

# Abstraction Through Game Play

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**Abstract** This paper examines the computer game play of an 11-year-old boy. In the course of building a virtual house he developed and used, without assistance, an artefact and an accompanying strategy to ensure that his house was symmetric. We argue that the creation and use of this artefact-strategy is a mathematical abstraction. The discussion contributes to knowledge on mathematical abstraction: of non-traditional knowledge; without teacher mediation; through game play. The paper also considers learning without instruction/instructional design and questions received distinctions between scientific and everyday knowledge.

**Keywords** Mathematical abstraction · Game play · Computers

## 1 Introduction

We examine the computer game play of an 11-year-old boy, Costas, who built, more or less without assistance, virtual houses in the computer game *The Sims 2*. This work began as exploration of what mathematics is involved in playing *The Sims 2*. The analysis of data collected revealed that quite a lot of mathematics was involved in Costas' playing *The Sims 2*. The interpretation of our data analysis led to some difficult (for us) questions about: what we regard as mathematics; the nature of abstraction; mediation (by people and by artefacts); the Vygotskian distinction between scientific and everyday concepts; and educational design and 'target knowledge'. Much of this paper focuses on an artefact and an accompanying strategy Costas developed to enable him to build a symmetric virtual house and we argue that the creation and use of this artefact and strategy is a mathematical abstraction. The paper is structured as follows: an integrated review of literature on abstraction and presentation of the theoretical framework; a brief review of learning through game play; the setting and methodology of the study; results pertinent to the focus of this paper; a discussion of issues arising.

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## 2 Theoretical Framework and Review of Literature on Abstraction

The overarching theoretical framework is socio-cultural in as much as we view context as paramount and learning as mediated activity. But there are many schools of thought within the socio-cultural church. Cultural historical activity theory (CHAT) is, perhaps, the best known socio-cultural school of thought but there different generations of CHAT (see Engeström (2001) for details). The notion of the activity can be ambiguous, is it the activity of schooling or can we speak of activity in a learning task? Wells (1993), in a celebrated paper, went so far as to use a lower and upper case letter ‘a’ to distinguish between activity as in classroom activity and Activity in activity theory. Related to this, the unit of analysis is a central concern for socio-cultural analysts but leading analysts differ in this respect, e.g. Wertsch (1998) focuses on mediated action whereas Engeström (2001) regards the activity system as the unit of analysis. Most socio-cultural mathematics education papers, to our knowledge, focus on mediated action rather than activity systems, and this goes even for those, e.g. Gresalfi et al. (2009), which invoke the term ‘activity system’. Roth (2009), writing with regard to a specific set of studies focused on mediated action, states “whereas these studies show us in great detail how some students participate and learn under certain ... conditions, I would ... seek more to provide accounts and theorize learning *generally*, including emotions”, p. 288. Our study is in the mediated action camp but, we will argue, after we have presented data and interpretation, there are aspects that are aligned with what Roth considers ‘learning *generally*, including emotions’.

Mediation can be viewed primarily as human-mediation but there are many forms of mediation, e.g. classrooms structures mediate classroom discourse. A central focus of this paper is tool-mediation. A focus on human-mediation can lead to statements that teaching (formal instruction) is essential for learning, e.g. “learning physics will not happen without formal instruction and without the mediation of a teacher, or a more informed peer, and mediational signs” (Tiberghien and Malkoun 2009, p. 43). We understand and largely agree with the position of the authors—advanced ideas in physics (e.g. Newton’s laws) and in mathematics (e.g. proof) are so very far removed from ‘nature’ that they require mediation by people who know more than the learner—but we would be more comfortable with the statement if “will not” was replaced by “will rarely”. We will return to this theme later in the paper, as it impinges on considerations of the divide between scientific and everyday concepts, but now consider mathematical abstraction.<sup>1</sup>

There is a current divide amongst mathematics education researchers about what abstraction is. Mitchelmore and White (2007), in the Editorial of a MERJ special edition on abstraction, refer to two approaches, empirical abstraction and theoretical abstraction. Skemp, building on the work of Dienes and extended by Tall, is cited with regard to empirical abstraction:

Abstracting is an activity by which we become aware of similarities ... among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class. (Skemp 1986, p. 21)

“Empirical abstraction focuses on the identification of underlying general features” but “is much deeper than the empirical abstraction described by Piaget, where the term denotes

<sup>1</sup> We suspect that abstraction differs over disciplines, hence reference to mathematical abstraction, but we shall, henceforth, leave the ‘mathematical’ as implicit and refer to abstraction.

the extraction of common properties of objects and extensional generalisations” (Mitchelmore and White 2007, p. 3).

