"Ch. 17—Queueing Theory" Excel Files:

Template for $M/M/s$ Model
Template for Finite Queue Variation of $M/M/s$ Model
Template for Finite Calling Population Variation of $M/M/s$ Model
Template for $M/G/1$ Model
Template for $M/D/1$ Model
Template for $M/E_k/1$ Model
Template for Nonpreemptive Priorities Model
Template for Preemptive Priorities Model
Template for $M/M/s$ Economic Analysis of Number of Servers

"Ch. 17—Queueing Theory" LINGO File for Selected Examples

Glossary for Chapter 17

See Appendix 1 for documentation of the software.

PROBLEMS[1]

To the left of each of the following problems (or their parts), we have inserted a T whenever one of the templates listed above can be helpful. An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

17.2-1.* Consider a typical barber shop. Demonstrate that it is a queueing system by describing its components.

17.2-2.* Newell and Jeff are the two barbers in a barber shop they own and operate. They provide two chairs for customers who are waiting to begin a haircut, so the number of customers in the shop varies between 0 and 4. For $n = 0, 1, 2, 3, 4$, the probability P_n that exactly n customers are in the shop is $P_0 = \frac{1}{16}$, $P_1 = \frac{4}{16}$, $P_2 = \frac{6}{16}$, $P_3 = \frac{4}{16}$, $P_4 = \frac{1}{16}$.

(a) Calculate L. How would you describe the meaning of L to Newell and Jeff?
(b) For each of the possible values of the number of customers in the queueing system, specify how many customers are in the queue. Then calculate L_q. How would you describe the meaning of L_q to Newell and Jeff?
(c) Determine the expected number of customers being served.
(d) Given that an average of 4 customers per hour arrive and stay to receive a haircut, determine W and W_q. Describe these two quantities in terms meaningful to Newell and Jeff.
(e) Given that Newell and Jeff are equally fast in giving haircuts, what is the average duration of a haircut?

17.2-3. Mom-and-Pop's Grocery Store has a small adjacent parking lot with three parking spaces reserved for the store's customers. During store hours, cars enter the lot and use one of the spaces at a mean rate of 2 per hour. For $n = 0, 1, 2, 3$, the probability P_n that exactly n spaces currently are being used is $P_0 = 0.2$, $P_1 = 0.3$, $P_2 = 0.3$, $P_3 = 0.2$.

(a) Describe how this parking lot can be interpreted as being a queueing system. In particular, identify the customers and the servers. What is the service being provided? What constitutes a service time? What is the queue capacity?
(b) Determine the basic measures of performance—L, L_q, W, and W_q—for this queueing system.
(c) Use the results from part (*b*) to determine the average length of time that a car remains in a parking space.

17.2-4. For each of the following statements about the queue in a queueing system, label the statement as true or false and then justify your answer by referring to a specific statement in the chapter.

(a) The queue is where customers wait in the queueing system until their service is completed.
(b) Queueing models conventionally assume that the queue can hold only a limited number of customers.
(c) The most common queue discipline is first-come-first-served.

17.2-5. Midtown Bank always has two tellers on duty. Customers arrive to receive service from a teller at a mean rate of 40 per hour. A teller requires an average of 2 minutes to serve a customer. When both tellers are busy, an arriving customer joins a single line to wait for service. Experience has shown that customers wait in line an average of 1 minute before service begins.

(a) Describe why this is a queueing system.
(b) Determine the basic measures of performance—W_q, W, L_q, and L—for this queueing system. (*Hint*: We don't know the probability distributions of interarrival times and service times for this queueing system, so you will need to use the relationships between these measures of performance to help answer the question.)

[1]See also the end of Chap. 26 (on the CD-ROM) for additional problems involving the application of queueing theory.

17.2-6. Explain why the utilization factor ρ for the server in a single-server queueing system must equal $1 - P_0$, where P_0 is the probability of having 0 customers in the system.

17.2-7. You are given two queueing systems, Q_1 and Q_2. The mean arrival rate, the mean service rate per busy server, and the steady-state expected number of customers for Q_2 are twice the corresponding values for Q_1. Let W_i = the steady-state expected waiting time in the system for Q_i, for $i = 1, 2$. Determine W_2/W_1.

17.2-8. Consider a single-server queueing system with *any* service-time distribution and *any* distribution of interarrival times (the *GI/G/*1 model). Use only basic definitions and the relationships given in Sec. 17.2 to verify the following general relationships:
(a) $L = L_q + (1 - P_0)$.
(b) $L = L_q + \rho$.
(c) $P_0 = 1 - \rho$.

17.2-9. Show that

$$L = \sum_{n=0}^{s-1} nP_n + L_q + s\left(1 - \sum_{n=0}^{s-1} P_n\right)$$

by using the statistical definitions of L and L_q in terms of the P_n.

17.3-1. Identify the customers and the servers in the queueing system in each of the following situations:
(a) The checkout stand in a grocery store.
(b) A fire station.
(c) The toll booth for a bridge.
(d) A bicycle repair shop.
(e) A shipping dock.
(f) A group of semiautomatic machines assigned to one operator.
(g) The materials-handling equipment in a factory area.
(h) A plumbing shop.
(i) A job shop producing custom orders.
(j) A secretarial typing pool.

17.4-1. Suppose that a queueing system has two servers, an exponential interarrival time distribution with a mean of 2 hours, and an exponential service-time distribution with a mean of 2 hours for each server. Furthermore, a customer has just arrived at 12:00 noon.
(a) What is the probability that the next arrival will come (i) before 1:00 P.M., (ii) between 1:00 and 2:00 P.M., and (iii) after 2:00 P.M.?
(b) Suppose that no additional customers arrive before 1:00 P.M. Now what is the probability that the next arrival will come between 1:00 and 2:00 P.M.?
(c) What is the probability that the number of arrivals between 1:00 and 2:00 P.M. will be (i) 0, (ii) 1, and (iii) 2 or more?
(d) Suppose that both servers are serving customers at 1:00 P.M. What is the probability that *neither* customer will have service completed (i) before 2:00 P.M., (ii) before 1:10 P.M., and (iii) before 1:01 P.M.?

17.4-2.* The jobs to be performed on a particular machine arrive according to a *Poisson* input process with a mean rate of two per hour. Suppose that the machine breaks down and will require 1 hour to be repaired. What is the probability that the number of new jobs that will arrive during this time is (*a*) 0, (*b*) 2, and (*c*) 5 or more?

17.4-3. The time required by a mechanic to repair a machine has an exponential distribution with a mean of 4 hours. However, a special tool would reduce this mean to 2 hours. If the mechanic repairs a machine in less than 2 hours, he is paid \$100; otherwise, he is paid \$80. Determine the mechanic's expected increase in pay per machine repaired if he uses the special tool.

17.4-4. A three-server queueing system has a controlled arrival process that provides customers in time to keep the servers continuously busy. Service times have an exponential distribution with mean 0.5.

You observe the queueing system starting up with all three servers beginning service at time $t = 0$. You then note that the first completion occurs at time $t = 1$. Given this information, determine the expected amount of time after $t = 1$ until the next service completion occurs.

17.4-5. A queueing system has three servers with expected service times of 20 minutes, 15 minutes, and 10 minutes. The service times have an exponential distribution. Each server has been busy with a current customer for 5 minutes. Determine the expected remaining time until the next service completion.

