varepsilon\rangle \\ \text{and } u^\varepsilon = (\mathcal{A}^\varepsilon)^{-1} f \rightharpoonup (\mathcal{A}^0)^{-1} f =: u^0 \end{gathered} \tag{5.28}$$

so, by weak lower semicontinuity of the norm

$$\alpha ||u^0||^2 \leq \langle f, u^0\rangle = \langle \mathcal{A}^0 u^0, u^0\rangle \tag{5.29}$$

for all $u^0 \in V$, since equations (5.28) and (5.29) hold for every $f \in V'$. □

Proof of Theorem 5.9. Let $\hat{\Omega}$ open bounded set such that $\Omega \subset \hat{\Omega}$ (for example $\hat{\Omega}$ is a big ball and we enlarge the domain so that we can have zero boundary conditions). Let $\hat{\psi} \in C_c^\infty(\hat{\Omega})$ such that $\hat{\psi} = 1$ on Ω.

In $\hat{\Omega}$, we consider the problem

$$\begin{cases} \hat{u}^\varepsilon \in H_0^1(\hat{\Omega}) \\ -div(\hat{A}^\varepsilon \nabla \hat{u}^\varepsilon) = \hat{f} \ \ , \text{ in } \ \mathcal{D}' \end{cases} \tag{5.30}$$

where

$$\hat{A}^\varepsilon(x) = \begin{cases} A^\varepsilon(x) \ \ , \ x \in \Omega \\ \alpha I \ \ , \ x \in \hat{\Omega} \setminus \Omega \end{cases} \tag{5.31}$$

or even $\hat{A}^\varepsilon$ is any matrix in $\mathcal{M}(\hat{\Omega}, \alpha, \beta)$ which coincides with A^ε in Ω. We consider the operator $\hat{\mathcal{A}}^\varepsilon$ defined in (5.30) by

$$\hat{\mathcal{A}}^\varepsilon \hat{u}^\varepsilon = \hat{f} \ \text{ in } \ H^{-1}(\hat{\Omega}) \tag{5.32}$$

and we apply to $\hat{\mathcal{A}}^\varepsilon$ the abstract setting of Lax Milgram in $(H_0^1(\hat{\Omega}), H^{-1}(\hat{\Omega}))$, i.e. Proposition 5.12 and thus there exist $\hat{\mathcal{A}}^0$ and a subsequence $(\hat{\mathcal{A}}^{\varepsilon'})$ such that

$$\hat{u}^{\varepsilon'} \rightharpoonup u^0 \quad (\hat{u}^{\varepsilon'} = (\hat{\mathcal{A}}^{\varepsilon'})^{-1} f) \tag{5.33}$$

We choose

$$\hat{h}_i = \hat{\mathcal{A}}^0(x_i \hat{\psi}(x)) \quad \text{and} \quad \hat{h}_i \in H^{-1}(\hat{\Omega}) \tag{5.34}$$

and note that $x_i \hat{\psi}(x) \in H_0^1(\hat{\Omega})$.

So we define $\hat{w}_i^{\varepsilon''}$ by

$$\begin{cases} \hat{w}_i^\varepsilon \in H_0^1(\hat{\Omega}) \\ -div(\hat{A}^\varepsilon \nabla \hat{w}_i^\varepsilon) = \hat{h}_i \ \text{ in } \ \mathcal{D}'(\hat{\Omega}) \end{cases} \tag{5.35}$$

up to subsequnce that we still denote as $\hat{w}_i^\varepsilon$. That is, $\hat{\mathcal{A}}^\varepsilon \hat{w}_i^\varepsilon = \hat{h}_i$. Then by Proposition 5.12,

$$\begin{gathered} \hat{w}_i^\varepsilon \rightharpoonup (\mathcal{A}^0)^{-1} \hat{h}_i \ \text{ in } \ H_0^1(\hat{\Omega}) \\ \text{and } \ (\hat{\mathcal{A}}^0)^{-1}(\mathcal{A}^0(x_i \hat{\psi}(x))) = x_i \hat{\psi}(x) \ \text{ in } \ H_0^1(\hat{\Omega}) \ \text{ (which is } \ x_i \ \text{ in } \ \Omega) \\ \hat{w}_i^\varepsilon|_\Omega \rightharpoonup x_i \hat{\psi}(x)|_\Omega = x_i \ \text{ weakly in } \ H^1(\Omega) \end{gathered} \tag{5.36}$$

and $w_i^\varepsilon := \hat{w}_i^\varepsilon|_\Omega \rightharpoonup x_i$ weakly in $H^1(\Omega)$, therefore equation (5.35) reads

$$\begin{cases} w_i^\varepsilon \in H_0^1(\Omega) \\ -div(A^\varepsilon \nabla w_i^\varepsilon) = \hat{h}_i|_\Omega \ \text{ in } \ \mathcal{D}'(\Omega) \end{cases} \tag{5.37}$$

Finally, it remains to prove that A^0 defined in (5.15) is $\alpha-$coercive and it's inverse is $\frac{1}{\beta}-$coercive and so in $L^\infty(\Omega)^{n\times n}$. We have $A^0 e_i = \xi_i$ where recall that ξ are the functions such that $A^{\varepsilon''}\nabla w^{\varepsilon''} \rightharpoonup \xi$ weakly in L^2.