Theoretical abstraction owes its genesis to Vygotsky’s distinction between ‘everyday’ and ‘scientific’ concepts and “Everyday concepts are formed by empirical abstraction” but scientific concepts are formed by establishing “a system of relations amongst concepts, an awareness of one’s own mental activity” and “penetration to the object’s essence” (Mitchelmore and White 2007, p. 3).

Moving beyond Mitchelmore and White’s valid (in our opinion) exposition of the two approaches we note a similarity and two differences. Similarity: both regard abstraction as a process and a product (the result of the abstracting process). Difference 1: empirical abstraction goes from the concrete to the abstract (Skemp’s, above, “collecting together our experiences on the basis of these similarities”) but theoretical abstraction involves interactions between our experiences, the concrete, and “the system of relations amongst concepts” based on “the object’s essence”, the abstract. Difference 2: empirical abstraction appears to view abstraction and generalisation as the same thing (read the Skemp quote above with ‘generalisation/generalising’ in place of ‘abstraction/abstracting’) whilst theoretical abstraction differentiates between them:

... abstraction and generalization function as two single aspects of the ascent of thought to the concrete. By *abstracting*, man isolates and, in the process of ascent, mentally retains the specific nature of the real relationship of things that determines the formation and integrity of assorted phenomena. In *generalization* he establishes real connections between this isolated relationship and the particular, individual phenomena that arise on its basis. (Davydov 1972/1990, p. 294)

As an aside, but an important aside for this paper, we would like to note that the discussion above appears to suggest that abstractions, in both empirical and theoretical approaches, are concepts. Whilst we hold that the end product of an abstraction process may be a concept, we, like Dreyfus and Tsamir (2004), would not wish to restrict abstractions just to concepts. In this paper we shall argue that a constructed artefact together with a strategy<sup>2</sup> for using this artefact is an abstraction.

Although we end this section by questioning one aspect of the Vygotskian divide between ‘everyday’ and ‘scientific’ concepts, we still position ourselves in the theoretical abstraction camp. One reason for this is an apparent contradiction in the Skemp quote above (but other empiricists shared this apparent contradiction—see Ozmantar and Monaghan 2007, for details). Skemp says that by abstracting “we become aware of similarities” but how do we see things as similar unless we already have a criteria for similarity, i.e. the abstraction? Theoretical abstraction overcomes this problem by arguing that an abstraction develops through interaction between the concrete and early formulations of the abstraction (Vygotsky’s ‘system of relations’ and ‘essence’).

Theoretical abstraction has a number of manifestations, e.g. Mason (1989), Davydov (1990), Noss and Hoyles (1996a), Ohlsson and Lehtinen (1997), van Oers (2001), Pratt and Noss (2002) and Schwarz et al. (2009). These accounts of abstractions have differences but all can be related to Davydov’s (1990) exposition of theoretical abstraction.<sup>3</sup> Davydov,

<sup>2</sup> ‘Strategy’ in this paper may be thought of as a procedure for rigorously attaining a goal or as an heuristic, as in ‘strategy games’ (see <http://nrch.maths.org/2546>).

<sup>3</sup> We do not go into details of similarities here because extended arguments for similarities between Davydov’s approach and many of these approaches have been made in co-authored papers with the second author: Dreyfus and Monaghan (2009); Monaghan and Ozmantar (2006b); Ozmantar and Monaghan (2007).

following Vygotsky, posits a cognitive divide between empirical thought and theoretical thought; empirical thought concerns observable reality, e.g. the wheels goes round and the vehicle moves, whilst theoretical thought aims to reproduce reality, e.g. why the car moves when the wheels go round. Davydov was a Marxist and employed Marxist ideas and language. An abstraction starts with the isolation of the ‘essence’, the “single source for a variety of phenomena” (p. 289). Cognitive development during abstraction is not an ascent from the concrete but a dialectical relationship between the concrete and the abstract where the concrete refers to:

some developed whole, interconnection ...synonym for the determining role of the whole with respect to its parts, features, and aspects. “The abstract” usually has several characteristics – it is something simple, devoid of differences, fragmentary, and undeveloped (p. 283).