17.4-6. Consider a queueing system with two types of customers. Type 1 customers arrive according to a Poisson process with a mean rate of 5 per hour. Type 2 customers also arrive according to a Poisson process with a mean rate of 5 per hour. The system has two servers, both of which serve both types of customers. For both types, service times have an exponential distribution with a mean of 10 minutes. Service is provided on a first-come-first-served basis.
(a) What is the probability distribution (including its mean) of the time between consecutive arrivals of customers of any type?
(b) When a particular type 2 customer arrives, she finds two type 1 customers there in the process of being served but no other customers in the system. What is the probability distribution (including its mean) of this type 2 customer's waiting time in the queue?

17.4-7. Consider a two-server queueing system where all service times are independent and identically distributed according to an exponential distribution with a mean of 10 minutes. Service is provided on a first-come-first-served basis. When a particular customer arrives, he finds that both servers are busy and no one is waiting in the queue.
(a) What is the probability distribution (including its mean and standard deviation) of this customer's waiting time in the queue?
(b) Determine the expected value and standard deviation of this customer's waiting time in the system.
(c) Suppose that this customer still is waiting in the queue 5 minutes after its arrival. Given this information, how does this change the expected value and the standard deviation of this customer's total waiting time in the system from the answers obtained in part (*b*)?

17.4-8. For each of the following statements regarding service times modeled by the exponential distribution, label the statement as true or false and then justify your answer by referring to specific statements (with page citations) in the chapter.

(a) The expected value and variance of the service times are always equal.

(b) The exponential distribution always provides a good approximation of the actual service-time distribution when each customer requires the same service operations.

(c) At an s-server facility, $s > 1$, with exactly s customers already in the system, a new arrival would have an expected waiting time before entering service of $1/\mu$ time units, where μ is the mean service rate for each busy server.

17.4-9. As for Property 3 of the exponential distribution, let $T_1, T_2, \ldots, T_n$ be independent exponential random variables with parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$, respectively, and let $U = \min\{T_1, T_2, \ldots, T_n\}$. Show that the probability that a particular random variable T_j will turn out to be smallest of the n random variables is

$$P\{T_j = U\} = \alpha_j \Big/ \sum_{i=1}^{n} \alpha_i, \qquad \text{for } j = 1, 2, \ldots, n.$$

(*Hint*: $P\{T_j = U\} = \int_0^\infty P\{T_i > T_j \text{ for all } i \neq j \mid T_j = t\}\alpha_j e^{-\alpha_j t}dt$.)

17.5-1. Consider the birth-and-death process with all $\mu_n = 2$ ($n = 1, 2, \ldots$), $\lambda_0 = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$, and $\lambda_n = 0$ for $n = 3, 4, \ldots$.

(a) Display the rate diagram.

(b) Calculate P_0, P_1, P_2, P_3, and P_n for $n = 4, 5, \ldots$.

(c) Calculate L, L_q, W, and W_q.

17.5-2. Consider a birth-and-death process with just three attainable states (0, 1, and 2), for which the steady-state probabilities are P_0, P_1, and P_2, respectively. The birth-and-death rates are summarized in the following table:

State	Birth Rate	Death Rate
0	1	—
1	1	2
2	0	2

(a) Construct the rate diagram for this birth-and-death process.

(b) Develop the balance equations.

(c) Solve these equations to find P_0, P_1, and P_2.

(d) Use the general formulas for the birth-and-death process to calculate P_0, P_1, and P_2. Also calculate L, L_q, W, and W_q.

17.5-3. Consider the birth-and-death process with the following mean rates. The birth rates are $\lambda_0 = 2$, $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, and $\lambda_n = 0$ for $n > 3$. The death rates are $\mu_1 = 3$, $\mu_2 = 4$, $\mu_3 = 1$, and $\mu_n = 2$ for $n > 4$.

(a) Construct the rate diagram for this birth-and-death process.

(b) Develop the balance equations.

(c) Solve these equations to find the steady-state probability distribution $P_0, P_1, \ldots$.

(d) Use the general formulas for the birth-and-death process to calculate $P_0, P_1, \ldots$. Also calculate L, L_q, W, and W_q.

17.5-4. Consider the birth-and-death process with all $\lambda_n = 2$ ($n = 0, 1, \ldots$), $\mu_1 = 2$, and $\mu_n = 4$ for $n = 2, 3, \ldots$.

(a) Display the rate diagram.

(b) Calculate P_0 and P_1. Then give a general expression for P_n in terms of P_0 for $n = 2, 3, \ldots$.

(c) Consider a queueing system with two servers that fits this process. What is the mean arrival rate for this queueing system? What is the mean service rate for each server when it is busy serving customers?

17.5-5.* A service station has one gasoline pump. Cars wanting gasoline arrive according to a Poisson process at a mean rate of 15 per hour. However, if the pump already is being used, these potential customers may *balk* (drive on to another service station). In particular, if there are n cars already at the service station, the probability that an arriving potential customer will balk is $n/3$ for $n = 1, 2, 3$. The time required to service a car has an exponential distribution with a mean of 4 minutes.

(a) Construct the rate diagram for this queueing system.

(b) Develop the balance equations.

(c) Solve these equations to find the steady-state probability distribution of the number of cars at the station. Verify that this solution is the same as that given by the general solution for the birth-and-death process.

(d) Find the expected waiting time (including service) for those cars that stay.

17.5-6. A maintenance person has the job of keeping two machines in working order. The amount of time that a machine works before breaking down has an exponential distribution with a mean of 10 hours. The time then spent by the maintenance person to repair the machine has an exponential distribution with a mean of 8 hours.

(a) Show that this process fits the birth-and-death process by defining the states, specifying the values of the λ_n and μ_n, and then constructing the rate diagram.

(b) Calculate the P_n.

(c) Calculate L, L_q, W, and W_q.

(d) Determine the proportion of time that the maintenance person is busy.

(e) Determine the proportion of time that any given machine is working.

(f) Refer to the nearly identical example of a *continuous time Markov chain* given at the end of Sec. 16.8. Describe the relationship between continuous time Markov chains and the birth-and-death process that enables both to be applied to this same problem.

17.5-7. Consider a single-server queueing system where interarrival times have an exponential distribution with parameter λ and service times have an exponential distribution with parameter μ. In addition, customers *renege* (leave the queueing system without being served) if their waiting time in the queue grows too large. In particular, assume that the time each customer is willing to wait in the queue before reneging has an exponential distribution with a mean of $1/\theta$.

(a) Construct the rate diagram for this queueing system.

(b) Develop the balance equations.

17.5-8.* A certain small grocery store has a single checkout stand with a full-time cashier. Customers arrive at the stand "randomly" (i.e., a Poisson input process) at a mean rate of 30 per hour. When there is only one customer at the stand, she is processed by the cashier alone, with an expected service time of 1.5 minutes. However, the stock boy has been given standard instructions that whenever there is more than one customer at the stand, he is to help the cashier by bagging the groceries. This help reduces the expected time required to process a customer to 1 minute. In both cases, the service-time distribution is exponential.

(a) Construct the rate diagram for this queueing system.

(b) What is the steady-state probability distribution of the number of customers at the checkout stand?

(c) Derive L for this system. (*Hint*: Refer to the derivation of L for the $M/M/1$ model at the beginning of Sec. 17.6.) Use this information to determine L_q, W, and W_q.

17.5-9. A department has one word-processing operator. Documents produced in the department are delivered for word processing according to a Poisson process with an expected interarrival time of 20 minutes. When the operator has just one document to process, the expected processing time is 15 minutes. When she has more than one document, then editing assistance that is available reduces the expected processing time for each document to 10 minutes. In both cases, the processing times have an exponential distribution.