Let $\lambda \in \mathbb{R}^n$, take the function

$$w^\varepsilon = \sum_i \lambda_i w_i^\varepsilon \rightharpoonup \sum_i \lambda_i x_i = w^0 \tag{5.38}$$

up to subsequence and $\nabla w^0 = \lambda$, so

$$\int_\Omega A^\varepsilon \nabla w^\varepsilon \nabla w^\varepsilon \phi \geq \alpha \int_\Omega |\nabla w^\varepsilon|^2 \phi \quad , \quad \forall \phi \in C_c^\infty(\Omega)\ ,\ \phi \geq 0 \tag{5.39}$$

and

$$\int_\Omega |\nabla w^\varepsilon|^2 \phi \geq \int_\Omega |\nabla w^0|^2 \phi \quad , \quad \text{by weak lower semicontinuity.} \tag{5.40}$$

Now by the div-curl lemma, i.e. Theorem 5.8 with $X^\varepsilon = A^\varepsilon \nabla u^\varepsilon$, $Y^\varepsilon = \nabla w^\varepsilon$ we have

$$\int_\Omega A^0 \nabla w^0 \nabla w^0 \phi \geq \alpha \int_\Omega |\nabla w^0|^2 \phi \tag{5.41}$$

That is,

$$\int_\Omega A^0 \lambda\lambda\phi \geq \alpha \int_\Omega |\lambda|^2 \phi \quad , \quad \forall \phi \in C_c^\infty(\Omega) \text{ and } \forall \lambda \in \mathbb{R}^n. \tag{5.42}$$

Thus, A^0 is $\alpha-$coercive.

For the $\frac{1}{\beta}-$coercivity of $(A^0)^{-1}$, let $\lambda \in \mathbb{R}^n$ and consider again the functions w^ε , w^0 as in (5.38),

$$I^\varepsilon = \int_\Omega \phi (A^\varepsilon)^{-1}(A^\varepsilon \nabla w^\varepsilon)(A^\varepsilon \nabla w^\varepsilon) \geq \frac{1}{\beta}\int_\Omega |A^\varepsilon \nabla w^\varepsilon|^2 \phi \quad , \quad \forall \phi \in C_c^\infty(\Omega)\ ,\ \phi \geq 0 \tag{5.43}$$

and

$$\int_\Omega |A^\varepsilon \nabla w^\varepsilon|^2 \phi \geq \int_\Omega |A^0\lambda|^2 \phi \quad \text{by weak lower semicontinuity} \tag{5.44}$$

$I^\varepsilon = \int_\Omega \phi \nabla w^\varepsilon A^\varepsilon \nabla w^\varepsilon$ and by Theorem 5.8 we obtain

$$\begin{gathered} I^\varepsilon \to \int_\Omega \phi\lambda A^0 \lambda = \int_\Omega (A^0)^{-1}(A^0\lambda)(A^0\lambda)\phi \\ \Rightarrow \int_\Omega (A^0)^{-1}(A^0\lambda)(A^0\lambda)\phi \geq \frac{1}{\beta}\int_\Omega |A^0\lambda|^2\phi \quad , \quad \forall \phi \in C_c^\infty(\Omega) \end{gathered} \tag{5.45}$$

$$\Rightarrow (A^0)^{-1}(A^0\lambda)(A^0\lambda) \geq \frac{1}{\beta}|A^0\lambda|^2 \quad , \text{a.e. } x \in \Omega \tag{5.46}$$

Let $\mu \in \mathbb{R}^n$, since A^0 is invertible we can choose λ so that $A^0\lambda = \mu$ and by (5.46) we get

$$(A^0)^{-1}\mu\mu \geq \frac{1}{\beta}|\mu|^2 \quad , \text{a.e. } x \in \Omega \text{ and } \forall \mu \in \mathbb{R}^n \tag{5.47}$$

Therefore $(A^0)^{-1}$ is $\frac{1}{\beta}-$coercive and $A^0 \in L^\infty(\Omega)^{n\times n}$. □

The Corrector result

Definition 5.13. (Corrector) Suppose (A^ε) H-converges to A^0 and let u^ε be the solution of

$$\begin{cases} -div(A^\varepsilon \nabla u^\varepsilon) = f \quad , \text{ in } \Omega \\ u^\varepsilon \in H_0^1(\Omega) \end{cases} \tag{5.48}$$

The *corrector* matrix $P^\varepsilon = (P^\varepsilon_{ij})_{i,j\in\{1,\ldots,n\}} \in L^2(\Omega)^{n\times n}$ is defined by

$$P^\varepsilon_{ij} = \frac{\partial w^\varepsilon_j}{\partial x_i} \tag{5.49}$$

where w_j defined in Theorem 5.9 and it holds that

$$P^\varepsilon \rightharpoonup I \text{ weakly in } L^2(\Omega)^{n\times n} \ (I \, : \text{ the identity matrix}) \tag{5.50}$$