Hershkowitz et al. (2001) provide a detailed account of how Davydov’s account of abstraction inspired the epistemic actions account, abstraction in context (AiC). AiC has many similarities to Noss and Hoyles’ (1996a) situated abstractions<sup>4</sup> (see Monaghan and Ozmentar 2006b) and Mason’s (1989) view of abstraction as the result of a delicate shift of attention (see Dreyfus and Monaghan 2009); a fundamental feature of all of these workers is interconnecting (situated) knowledge.

We now provide an outline of AiC as we will later use elements of this theoretical account of abstraction to ‘measure’ the 11-year-old boy’s abstraction. Apart from close familiarity with AiC we choose this account of abstraction because it draws on the concept of ‘vertical mathematising’ (in the sense inspired by workers in the Freudenthal Institute) and we will raise the issue of ‘must vertical mathematisation link a learner’s constructed maths to a target mathematical idea?’ in the Discussion section. Further to this, as noted above, AiC, whilst not identical to other theoretical accounts of abstraction, has much in common with them. Schwarz et al. (2009) provide a comprehensive review of AiC studies *circa* 2009.

AiC views abstraction as vertically reorganising (in the sense of Freudenthal 1991) previously constructed mathematical knowledge through three nested epistemic actions: recognising (familiar mathematics), building-with (elements of familiar mathematics) and constructing (new mathematics); although the focus on “three nested epistemic actions” is peculiar to AiC the idea behind this approach is the common feature of theoretical accounts of interconnecting (situated) knowledge and is a realisation of Noss and Hoyles’ (1996a) claim that “Mathematics is more than action in the sense of activity-with-objects. It is activity-with-relationships” (p. 124). We will not, however, dwell on recognising, building-with and constructing analysis in this paper as we feel we can make our arguments without this analysis. Familiar and new mathematics will depend on the person(s)—what is new to one person may be familiar to another—and the cultural tools (Wertsch 1998) available to the individual or group; these points are also made by Davydov and by Noss and Hoyles. Three abstraction stages are posited: a need for new mathematics; the construction of new mathematics; the consolidation of the new mathematics, i.e. its use in further mathematical activity. It is possible to regard the formation of an abstraction as simply the middle stage, the construction of new mathematics, but we view the first and third stages as essential. The need for an abstraction is related to the object of the activity and relates to Roth’s point (above) about student participation; indeed, we would question whether it is possible to form an abstraction unless one needs to form one for some reason.

<sup>4</sup> See also Pratt and Noss (2002).

Consolidation in AiC-speak is not the consolidation of rote learning, i.e. students consolidating their multiplication table facts by repeatedly reciting them (see Monaghan and Ozmantar 2006a). Consolidation here concerns learners using what they have constructed, that if someone claims to know something but does not/can not use this knowledge in activity-with-relationships, then this claimed knowledge, as mathematical knowledge, is questionable.

As this paper focuses on a learner working without teacher intervention we consider the two (contrasting) AiC studies which report on learning without teacher intervention. Dreyfus and Kidron (2006) reports on a solitary learner, an adult mathematician, solving a problem concerned with bifurcations in dynamic processes that she had set herself and which required new, to her, mathematics. Abstraction was assisted by books, internet resources and *Mathematica*. Williams (2007) reports on “spontaneous” learning by individual school students in working on mathematics set by adults; Kerri, for example, worked on a problem in a test on the equation of a straight line. In both studies, however, and in contrast to the work of the boy we report on, the abstractions concerned ‘scholastic mathematics’, i.e. mathematics that might be formally taught.