(a) Construct the rate diagram for this queueing system.

(b) Find the steady-state distribution of the number of documents that the operator has received but not yet completed.

(c) Derive L for this system. (*Hint:* Refer to the derivation of L for the $M/M/1$ model at the beginning of Sec. 17.6.) Use this information to determine L_q, W, and W_q.

17.5-10. Customers arrive at a queueing system according to a Poisson process with a mean arrival rate of 2 customers per minute. The service time has an exponential distribution with a mean of 1 minute. An unlimited number of servers are available as needed so customers never wait for service to begin. Calculate the steady-state probability that exactly 1 customer is in the system.

17.5-11. Suppose that a single-server queueing system fits all the assumptions of the birth-and-death process *except* that customers always arrive in *pairs*. The mean arrival rate is 2 pairs per hour (4 customers per hour) and the mean service rate (when the server is busy) is 5 customers per hour.

(a) Construct the rate diagram for this queueing system.

(b) Develop the balance equations.

(c) For comparison purposes, display the rate diagram for the corresponding queueing system that completely fits the birth-and-death process, i.e., where customers arrive *individually* at a mean rate of 4 per hour.

17.5-12. Consider a single-server queueing system with a finite queue that can hold a maximum of 2 customers *excluding* any being served. The server can provide *batch service* to 2 customers simultaneously, where the service time has an exponential distribution with a mean of 1 unit of time regardless of the number being served. Whenever the queue is not full, customers arrive individually according to a Poisson process at a mean rate of 1 per unit of time.

(a) Assume that the server *must* serve 2 customers simultaneously. Thus, if the server is idle when only 1 customer is in the system, the server must wait for another arrival before beginning service. Formulate the queueing model as a continuous time Markov chain by defining the states and then constructing the rate diagram. Give the balance equations, but do not solve further.

(b) Now assume that the batch size for a service is 2 only if 2 customers are in the queue when the server finishes the preceding service. Thus, if the server is idle when only 1 customer is in the system, the server must serve this single customer, and any subsequent arrivals must wait in the queue until service is completed for this customer. Formulate the resulting queueing model as a continuous time Markov chain by defining the states and then constructing the rate diagram. Give the balance equations, but do not solve further.

17.5-13. Consider a queueing system that has two classes of customers, two clerks providing service, and *no queue.* Potential customers from each class arrive according to a Poisson process, with a mean arrival rate of 10 customers per hour for class 1 and 5 customers per hour for class 2, but these arrivals are lost to the system if they cannot immediately enter service.

Each customer of class 1 that enters the system will receive service from either one of the clerks that is free, where the service times have an exponential distribution with a mean of 5 minutes.

Each customer of class 2 that enters the system requires the *simultaneous use of both clerks* (the two clerks work together as a single server), where the service times have an exponential distribution with a mean of 5 minutes. Thus, an arriving customer of this kind would be lost to the system unless both clerks are free to begin service immediately.

(a) Formulate the queueing model as a continuous time Markov chain by defining the states and constructing the rate diagram.

(b) Now describe how the formulation in part (*a*) can be fitted into the format of the birth-and-death process.

(c) Use the results for the birth-and-death process to calculate the steady-state joint distribution of the number of customers of each class in the system.

(d) For each of the two classes of customers, what is the expected fraction of arrivals who are unable to enter the system?

17.6-1.* The 4M Company has a single turret lathe as a key work center on its factory floor. Jobs arrive at this work center according to a Poisson process at a mean rate of 2 per day. The processing time to perform each job has an exponential distribution with a mean of $\frac{1}{4}$ day. Because the jobs are bulky, those not being worked on are currently being stored in a room some distance from the machine. However, to save time in fetching the jobs, the production manager is proposing to add enough in-process storage space next to the turret lathe to accommodate 3 jobs in addition to the one being processed. (Excess jobs will continue to be stored temporarily in the distant room.) Under this proposal, what proportion of the time will this storage space next to the turret lathe be adequate to accommodate all waiting jobs?

(a) Use available formulas to calculate your answer.

T (b) Use the corresponding Excel template to obtain the probabilities needed to answer the question.

17.6-2. Customers arrive at a single-server queueing system according to a Poisson process at a mean rate of 10 per hour. If the server works continuously, the number of customers that can be served in an hour has a Poisson distribution with a mean of 15. Determine the proportion of time during which no one is waiting to be served.

17.6-3. Consider the *M*/*M*/1 model, with $\lambda < \mu$.

(a) Determine the steady-state probability that a customer's actual waiting time in the system is longer than the expected waiting time in the system, i.e., $P\{\mathcal{W} > W\}$.

(b) Determine the steady-state probability that a customer's actual waiting time in the queue is longer than the expected waiting time in the queue, i.e., $P\{\mathcal{W}_q > W_q\}$.

17.6-4. Verify the following relationships for an *M*/*M*/1 queueing system:

$$\lambda = \frac{(1-P_0)^2}{W_q P_0}, \qquad \mu = \frac{1-P_0}{W_q P_0}.$$

17.6-5. It is necessary to determine how much in-process storage space to allocate to a particular work center in a new factory. Jobs arrive at this work center according to a Poisson process with a mean rate of 3 per hour, and the time required to perform the necessary work has an exponential distribution with a mean of 0.25 hour. Whenever the waiting jobs require more in-process storage space than has been allocated, the excess jobs are stored temporarily in a less convenient location. If each job requires 1 square foot of floor space while it is in in-process storage at the work center, how much space must be provided to accommodate all waiting jobs (*a*) 50 percent of the time, (*b*) 90 percent of the time, and (*c*) 99 percent of the time? Derive an analytical expression to answer these three questions. *Hint:* The sum of a geometric series is

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}.$$

17.6-6. Consider the following statements about an *M*/*M*/1 queueing system and its utilization factor ρ. Label each of the statements as true or false, and then justify your answer.

(a) The probability that a customer has to wait before service begins is proportional to ρ.

(b) The expected number of customers in the system is proportional to ρ.

(c) If ρ has been increased from $\rho = 0.9$ to $\rho = 0.99$, the effect of any further increase in ρ on L, L_q, W, and W_q will be relatively small as long as $\rho < 1$.

17.6-7. Customers arrive at a single-server queueing system in accordance with a Poisson process with an expected interarrival time of 25 minutes. Service times have an exponential distribution with a mean of 30 minutes.

Label each of the following statements about this system as true or false, and then justify your answer.

(a) The server definitely will be busy forever after the first customer arrives.

(b) The queue will grow without bound.

(c) If a second server with the same service-time distribution is added, the system can reach a steady-state condition.

17.6-8. For each of the following statements about an *M*/*M*/1 queueing system, label the statement as true or false and then justify your answer by referring to specific statements (with page citations) in the chapter.

(a) The waiting time in the system has an exponential distribution.

(b) The waiting time in the queue has an exponential distribution.

(c) The conditional waiting time in the system, given the number of customers already in the system, has an Erlang (gamma) distribution.

17.6-9. The Friendly Neighbor Grocery Store has a single checkout stand with a full-time cashier. Customers arrive randomly at the stand at a mean rate of 30 per hour. The service-time distribution is exponential, with a mean of 1.5 minutes. This situation has resulted in occasional long lines and complaints from customers. Therefore, because there is no room for a second checkout stand, the manager is considering the alternative of hiring another person to help the cashier by bagging the groceries. This help would reduce the expected time required to process a customer to 1 minute, but the distribution still would be exponential.