Then, H-convergence implies in particular that

$$\nabla u^\varepsilon - P^\varepsilon \nabla u^0 \rightharpoonup 0 \, , \text{ weakly in } L^1(\Omega)^n. \tag{5.51}$$

The main corrector result is

Theorem 5.14. *Suppose (A^ε) H-converges to A^0 and u^ε be the solution of (5.48). Let (P^ε) be any sequence of corrector matrices given by definition 5.13.*
Then

$$\nabla u^\varepsilon - P^\varepsilon \nabla u^0 \to 0 \, , \text{ strongly in } L^1(\Omega)^n. \tag{5.52}$$

In other words, we have the expression:

$$\begin{aligned} \nabla u^\varepsilon = P^\varepsilon \nabla u^0 + \tau^\varepsilon \quad &(\text{definition of the remainder of } u^\varepsilon) \\ \text{and } \quad \tau^\varepsilon \to 0 \quad &\text{strongly in } L^1(\Omega) \end{aligned} \tag{5.53}$$

so ∇u^ε is "very close" to $P^\varepsilon \nabla u^0$ (similar to a 1^{st} order Taylor expansion in some sense).

Proof. We consider for simplicity the case where $u^0 \in C_c^\infty(\Omega)$. Let

$$\mathcal{E}_\varepsilon = \int_\Omega \varphi A^\varepsilon(\nabla u^\varepsilon - P^\varepsilon\phi)(\nabla u^\varepsilon - P^\varepsilon\phi)dx \quad , \quad \varphi \in C_c^\infty(\Omega) \ , \ \varphi \geq 0$$

and $\phi \in C_c^\infty(\Omega)^n$. So, we have

$$\mathcal{E}_\varepsilon \geq \int_\Omega \varphi\alpha|\nabla u^\varepsilon - P^\varepsilon\phi|^2 dx$$

and we apply Theorem 5.8 with $X^\varepsilon = A^\varepsilon(\nabla u^\varepsilon - P^\varepsilon\phi)$ and $Y^\varepsilon = \nabla u^\varepsilon - P^\varepsilon\phi$,

$$\begin{aligned} &\Rightarrow \int_\Omega X^\varepsilon Y^\varepsilon \to \int_\Omega \varphi A^0(\nabla u^0 - \phi)(\nabla u^0 - \phi) \\ \Rightarrow \limsup_\varepsilon \alpha \int_\Omega \varphi|\nabla u^\varepsilon - P^\varepsilon\phi|^2 dx &\leq \int_\Omega \varphi A^0(\nabla u^0 - \phi)(\nabla u^0 - \phi) \\ &\leq \int_\Omega \varphi\beta|\nabla u^\varepsilon - P^\varepsilon\phi|^2 \ , \ \forall\phi \in C_c^\infty(\Omega)^n \end{aligned} \tag{5.54}$$

so since $u^0 \in C_c^\infty(\Omega)$, we take $\phi = \nabla u^0$ and we conclude. In this case, actually we have strong convergence in $L^2(\Omega)^n$.

In the general case, we consider $\tau^\varepsilon = \nabla u^\varepsilon - P^\varepsilon \nabla u^0$ and we write

$$\tau^\varepsilon = (\nabla u^\varepsilon - P^\varepsilon\phi) + P^\varepsilon(\phi - \nabla u^0)$$

The first term is handled as,

$$\begin{aligned} \limsup_\varepsilon \int_\Omega |\nabla u^\varepsilon - P^\varepsilon\phi| &\leq |\Omega|^{1/2}(\limsup_\varepsilon \int_\Omega |\nabla u^\varepsilon - P^\varepsilon\phi|^2)^{1/2} \\ &\leq \beta|\Omega|^{1/2}||\nabla u^0 - \phi||_{L^2(\Omega)^n} \end{aligned} \tag{5.55}$$

arguing as in the previous simpler case.

For the second term,

$$\begin{aligned} ||P^\varepsilon(\nabla u^0 - \phi)||_{L^1(\Omega)^n} &\leq ||P^\varepsilon||_{L^2(\Omega)^{n\times n}}||\nabla u^0 - \phi||_{L^2(\Omega)^n} \\ &\leq (\limsup_\varepsilon ||P^\varepsilon||_{L^2(\Omega)^{n\times n}})||\nabla u^0 - \phi||_{L^2(\Omega)^n} \end{aligned} \tag{5.56}$$

Thus we obtain

$$\begin{gathered} \limsup_\varepsilon ||\tau^\varepsilon||_{L^1} \leq C||\nabla u^0 - \phi||_{L^2} \\ \text{where } \ C = C(\beta, |\Omega|^{1/2}, \limsup_\varepsilon ||P^\varepsilon||_{L^2}) \end{gathered} \tag{5.57}$$

so we take ϕ such that given $\delta > 0$, $||\nabla u^0 - \phi||_{L^2} < \frac{\delta}{C}$ and we conclude. □

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