We draw this section to a close with a consideration of design and the divide between ‘everyday’ and ‘scientific’ concepts. AiC, situated abstraction and abstraction as the result of a delicate shift of attention are all concerned with instruction and/or instructional design. AiC originated from an attempt to understand students’ abstractions in instructional situations, “Our approach ... took shape in the course of research that accompanied curriculum development” (Schwarz et al. 2009, p. 17). Mason’s work cannot be separated from his (laudable) missionary zeal in leading people to see the general in the particular and the particular in the general. Noss and Hoyles (1996a) are less obviously tied to instruction but they are deeply committed to instructional design in “an autoexpressive computation medium” where it “is possible to design settings involving the requirement to express generality as a result of manipulating computational objects” (1996, p. 125). Similar remarks could be made concerning Pratt and Noss’ (2002) approach to abstraction, which builds on that of Noss and Hoyle. These observations are not criticisms, instruction and instructional design are central to most of our work too; we draw attention to them because the empirical work reported on in this paper took place with only an *epsilon* of instruction/instructional design. Is an analysis informed by theoretical abstraction appropriate in such a case? We return to this matter in the Discussion subsection on ‘instructional intent and design’.

We have spoken above of Vygotsky’s distinction between ‘everyday’ and ‘scientific’ concepts and Davydov’s, following Vygotsky, distinction between empirical thought and theoretical thought. We regard this/these distinction(s) as important but adopting a Vygotskian perspective does not, to us, entail blind faith in every interpretation of Vygotsky. As Scott et al. (2011, p. 6) point out, Vygotsky’s ‘scientific’ concerns scientific communities:

... scientific concepts are taken to be the products of specific scientific communities and constitute part of the disciplinary knowledge of that community. The term ‘scientific’ as used by Vygotsky is not restricted to the natural sciences, but covers all comparable communities such as those of history, philosophy, art and so on. As the agreed upon products of specific communities, scientific concepts are not open to ‘discovery’ by the individual but can only be learned through some form of tuition.

This, presumably, is the idea behind Tiberghien and Malkoun’s comment on “learning physics” cited earlier in this section. But we are not completely happy with this stance with

regard to ‘learning mathematics’ (and, indeed, learning through game play). We compare mathematics and physics. Phil Scott, (first author, above) was fond of citing ‘gravity’ as an example of a scientific concept that students would not discover by themselves (and Phil would then wax lyrical about the need for teacher-student dialogic interactions which link scientific and everyday concepts). We do not disagree with this and there are many parallels in mathematics, e.g. students would not discover the current concept of a function by themselves. But mathematics is open to inventions, especially strategies such as: sum the consecutive numbers from 1 to  $n$  by  $\frac{n}{2}(n+1)$ ; check if a number is prime by dividing by all primes below the square root of the number. These examples are strategies actually devised by school children; we would also call them abstractions and claim they warrant the name ‘scientific’. We further believe that Davydov would accept them as examples of his theoretical thought, even if they are not scientific in the strict Vygotskian ‘scientific community’ sense. We return to these matters in the Discussion section.

### 3 Learning Through Game Play

Mathematics educators often recommend mathematical game play, “Mathematical games can foster mathematical communication... can motivate students and engage them in thinking about and applying concepts and skills” (NCTM 2004). An early study on games in mathematics (Bright et al. 1985, p. 133) concluded “games can be used to teach a variety of content in a variety of instructional settings... there is no guarantee that every game will be effective... But many games are effective”. Bragg (2006, p. 233), however, in a study of students’ perceptions of game play concludes that “it appears that assumptions that students will see the usefulness of mathematics games in classrooms are problematic”. It depends, to us, on the game, the students and the context of game play. Wei and Hendrix (2009) suggest that older boys are more likely to retain knowledge learnt through non-competitive games as when competition is involved, the strongest memories tend to be of winning or losing.

Shaffer (2005) writes about ‘epistemic games’, games which help players to develop new ways of ‘knowing’. In some cases these games might be designed for the purpose of teaching/learning but in other cases, they may be games which have been developed for a general market. *The Sims* games series are popular life simulation computer games that allows players to control the lives and relationships of game characters and create houses and neighbourhoods for them. *The Sims* series are not games in the conventional sense in that they do not involve competition (neither with another player nor with oneself) although successive iterations have included more goal-based activities that a player can pursue. However, the player is free to ignore these and to engage with the games on an open-ended basis. Gee and Hayes (2010) cite a number of innovative approaches that players take towards *The Sims* including a community of players who make and recreate families living at minimum wage level (recreating Barbara Ehrenreich’s book ‘Nickel and Dimed’). *The Sims* series are examples of *sandbox* games which means that players can experiment with building the onscreen environment. There are a number of games which allow this type of activity. *Grand Theft Auto*, for example, allows players to create cityscapes. In most cases, as with *Grand Theft Auto*, the sandbox activity is incidental to the main purpose of the game. In *The Sims* series, building is an integral part of the game but for most players is unlikely to be the main focus of play. Players can choose to put their families into ready-made houses which they may subsequently adapt or they can build