The manager would like to have the percentage of time that there are more than two customers at the checkout stand down below 25 percent. She also would like to have no more than 5 percent of the customers needing to wait at least 5 minutes before beginning service, or at least 7 minutes before finishing service.

(a) Use the formulas for the *M*/*M*/1 model to calculate L, W, W_q, L_q, P_0, P_1, and P_2 for the current mode of operation. What is the probability of having more than two customers at the checkout stand?

T (b) Use the Excel template for this model to check your answers in part (*a*). Also find the probability that the waiting time before beginning service exceeds 5 minutes, and the probability that the waiting time before finishing service exceeds 7 minutes.

(c) Repeat part (*a*) for the alternative being considered by the manager.

(d) Repeat part (*b*) for this alternative.

(e) Which approach should the manager use to satisfy her criteria as closely as possible?

T **17.6-10.** The Centerville International Airport has two runways, one used exclusively for takeoffs and the other exclusively for landings. Airplanes arrive in the Centerville air space to request landing instructions according to a Poisson process at a mean rate of

10 per hour. The time required for an airplane to land after receiving clearance to do so has an exponential distribution with a mean of 3 minutes, and this process must be completed before giving clearance to do so to another airplane. Airplanes awaiting clearance must circle the airport.

The Federal Aviation Administration has a number of criteria regarding the safe level of congestion of airplanes waiting to land. These criteria depend on a number of factors regarding the airport involved, such as the number of runways available for landing. For Centerville, the criteria are (1) the average number of airplanes waiting to receive clearance to land should not exceed 1, (2) 95 percent of the time, the actual number of airplanes waiting to receive clearance to land should not exceed 4, (3) for 99 percent of the airplanes, the amount of time spent circling the airport before receiving clearance to land should not exceed 30 minutes (since exceeding this amount of time often would require rerouting the plane to another airport for an emergency landing before its fuel runs out).

(a) Evaluate how well these criteria are currently being satisfied.
(b) A major airline is considering adding this airport as one of its hubs. This would increase the mean arrival rate to 15 airplanes per hour. Evaluate how well the above criteria would be satisfied if this happens.
(c) To attract additional business [including the major airline mentioned in part (*b*)], airport management is considering adding a second runway for landings. It is estimated that this eventually would increase the mean arrival rate to 25 airplanes per hour. Evaluate how well the above criteria would be satisfied if this happens.

T **17.6-11.** The Security & Trust Bank employs 4 tellers to serve its customers. Customers arrive according to a Poisson process at a mean rate of 2 per minute. However, business is growing and management projects that the mean arrival rate will be 3 per minute a year from now. The transaction time between the teller and customer has an exponential distribution with a mean of 1 minute.

Management has established the following guidelines for a satisfactory level of service to customers. The average number of customers waiting in line to begin service should not exceed 1. At least 95 percent of the time, the number of customers waiting in line should not exceed 5. For at least 95 percent of the customers, the time spent in line waiting to begin service should not exceed 5 minutes.

(a) Use the *M/M/s* model to determine how well these guidelines are currently being satisfied.
(b) Evaluate how well the guidelines will be satisfied a year from now if no change is made in the number of tellers.
(c) Determine how many tellers will be needed a year from now to completely satisfy these guidelines.

17.6-12. Consider the *M/M/s* model.

T **(a)** Suppose there is one server and the expected service time is exactly 1 minute. Compare L for the cases where the mean arrival rate is 0.5, 0.9, and 0.99 customers per minute, respectively. Do the same for L_q, W, W_q, and $P\{\mathcal{W} > 5\}$. What conclusions do you draw about the impact of increasing the utilization factor ρ from small values (e.g., $\rho = 0.5$) to fairly large values (e.g., $\rho = 0.9$) and then to even larger values very close to 1 (e.g., $\rho = 0.99$)?
(b) Now suppose there are two servers and the expected service time is exactly 2 minutes. Follow the instructions for part (*a*).

T **17.6-13.** Consider the *M/M/s* model with a mean arrival rate of 10 customers per hour and an expected service time of 5 minutes. Use the Excel template for this model to obtain and print out the various measures of performance (with $t = 10$ and $t = 0$, respectively, for the two waiting time probabilities) when the number of servers is 1, 2, 3, 4, and 5. Then, for each of the following possible criteria for a satisfactory level of service (where the unit of time is 1 minute), use the printed results to determine how many servers are needed to satisfy this criterion.

(a) $L_q \le 0.25$
(b) $L \le 0.9$
(c) $W_q \le 0.1$
(d) $W \le 6$
(e) $P\{\mathcal{W}_q > 0\} \le 0.01$
(f) $P\{\mathcal{W} > 10\} \le 0.2$
(g) $\sum_{n=0}^{s} P_n \ge 0.95$

17.6-14. A gas station with only one gas pump employs the following policy: If a customer has to wait, the price is $1 per gallon; if she does not have to wait, the price is $1.20 per gallon. Customers arrive according to a Poisson process with a mean rate of 15 per hour. Service times at the pump have an exponential distribution with a mean of 3 minutes. Arriving customers always wait until they can eventually buy gasoline. Determine the expected price of gasoline per gallon.

17.6-15. You are given an *M/M/*1 queueing system with mean arrival rate λ and mean service rate μ. An arriving customer receives n dollars if n customers are already in the system. Determine the expected cost in dollars per customer.

17.6-16. Section 17.6 gives the following equations for the *M/M/*1 model:

$$(1)\quad P\{\mathcal{W} > t\} = \sum_{n=0}^{\infty} P_n P\{S_{n+1} > t\}.$$

$$(2)\quad P\{\mathcal{W} > t\} = e^{-\mu(1-\rho)t}.$$

Show that Eq. (1) reduces algebraically to Eq. (2). (*Hint:* Use differentiation, algebra, and integration.)

17.6-17. Derive W_q directly for the following cases by developing and reducing an expression analogous to Eq. (1) in Prob. 17.6-16. (*Hint:* Use the *conditional* expected waiting time in the queue given that a random arrival finds n customers already in the system.)

(a) The *M/M/*1 model
(b) The *M/M/s* model

T **17.6-18.** Consider an *M/M/*2 queueing system with $\lambda = 4$ and $\mu = 3$. Determine the mean rate at which service completions occur during the periods when no customers are waiting in the queue.

T 17.6-19. You are given an $M/M/2$ queueing system with $\lambda = 4$ per hour and $\mu = 6$ per hour. Determine the probability that an arriving customer will wait more than 30 minutes in the queue, given that at least 2 customers are already in the system.

17.6-20.* In the Blue Chip Life Insurance Company, the deposit and withdrawal functions associated with a certain investment product are separated between two clerks, Clara and Clarence. Deposit slips arrive randomly (a Poisson process) at Clara's desk at a mean rate of 16 per hour. Withdrawal slips arrive randomly (a Poisson process) at Clarence's desk at a mean rate of 14 per hour. The time required to process either transaction has an exponential distribution with a mean of 3 minutes. To reduce the expected waiting time in the system for both deposit slips and withdrawal slips, the actuarial department has made the following recommendations: (1) Train each clerk to handle both deposits and withdrawals, and (2) put both deposit and withdrawal slips into a single queue that is accessed by both clerks.

(a) Determine the expected waiting time in the system under current procedures for each type of slip. Then combine these results to calculate the expected waiting time in the system for a random arrival of either type of slip.

T (b) If the recommendations are adopted, determine the expected waiting time in the system for arriving slips.