from scratch on empty lots. *Sims* players may also offer their houses for other players to download or purchase.

In the ‘game play’ considered in this paper the sandbox activity is the focus of the game. Is this still ‘game play’? Our view is ‘yes, it is still game play’. In support of this we cite Salen and Zimmerman (2003) who define a game as “a system in which players engage in an artificial conflict, defined by rules, that results in a quantifiable outcome” (p.80). In other words, a game is interactive, challenging, it has rules and boundaries and has consequences. In Costas’ case, playing the sandbox ‘building mode’ of the *Sims* and his activity thereafter was his own choice. His activity within this mode was interactive, engaging and challenging because, as shall be seen, in each abandoned house Costas had to compete with himself, working with the game’s rules and constraints in order to build a house which was his quantifiable outcome.

Gee and Hayes (2010) point out that designing and building in *The Sims* requires considerable knowledge of geometry, arguing that the “tools require one to use a good deal of geometry to get all the angles and shapes to fit perfectly together” (p. 114). However, unlike much school learning of geometry, geometrical practice in *The Sims* is situated and contextualised. Lave (1988) showed how people are more able to solve real-world mathematical problems when they are situated in a real-world context (in that case, supermarket shopping) than when expressed as school-type problems. Similarly, Prensky (2006) describes anecdotally how players in *The Sims* can learn about everyday and financial household issues through game play. Gee and Hayes (2010) also show how an individual can develop geometry knowledge and skills by building in a virtual world (in their case *Second Life*). They argue that in contexts such as *The Sims* and *Second Life* the concepts are not only situated but also embodied, i.e. players can walk their virtual selves through the buildings and so understand how the abstract numbers and angles relate to the three-dimensionality of the physical world.

#### 4 Setting and Methodology

The research focused on the mathematics used in a child’s game play of a popular video game, *The Sims 2* (2006). This paper reports on an exploratory case study (Yin 2003) of one 11-year-old Cypriot boy, Costas, who satisfied three a priori criteria: prior experience of computer game play but not of *The Sims 2*; ability to read and understand English (the language of the game); perceived willingness and ability to express himself to the researcher (first author, a Cypriot). Costas was told that the game play was for research purposes but was not informed of the mathematical focus until after data had been collected. The study was conducted in Costas’ bedroom and on his computer; case studies should be conducted “within its real-life context” (Yin 2003, p. 13). Five meetings took place. The first was ‘off the record’ and the other four were recorded (procedures are explained below). Approximately 3 h of data was recorded. The intention was for Costas to build two houses: the first to be built without budget constraints, the second to be built for a specific family and to a strict budget. The first meeting, for which researcher’s post observation notes were the only data, was free play familiarisation, assisted by the researcher, on any aspect of the game. At the end of this meeting Costas, fortuitously, asked to build his own *Sims* house as his task.

A note on building a house in *The Sims 2*. Figure 1 shows Costas’ partially completed second house with swimming pool. The player is constrained to work in a rectangular grid and with predetermined building tools and extras (doors, windows, swimming pool) but is

**Fig. 1** Partially completed house with swimming pool



free to choose how to assemble these. Floor space, wall units and extras cost *Sims* money. Knocking down and rebuilding a part of a house costs money because the refund on floor units (called “cubes” in *The Sims 2* and, hereafter, in this paper), wall units and extras is less than the original cost.

The next four meetings were recorded using screen capture recording software which recorded all screen activity and discourse; this formed the primary data for analysis. Discourse was in Greek but was transcribed and translated into English. The researcher acted as observer participant but all decisions regarding the house were made by Costas, the researcher simply encouraged Costas to express his thoughts and occasionally helped him with purely technical matters. Costas built, at his request, three houses; the first two without constraints. He called the third house his “dream house” for his family and he had a modest budget of 40,000 in *Sims* money which he regarded as Cypriot pounds. He was extremely motivated—the house was to be perfect and the meetings ended when he said, with satisfaction, “Seems good. OK, I’m done. I think the family should take it from here.”

Data analysis was conducted in three stages. The first author carried out the data analysis (from Greek transcripts) and the second author conducted independent analysis on selected data (from English translations). The first stage produced open codes à la Strauss and Corbin (1998) with regard to Costas’ actions during game play. The second stage of data analysis, isolating problems, arose from an observation in conducting the first stage analysis—a pattern of work was detected, Costas usually planned to include a feature, e.g. a door, then executed his plan using calculations and then considered the appearance of what he had done. We referred to these sequences as mini-problems. Many of these mini-problems were nested in a manner that will be described in the next section. The third stage of data analysis arose from a question in stage two, how were mini-problems initiated and how did they end? In the course of addressing these questions we looked at the goals<sup>5</sup> Costas needed to accomplish to build his house. Some mini-problems had a single goal but some had several goals.

<sup>5</sup> ‘Goals’ is used in two distinct ways in mathematics education literature: the goal of activity, e.g. Leont’ev (1978), and emergent goals, ‘must do’ things that come into being during activity and often pass away, e.g. Saxe (1991). Reference to goals in this paper are to emergent goals.

## 5 Results

We present selected results from the three stages of data analysis and a description of an artefact-strategy Costas created to ensure symmetry. The description of the artefact-strategy is central to our later argument that Costas constructed an abstraction in his game play. The results from the second and third stages of data analysis are necessary to understand why and how the artefact-strategy was constructed. The first stage data presented is a summary and establishes that a lot of mathematics arose in Costas' game play.

The first stage of data analysis produced 14 categories which are illustrated in Table 1. Categories with consistent 'visible' mathematical content were calculations, symmetry, size/space comparisons, money matters and mathematical terms but mathematics also featured in other categories, e.g. planning. The examples in Table 1 illustrate recognising and building-with epistemic actions though, as noted above, we do not provide fine

**Table 1** Categories from the first stage of data analysis

| Category                                 | Description of action                                                      | Example                                                                                                                 |
|------------------------------------------|----------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------|
| 1. Calculations                          | Adds, subtracts, multiplies, divides, counts (verbally and/or by gestures) | Hmm, 144 pounds. That means 36 cubes. And the 4 that are not selected, that means 40 cubes the length                   |
| 2. Symmetry                              | Ensuring symmetry: refers to 'middle', 'same' and 'opposite'               | I can have 4 cubes from left and 4 from right. There, right in the middle                                               |
| 3. Concreteness                          | Seeks observable features                                                  | I need to colour the tiles so that I can see the cubes                                                                  |
| 4. Size/space comparisons                | Considerations of size and space<br>Comparisons of different objects       | The entire table takes only two cubes. The house is too big I should have made it smaller                               |
| 5. Questioning                           | Ask questions (to himself) about language/game difficulties                | Hey I've deleted it and instead of giving me 4 pounds back they gave me 3. Why is that?                                 |
| 6. Money matters                         | Performs actions in a way to save money for the family                     | 275 pounds for a bar? No way I think I'll take these ones Much cheaper                                                  |
| 7. Previous houses                       | Refers to previous houses, strategies, designs                             | I want to put the pool in the middle of the house like the other house. ... I'll choose the same doors like before      |
| 8. Appearance considerations             | Considers the way his constructions looked like                            | Seems nice, this looks better than before                                                                               |
| 9. Mathematical terms                    | Uses mathematics language                                                  | Equal ... I've added ... area of $18 \times 18$                                                                         |
| 10. Errors and feedback                  | Reacts to feedback from the game                                           | Error: needs something to support the attic<br>Reaction: moves the attic towards the roof                               |
| 11. Planning                             | Plans his next move, design, calculations                                  | So I was thinking of colouring the cubes that I want to delete to see what to delete                                    |
| 12. Difficulties in using the SW         | Experiences difficulties in using the software                             | I must get used to the cameras for viewing                                                                              |
| 13. Experiencing the reality of the game | Makes considerations about the family's wellbeing, interests, safety       | I'll leave some tiles (upper veranda) so that people can be protected from the sun when they are at the pool downstairs |
| 14. Gestures                             | Uses hands or mouse to show something                                      | I think I will put it here (moves the mouse over the area)                                                              |