T (c) Now suppose that adopting the recommendations would result in a slight increase in the expected processing time. Use the Excel template for the $M/M/s$ model to determine by trial and error the expected processing time (within 0.001 hour) that would cause the expected waiting time in the system for a random arrival to be essentially the same under current procedures and under the recommendations.

17.6-21. People's Software Company has just set up a call center to provide technical assistance on its new software package. Two technical representatives are taking the calls, where the time required by either representative to answer a customer's questions has an exponential distribution with a mean of 8 minutes. Calls are arriving according to a Poisson process at a mean rate of 10 per hour.

By next year, the mean arrival rate of calls is expected to decline to 5 per hour, so the plan is to reduce the number of technical representatives to one then.

T (a) Assuming that μ will continue to be 7.5 calls per hour for next year's queueing system, determine L, L_q, W, and W_q for both the current system and next year's system. For each of these four measures of performance, which system yields the smaller value?

(b) Now assume that μ will be adjustable when the number of technical representatives is reduced to one. Solve algebraically for the value of μ that would yield the same value of W as for the current system.

(c) Repeat part (b) with W_q instead of W.

17.6-24

17.6-22. Consider a generalization of the $M/M/1$ model where the server needs to "warm up" at the beginning of a busy period, and so serves the first customer of a busy period at a slower rate than other customers. In particular, if an arriving customer finds the server idle, the customer experiences a service time that has an exponential distribution with parameter μ_1. However, if an arriving customer finds the server busy, that customer joins the queue and subsequently experiences a service time that has an exponential distribution with parameter μ_2, where $\mu_1 < \mu_2$. Customers arrive according to a Poisson process with mean rate λ.

(a) Formulate this model as a continuous time Markov chain by defining the states and constructing the rate diagram accordingly.

(b) Develop the balance equations.

(c) Suppose that numerical values are specified for μ_1, μ_2, and λ, and that $\lambda < \mu_2$ (so that a steady-state distribution exists). Since this model has an infinite number of states, the steady-state distribution is the simultaneous solution of an infinite number of balance equations (plus the equation specifying that the sum of the probabilities equals 1). Suppose that you are unable to obtain this solution analytically, so you wish to use a computer to solve the model numerically. Considering that it is impossible to solve an infinite number of equations numerically, briefly describe what still can be done with these equations to obtain an approximation of the steady-state distribution. Under what circumstances will this approximation be essentially exact?

(d) Given that the steady-state distribution has been obtained, give explicit expressions for calculating L, L_q, W, and W_q.

(e) Given this steady-state distribution, develop an expression for $P\{\mathcal{W} > t\}$ that is analogous to Eq. (1) in Prob. 17.6-16.

17.6-23. For each of the following models, write the balance equations and show that they are satisfied by the solution given in Sec. 17.6 for the steady-state distribution of the number of customers in the system.

(a) The $M/M/1$ model.

(b) The finite queue variation of the $M/M/1$ model, with $K = 2$.

(c) The finite calling population variation of the $M/M/1$ model, with $N = 2$.

17.6-28

T 17.6-24. Consider a telephone system with three lines. Calls arrive according to a Poisson process at a mean rate of 6 per hour. The duration of each call has an exponential distribution with a mean of 15 minutes. If all lines are busy, calls will be put on hold until a line becomes available.

(a) Print out the measures of performance provided by the Excel template for this queueing system (with $t = 1$ hour and $t = 0$, respectively, for the two waiting time probabilities).

(b) Use the printed result giving $P\{\mathcal{W}_q > 0\}$ to identify the steady-state probability that a call will be answered immediately (not put on hold). Then verify this probability by using the printed results for the P_n.

(c) Use the printed results to identify the steady-state probability distribution of the number of calls on hold.

(d) Print out the new measures of performance if arriving calls are lost whenever all lines are busy. Use these results to identify the steady-state probability that an arriving call is lost.

17.6-25.* Janet is planning to open a small car-wash operation, and she must decide how much space to provide for waiting cars.

Janet estimates that customers would arrive randomly (i.e., a Poisson input process) with a mean rate of 1 every 4 minutes, unless the waiting area is full, in which case the arriving customers would take their cars elsewhere. The time that can be attributed to washing one car has an exponential distribution with a mean of 3 minutes. Compare the expected fraction of potential customers that will be *lost* because of inadequate waiting space if (*a*) 0 spaces (not including the car being washed), (*b*) 2 spaces, and (*c*) 4 spaces were provided.

17.6-26. Consider the finite queue variation of the *M/M/s* model. Derive the expression for L_q given in Sec. 17.6 for this model.

17.6-27. For the finite queue variation of the *M/M/*1 model, develop an expression analogous to Eq. (1) in Prob. 17.6-16 for the following probabilities:

(a) $P\{\mathcal{W} > t\}$.

(b) $P\{\mathcal{W}_q > t\}$.

[*Hint:* Arrivals can occur only when the system is not full, so the probability that a random arrival finds n customers already there is $P_n/(1 - P_K)$.]

17.6-28. George is planning to open a drive-through photo-developing booth with a single service window that will be open approximately 200 hours per month in a busy commercial area. Space for a drive-through lane is available for a rental of $200 per month per car length. George needs to decide how many car lengths of space to provide for his customers.

Excluding this rental cost for the drive-through lane, George believes that he will average a profit of $4 per customer served (nothing for a drop off of film and $8 when the photographs are picked up). He also estimates that customers will arrive randomly (a Poisson process) at a mean rate of 20 per hour, although those who find the drive-through lane full will be forced to leave. Half of the customers who find the drive-through lane full wanted to drop off film, and the other half wanted to pick up their photographs. The half who wanted to drop off film will take their business elsewhere instead. The other half of the customers who find the drive-through lane full will not be lost because they will keep trying later until they can get in and pick up their photographs. George assumes that the time required to serve a customer will have an exponential distribution with a mean of 2 minutes.

T **(a)** Find L and the mean rate at which customers are lost when the number of car lengths of space provided is 2, 3, 4, and 5.

(b) Calculate W from L for the cases considered in part (*a*).

(c) Use the results from part (*a*) to calculate the decrease in the mean rate at which customers are lost when the number of car lengths of space provided is increased from 2 to 3, from 3 to 4, and from 4 to 5. Then calculate the increase in expected profit per hour (excluding space rental costs) for each of these three cases.

(d) Compare the increases in expected profit found in part (*c*) with the cost per hour of renting each car length of space. What conclusion do you draw about the number of car lengths of space that George should provide?

17.6-29. At the Forrester Manufacturing Company, one repair technician has been assigned the responsibility of maintaining three machines. For each machine, the probability distribution of the running time before a breakdown is exponential, with a mean of 9 hours. The repair time also has an exponential distribution, with a mean of 2 hours.

(a) Which queueing model fits this queueing system?

T **(b)** Use this queueing model to find the probability distribution of the number of machines not running, and the mean of this distribution.

(c) Use this mean to calculate the expected time between a machine breakdown and the completion of the repair of that machine.

(d) What is the expected fraction of time that the repair technician will be busy?

T **(e)** As a crude approximation, assume that the calling population is infinite and that machine breakdowns occur randomly at a mean rate of 3 every 9 hours. Compare the result from part (*b*) with that obtained by making this approximation while using (i) the *M/M/s* model and (ii) the finite queue variation of the *M/M/s* model with $K = 3$.

T **(f)** Repeat part (*b*) when a second repair technician is made available to repair a second machine whenever more than one of these three machines require repair.