The columns give, respectively, the code names of the categories, descriptions of the categories and translated protocol excerpts to illustrate the categories

analysis of the development of these actions in this paper. The examples, bar that for ‘mathematical terms’, show that Costas’ actions emerged in game play, i.e. that the technology was an essential, not an ancillary, part of his activity and that thought and game play were irreducibly linked.

The second stage of data analysis isolated 42 mini-problems. We present a nested set of mini-problems below which were important to his later abstraction. Costas had already created the foundations of the house and the walls of the lower floor. He also managed to create the swimming pool in the middle of the house. Then he wanted to create a door for the family to enter the house and he wanted it to be in the middle of the main entrance’s wall. The length of the main entrance’s side of the house was 15 cubes (see Fig. 2). However, when Costas tried to create the door he realised that:

Costas Ah! The door takes 2 cubes (in length) and the house is 15! I can’t put it in the middle. I think I will delete a column and in this way there will be 14 cubes

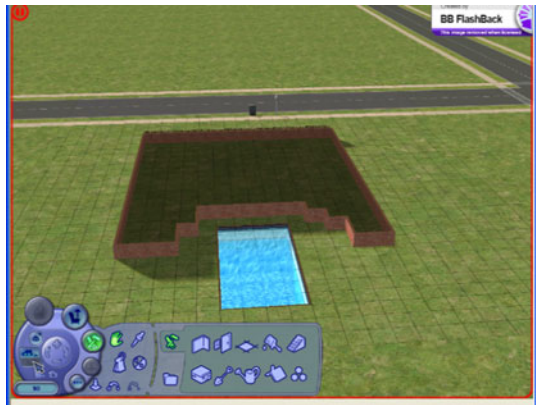
But when Costas deleted the column he realised that:

Costas The right side of the house is different from the left ... on the left side there are 3 cubes but on the right side only two (see Fig. 3)

**Fig. 2** Ensuring a door is centrally placed



**Fig. 3** House is not symmetric



**Fig. 4** Adding cubes to ensure symmetry



After thinking about his next move he said:

Costas I'll add two more cubes here ( $2 \times 1$  row) and then they will be equal. (see Fig. 4)

When Costas finished with the back of the house, he returned to his original problem, which was creating the door in the middle of the main entrance and said:

Costas Since it's 14 cubes now... then the door must be put after the 7th. One, two, three, four, five, six, seven... I think this is the place, it looks in the middle

Costas started with the problem of getting the door in the middle of the house, mini-problem 11 (MP11). However, in his effort to do so, he came across two other problems: (a) he had to change the 'cube length of the wall for the 'two cube' door to be centrally place (MP12) and (b) making the left and right side of the house the same (MP13), which he needed to solve in order to get the first problem (MP11) solved. These problems occurred in building the second house and the loss of money was a reason the house was discarded and for designing an artefact that would ensure symmetry.

The excerpts presented above show that Costas' actions emerged in game play, that thought and game play were irreducibly linked. As mentioned, the third stage of data analysis arose from the question, how were mini-problems initiated and how did they end? In the course of addressing this question we looked at the goals Costas needed to accomplish to build his house. Some mini-problems had a single goal but some had several goals; we isolated 55 goals in the 42 mini-problems.

Each goal had a start point from which it was initiated, a procedure by which Costas tried to accomplish the goal and an end which indicated the accomplishment (or not) of it. Through discussion between the first two authors coupled with revisiting the data on the basis of initial codes and influenced to some degree by Saxe's (1991) four parameter model, the data was coded into five categories: (1) *computer artefacts and conventions* (C), both artefacts that Costas created and features of *The Sims 2*, e.g. ready made tools in *The Sims 2*; (2) *prior mathematical knowledge* (M); (3) *prior everyday knowledge* (E), e.g. considerations about where to position rooms; (4) Combinations of (1)–(3), e.g. C/M/E, C/M; and (5) *others* (unspecified actions (U), restart, undo). Most coding were combinations of two of C, M and E as it was difficult to separate the components, as they seemed to 'blend together'; we will return to this point in the Discussion section. Table 2 shows a subset of the codes produced. We present selected mini-problems in Table 2 to illustrate: the codes associated