17.6-30. Reconsider the specific birth-and-death process described in Prob. 17.5-1.

(a) Identify a queueing model (and its parameter values) in Sec. 17.6 that fits this process.

T **(b)** Use the corresponding Excel template to obtain the answers for parts (*b*) and (*c*) of Prob. 17.5-1.

T **17.6-31.*** The Dolomite Corporation is making plans for a new factory. One department has been allocated 12 semiautomatic machines. A small number (yet to be determined) of operators will be hired to provide the machines the needed occasional servicing (loading, unloading, adjusting, setup, and so on). A decision now needs to be made on how to organize the operators to do this. Alternative 1 is to assign each operator to her own machines. Alternative 2 is to pool the operators so that any idle operator can take the next machine needing servicing. Alternative 3 is to combine the operators into a single crew that will work together on any machine needing servicing.

The running time (time between completing service and the machine's requiring service again) of each machine is expected to have an exponential distribution, with a mean of 150 minutes. The service time is assumed to have an exponential distribution, with a mean of 15 minutes (for Alternatives 1 and 2) or 15 minutes divided by the number of operators in the crew (for Alternative 3). For the department to achieve the required production rate, the machines must be running at least 89 percent of the time on average.

(a) For Alternative 1, what is the maximum number of machines that can be assigned to an operator while still achieving the required production rate? What is the resulting utilization of each operator?

(b) For Alternative 2, what is the minimum number of operators needed to achieve the required production rate? What is the resulting utilization of the operators?

(c) For Alternative 3, what is the minimum size of the crew needed to achieve the required production rate? What is the resulting utilization of the crew?

17.6-32. A shop contains three identical machines that are subject to a failure of a certain kind. Therefore, a maintenance system is provided to perform the maintenance operation (recharging) required by a failed machine. The time required by each operation has an exponential distribution with a mean of 30 minutes. However, with probability $\frac{1}{3}$, the operation must be performed a second time (with the same distribution of time) in order to bring the failed machine back to a satisfactory operational state. The maintenance system works on only one failed machine at a time, performing all the operations (one or two) required by that machine, on a first-come-first-served basis. After a machine is repaired, the time until its next failure has an exponential distribution with a mean of 3 hours.

(a) How should the states of the system be defined in order to formulate this queueing system as a continuous time Markov chain? (*Hint:* Given that a first operation is being performed on a failed machine, completing this operation *successfully* and completing it *unsuccessfully* are two separate events of interest. Then use Property 6 regarding disaggregation for the exponential distribution.)

(b) Construct the corresponding rate diagram.

(c) Develop the balance equations.

17.7-1.* Consider the $M/G/1$ model.

(a) Compare the expected waiting time in the queue if the service-time distribution is (i) exponential, (ii) constant, (iii) Erlang with the amount of variation (i.e., the standard deviation) halfway between the constant and exponential cases.

(b) What is the effect on the expected waiting time in the queue and on the expected queue length if both λ and μ are doubled and the scale of the service-time distribution is changed accordingly?

17.7-2. Consider the $M/G/1$ model with $\lambda = 0.2$ and $\mu = 0.25$.

T (a) Use the Excel template for this model (or hand calculations) to find the main measures of performance—L, L_q, W, W_q—for each of the following values of σ: 4, 3, 2, 1, 0.

(b) What is the ratio of L_q with $\sigma = 4$ to L_q with $\sigma = 0$? What does this say about the importance of reducing the variability of the service times?

(c) Calculate the reduction in L_q when σ is reduced from 4 to 3, from 3 to 2, from 2 to 1, and from 1 to 0. Which is the largest reduction? Which is the smallest?

(d) Use trial and error with the template to see approximately how much μ would need to be increased with $\sigma = 4$ to achieve the same L_q as with $\mu = 0.25$ and $\sigma = 0$.

17.7-3. Consider the following statements about an $M/G/1$ queueing system, where σ^2 is the variance of service times. Label each statement as true or false, and then justify your answer.

(a) Increasing σ^2 (with fixed λ and μ) will increase L_q and L, but will not change W_q and W.

(b) When choosing between a tortoise (small μ and σ^2) and a hare (large μ and σ^2) to be the server, the tortoise always wins by providing a smaller L_q.

(c) With λ and μ fixed, the value of L_q with an exponential service-time distribution is twice as large as with constant service times.

(d) Among all possible service-time distributions (with λ and μ fixed), the exponential distribution yields the largest value of L_q.

17.7-4. Marsha operates an expresso stand. Customers arrive according to a Poisson process at a mean rate of 30 per hour. The time needed by Marsha to serve a customer has an exponential distribution with a mean of 75 seconds.

(a) Use the $M/G/1$ model to find L, L_q, W, and W_q.

(b) Suppose Marsha is replaced by an expresso vending machine that requires exactly 75 seconds for each customer to operate. Find L, L_q, W, and W_q.

(c) What is the ratio of L_q in part (*b*) to L_q in part (*a*)?

T (d) Use trial and error with the Excel template for the $M/G/1$ model to see approximately how much Marsha would need to reduce her expected service time to achieve the same L_q as with the expresso vending machine.

17.7-5. E Antonio runs a shoe repair store by himself. Customers arrive to bring a pair of shoes to be repaired according to a Poisson process at a mean rate of 1 per hour. The time Antonio requires to repair each individual shoe has an exponential distribution with a mean of 15 minutes.

(a) Consider the formulation of this queueing system where the individual shoes (not pairs of shoes) are considered to be the customers. For this formulation, construct the rate diagram and develop the balance equations, but do not solve further.

(b) Now consider the formulation of this queueing system where the pairs of shoes are considered to be the customers. Identify the specific queueing model that fits this formulation.

(c) Calculate the expected number of pairs of shoes in the shop.

(d) Calculate the expected amount of time from when a customer drops off a pair of shoes until they are repaired and ready to be picked up.

T (e) Use the corresponding Excel template to check your answers in parts (*c*) and (*d*).

17.7-6.* The maintenance base for Friendly Skies Airline has facilities for overhauling only one airplane engine at a time. Therefore, to return the airplanes to use as soon as possible, the policy has been to stagger the overhauling of the four engines of each airplane. In other words, only one engine is overhauled each time an airplane comes into the shop. Under this policy, airplanes have arrived according to a Poisson process at a mean rate of 1 per day. The time required for an engine overhaul (once work has begun) has an exponential distribution with a mean of $\frac{1}{2}$ day.

A proposal has been made to change the policy so that all four engines are overhauled consecutively each time an airplane comes into the shop. Although this would quadruple the expected service time, each plane would need to come to the maintenance base only one-fourth as often.

Management now needs to decide whether to continue the status quo or adopt the proposal. The objective is to minimize the average amount of flying time lost by the entire fleet per day due to engine overhauls.

(a) Compare the two alternatives with respect to the average amount of flying time lost by an airplane each time it comes to the maintenance base.

(b) Compare the two alternatives with respect to the average number of airplanes losing flying time due to being at the maintenance base.

(c) Which of these two comparisons is the appropriate one for making management's decision? Explain.

17.7-7. Reconsider Prob. 17.7-6. Management has adopted the proposal but now wants further analysis conducted of this new queueing system.

(a) How should the state of the system be defined in order to formulate the queueing model as a continuous time Markov chain?

(b) Construct the corresponding rate diagram.

17.7-8. The McAllister Company factory currently has *two* tool cribs, each with a *single* clerk, in its manufacturing area. One tool crib handles only the tools for the heavy machinery; the second one handles all other tools. However, for each crib the mechanics arrive to obtain tools at a mean rate of 24 per hour, and the expected service time is 2 minutes.

Because of complaints that the mechanics coming to the tool crib have to wait too long, it has been proposed that the two tool cribs be combined so that either clerk can handle either kind of tool as the demand arises. It is believed that the mean arrival rate to the combined two-clerk tool crib would double to 48 per hour and that the expected service time would continue to be 2 minutes. However, information is not available on the *form* of the probability distributions for interarrival and service times, so it is not clear which queueing model would be most appropriate.

Compare the status quo and the proposal with respect to the total expected number of mechanics at the tool crib(s) and the expected waiting time (including service) for each mechanic. Do this by tabulating these data for the four queueing models considered in Figs. 17.6, 17.8, 17.10, and 17.11 (use $k = 2$ when an Erlang distribution is appropriate).

17.7-9.* E Consider a single-server queueing system with a Poisson input, Erlang service times, and a finite queue. In particular, suppose that $k = 2$, the mean arrival rate is 2 customers per hour, the expected service time is 0.25 hour, and the maximum permissible number of customers in the system is 2. This system can be formulated as a continuous time Markov chain by dividing each service time into two consecutive phases, each having an exponential distribution with a mean of 0.125 hour, and then defining the state of the system as (n, p), where n is the number of customers in the system ($n = 0, 1, 2$), and p indicates the phase of the customer being served ($p = 0, 1, 2$, where $p = 0$ means that no customer is being served).

(a) Construct the corresponding rate diagram. Write the balance equations, and then use these equations to solve for the steady-state distribution of the state of this Markov chain.

(b) Use the steady-state distribution obtained in part (*a*) to identify the steady-state distribution of the number of customers in the system (P_0, P_1, P_2) and the steady-state expected number of customers in the system (L).

(c) Compare the results from part (*b*) with the corresponding results when the service-time distribution is exponential.

17.7-10. Consider the $E_2/M/1$ model with $\lambda = 4$ and $\mu = 5$. This model can be formulated as a continuous time Markov chain by dividing each interarrival time into two consecutive phases, each having an exponential distribution with a mean of $1/(2\lambda) = 0.125$, and then defining the state of the system as (n, p), where n is the number of customers in the system ($n = 0, 1, 2, \ldots$) and p indicates the phase of the *next* arrival (not yet in the system) ($p = 1, 2$).

Construct the corresponding rate diagram (but do not solve further).

17.7-11. A company has one repair technician to keep a large group of machines in running order. Treating this group as an infinite calling population, individual breakdowns occur according to a Poisson process at a mean rate of 1 per hour. For each breakdown, the probability is 0.9 that only a minor repair is needed, in which case the repair time has an exponential distribution with a mean of $\frac{1}{2}$ hour. Otherwise, a major repair is needed, in which case the repair time has an exponential distribution with a mean of 5 hours. Because both of these *conditional* distributions are exponential, the *unconditional* (combined) distribution of repair times is *hyperexponential.*

(a) Compute the mean and standard deviation of this hyperexponential distribution. [*Hint:* Use the general relationships from probability theory that, for any random variable X and any pair of mutually exclusive events E_1 and E_2, $E(X) = E(X \mid E_1)P(E_1) + E(X \mid E_2)P(E_2)$ and $\text{var}(X) = E(X^2) - E(X)^2$.] Compare this standard deviation with that for an exponential distribution having this mean.

(b) What are P_0, L_q, L, W_q, and W for this queueing system?

(c) What is the conditional value of W, given that the machine involved requires major repair? A minor repair? What is the division of L between machines requiring the two types of repairs? (*Hint:* Little's formula still applies for the individual categories of machines.)

(d) How should the states of the system be defined in order to formulate this queueing system as a continuous time Markov chain? (*Hint:* Consider what additional information must be given, besides the number of machines down, for the conditional distribution of the time remaining until the next event of each kind to be exponential.)

(e) Construct the corresponding rate diagram.

17.7-12. Consider the finite queue variation of the $M/G/1$ model, where K is the maximum number of customers allowed in the system. For $n = 1, 2, \ldots$, let the random variable X_n be the number of customers in the system at the moment t_n when the nth customer has just finished being served. (Do not count the departing customer.) The times $\{t_1, t_2, \ldots\}$ are called *regeneration points.* Furthermore, $\{X_n\}$ ($n = 1, 2, \ldots$) is a discrete time Markov chain and is known as an *embedded Markov chain.* Embedded Markov chains are useful for studying the properties of continuous time stochastic processes such as for an $M/G/1$ model.

Now consider the particular special case where $K = 4$, the service time of successive customers is a fixed constant, say, 10 minutes, and the mean arrival rate is 1 every 50 minutes. Therefore, $\{X_n\}$ is an embedded Markov chain with states 0, 1, 2, 3. (Because there are never more than 4 customers in the system, there can never be more than 3 in the system at a regeneration point.) Because the system is observed at successive departures, X_n can never decrease by more than 1. Furthermore, the probabilities of transitions that result in increases in X_n are obtained directly from the Poisson distribution.

(a) Find the one-step transition matrix for the embedded Markov chain. (*Hint:* In obtaining the transition probability from state 3 to state 3, use the probability of 1 or more arrivals rather than just 1 arrival, and similarly for other transitions to state 3.)

(b) Use the corresponding routine in the Markov chains area of your OR Courseware to find the steady-state probabilities for the number of customers in the system at regeneration points.

(c) Compute the expected number of customers in the system at regeneration points, and compare it to the value of L for the $M/D/1$ model (with $K = \infty$) in Sec. 17.7.

17.8-1.* Southeast Airlines is a small commuter airline serving primarily the state of Florida. Their ticket counter at a certain airport is staffed by a single ticket agent. There are two separate lines—one for first-class passengers and one for coach-class passengers. When the ticket agent is ready for another customer, the next first-class passenger is served if there are any in line. If not, the next coach-class passenger is served. Service times have an exponential distribution with a mean of 3 minutes for both types of customers. During the 12 hours per day that the ticket counter is open, passengers arrive randomly at a mean rate of 2 per hour for first-class passengers and 10 per hour for coach-class passengers.

(a) What kind of queueing model fits this queueing system?

T (b) Find the main measures of performance—L, L_q, W, and W_q—for both first-class passengers and coach-class passengers.

(c) What is the expected waiting time before service begins for first-class customers as a fraction of this waiting time for coach-class customers?

(d) Determine the average number of hours per day that the ticket agent is busy.

T **17.8-2.** Consider the model with nonpreemptive priorities presented in Sec. 17.8. Suppose there are two priority classes, with $\lambda_1 = 4$ and $\lambda_2 = 4$. In designing this queueing system, you are offered the choice between the following alternatives: (1) one fast server ($\mu = 10$) and (2) two slow servers ($\mu = 5$).

Compare these alternatives with the usual four mean measures of performance (W, L, W_q, L_q) for the individual priority classes (W_1, W_2, L_1, L_2, and so forth). Which alternative is preferred if your primary concern is expected waiting time in the *system* for priority class 1 (W_1)? Which is preferred if your primary concern is expected waiting time in the *queue* for priority class 1?

17.8-3. Consider the single-server variation of the nonpreemptive priorities model presented in Sec. 17.8. Suppose there are three priority classes, with $\lambda_1 = 1$, $\lambda_2 = 1$, and $\lambda_3 = 1$. The expected service times for priority classes 1, 2, and 3 are 0.4, 0.3, and 0.2, respectively, so $\mu_1 = 2.5$, $\mu_2 = 3\frac{1}{3}$, and $\mu_3 = 5$.

(a) Calculate W_1, W_2, and W_3.

(b) Repeat part (*a*) when using the approximation of applying the general model for nonpreemptive priorities presented in Sec. 17.8 instead. Since this general model assumes that the expected service time is the same for all priority classes, use an expected service time of 0.3 so $\mu = 3\frac{1}{3}$. Compare the results with those obtained in part (*a*) and evaluate how good an approximation is provided by making this assumption.

T **17.8-4.*** A particular work center in a job shop can be represented as a single-server queueing system, where jobs arrive according to a Poisson process, with a mean rate of 8 per day. Although the arriving jobs are of three distinct types, the time required to perform any of these jobs has the same exponential distribution, with a mean of 0.1 working day. The practice has been to work on arriving jobs on a first-come-first-served basis. However, it is important that jobs of type 1 not wait very long, whereas the wait is only moderately important for jobs of type 2 and is relatively unimportant for jobs of type 3. These three types arrive with a mean rate of 2, 4, and 2 per day, respectively. Because all three types have experienced rather long delays on average, it has been proposed that the jobs be selected according to an appropriate priority discipline instead.

Compare the expected waiting time (including service) for each of the three types of jobs if the queue discipline is (*a*) first-come-first-served, (*b*) nonpreemptive priority, and (*c*) preemptive priority.

T **17.8-5.** Reconsider the *County Hospital* emergency room problem as analyzed in Sec. 17.8. Suppose that the definitions of the three categories of patients are tightened somewhat in order to move marginal cases into a lower category. Consequently, only 5 percent of the patients will qualify as critical cases, 20 percent as serious cases, and 75 percent as stable cases. Develop a table showing the data presented in Table 17.3 for this revised problem.

17.8-6. Reconsider the queueing system described in Prob. 17.4-6. Suppose now that type 1 customers are more important than type 2 customers. If the queue discipline were changed from first-come-first-served to a priority system with type 1 customers being given nonpreemptive priority over type 2 customers, would this increase, decrease, or keep unchanged the expected total number of customers in the system?

(a) Determine the answer without any calculations, and then present the reasoning that led to your conclusion.

T (b) Verify your conclusion in part (*a*) by finding the expected total number of customers in the system under each of these two queue disciplines.

17.8-7. Consider the queueing model with a preemptive priority queue discipline presented in Sec. 17.8. Suppose that $s = 1$, $N = 2$, and $(\lambda_1 + \lambda_2) < \mu$; and let P_{ij} be the steady-state probability that there are i members of the higher-priority class and j members of the lower-priority class in the queueing system ($i = 0, 1, 2, \ldots; j = 0, 1, 2, \ldots$). Use a method analogous to that presented in Sec. 17.5 to derive a system of linear equations whose simultaneous solution is the P_{ij}. Do not actually obtain this solution.

17.9-1. Consider a queueing system with two servers, where the customers arrive from two different sources. From source 1, the customers always arrive 2 at a time, where the time between consecutive arrivals of pairs of customers has an exponential distribution with a mean of 20 minutes. Source 2 is itself a two-server queueing system, which has a Poisson input process with a mean rate of 7 customers per hour, and the service time from each of these two servers has an exponential distribution with a mean of 15 minutes. When a customer completes service at source 2, he or she immediately enters the queueing system under consideration for another type of service. In the latter queueing system, the queue discipline is preemptive priority where customers from source 1 always have preemptive priority over customers from source 2. However, service times are independent and identically distributed for both types of customers according to an exponential distribution with a mean of 6 minutes.

(a) First focus on the problem of deriving the steady-state distribution of *only* the number of source 1 customers in the queueing system under consideration. Using a continuous time Markov chain formulation, define the states and construct the rate diagram for most efficiently deriving this distribution (but do not actually derive it).

(b) Now focus on the problem of deriving the steady-state distribution of the *total* number of customers of both types in the queueing system under consideration. Using a continuous time Markov chain formulation, define the states and construct the rate diagram for most efficiently deriving this distribution (but do not actually derive it).

(c) Now focus on the problem of deriving the steady-state *joint* distribution of the number of customers of each type in the queueing system under consideration. Using a continuous time Markov chain formulation, define the states and construct the rate diagram for deriving this distribution (but do not actually derive it).

17.9-2. Consider a system of two infinite queues in series, where each of the two service facilities has a single server. All service times are independent and have an exponential distribution, with a mean of 3 minutes at facility 1 and 4 minutes at facility 2. Facility 1 has a Poisson input process with a mean rate of 10 per hour.

(a) Find the steady-state distribution of the number of customers at facility 1 and then at facility 2. Then show the product form solution for the *joint* distribution of the number at the respective facilities.

(b) What is the probability that both servers are idle?

(c) Find the expected *total* number of customers in the system and the expected *total* waiting time (including service times) for a customer.

17.9-3. Under the assumptions specified in Sec. 17.9 for a system of infinite queues in series, this kind of queueing network actually is a special case of a Jackson network. Demonstrate that this is true by describing this system as a Jackson network, including specifying the values of the a_j and the p_{ij}, given λ for this system.

17.9-4. Consider a Jackson network with three service facilities having the parameter values shown below.

Facility j	s_j	μ_j	a_j	p_{ij}: $i = 1$	$i = 2$	$i = 3$
$j = 1$	1	40	10	0	0.3	0.4
$j = 2$	1	50	15	0.5	0	0.5
$j = 3$	1	30	3	0.3	0.2	0

T **(a)** Find the total arrival rate at each of the facilities.

(b) Find the steady-state distribution of the number of customers at facility 1, facility 2, and facility 3. Then show the product form solution for the joint distribution of the number at the respective facilities.

(c) What is the probability that all the facilities have empty queues (no customers waiting to begin service)?

(d) Find the expected total number of customers in the system.

(e) Find the expected total waiting time (including service times) for a customer.

T **17.10-1.** When describing economic analysis of the number of servers to provide in a queueing system, Sec. 17.10 introduces a basic cost model where the objective is to minimize $E(\text{TC}) = C_s s + C_w L$. The purpose of this problem is to enable you to explore the effect that the relative sizes of C_s and C_w have on the optimal number of servers.

Suppose that the queueing system under consideration fits the $M/M/s$ model with $\lambda = 8$ customers per hour and $\mu = 10$ customers per hour. Use the Excel template in your OR Courseware for economic analysis with the $M/M/s$ model to find the optimal number of servers for each of the following cases.

(a) $C_s = \$100$ and $C_w = \$10$.

(b) $C_s = \$100$ and $C_w = \$100$.

(c) $C_s = \$10$ and $C_w = \$100$.

T **17.10-2.*** Jim McDonald, manager of the fast-food hamburger restaurant McBurger, realizes that providing fast service is a key to the success of the restaurant. Customers who have to wait very long are likely to go to one of the other fast-food restaurants in town next time. He estimates that each minute a customer has to wait in line before completing service costs him an average of 30 cents in lost future business. Therefore, he wants to be sure that enough cash registers always are open to keep waiting to a minimum. Each cash register is operated by a part-time employee who obtains the food ordered by each customer and collects the payment. The total cost for each such employee is \$9 per hour.

During lunch time, customers arrive according to a Poisson process at a mean rate of 66 per hour. The time needed to serve a customer is estimated to have an exponential distribution with a mean of 2 minutes.

Determine how many cash registers Jim should have open during lunch time to minimize his expected total cost per hour.