**Table 2** A subset of the outcome of the third stage of data analysis

| Problem | Goal |                                             | Start | Procedure | End   |
|---------|------|---------------------------------------------|-------|-----------|-------|
| 11      | 12   | Put the door in the middle of the house     | C/M   | C/M       | C/M   |
| 12      | 13   | Make the cubes 14                           | C/M   | C/M       | C/M   |
| 13      | 14   | Make the right and left hand sides the same | C/M   | C/M       | C/M   |
| 41      | 52   | Choosing the appropriate doors              | C/E/M | C/E/M     | C/E/M |
| 42      | 53   | Create the family's bedrooms                | C/E   | C         | C     |
|         | 54   | Decide the number of the bedrooms           | M     | E         | C/M/E |
|         | 55   | Decide where to put them                    | C/E   | C/M/E     | C     |

The table shows, from left to right, mini-problem number, goal (number and description), codes for how the mini-problems initiated, how they were processed and how they ended

with the nested mini-problems (MP11–MP13) discussed above; MP41, which will be referred to in the Discussion section; to show one mini-problem (MP42) which generated more than one goal. The following quote from Costas accompanied MP41 game play:

What? 350 pounds for the door [glass-door] Oh! ... that's expensive... well... there are more expensive ones, but... there are also cheaper ones... but, I want them [the family] to see the pool from the living room. Well, it's three doors for the lower floor and one for the master bedroom upstairs... That's up to 1500<sup>6</sup> pounds [he sighs]. I guess it's OK.

In this quote we can see the interplay between C, M and E: *The Sims 2* (C) provides a fixed choice of doors; Costas picks a door from this choice from his everyday experience (E) of what the family would like; he calculates (M) the cost of four such doors; he judges (E) that this cost is reasonable.

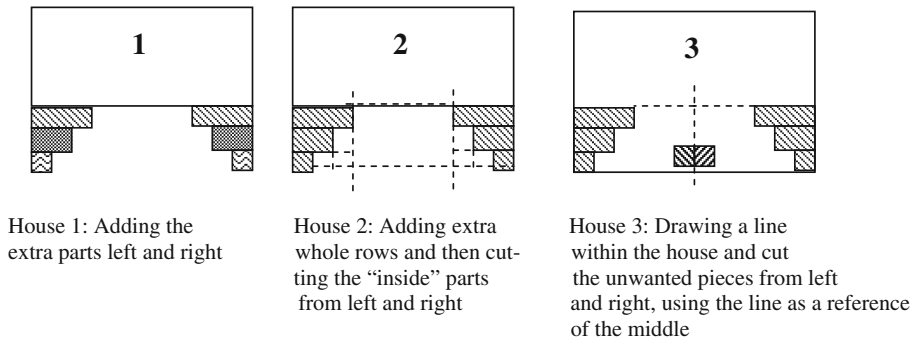
*The Sims 2* (which is included in the code 'C') was important with regard to goals: seven goals started and 13 ended as a result of its features and a further 44 started (34 ended) as a results of a combination of *The Sims 2* features, mathematics and everyday knowledge. For example, as mentioned, the goal of avoiding losing money by deleting cubes was initiated by the fact that the game did not return the full money value of deleted cubes but this is only important if this financial loss (mathematics) is regarded as important (everyday knowledge). With regard to goals and the final house, the experience of losing money by deleting cubes was clearly important to him. His first mini-problem in building the third house was *How to build the foundations of the house?* He had two goals: to make the foundation  $18 \times 18$  cubes; to ensure that this would allow a centrally placed door. We use the term *episode* to describe a set of related (including nested) mini-problems and goals.

Thirteen of the 55 goals were related to making his house symmetrical (though he did not use this term, instead he used the terms 'middle', 'same' and 'opposite'). We end this section by reporting on five episodes of Costas's work, in the order in which they occurred during his activity, where he attempted to ensure symmetry.

### **Episode 1** *Making the swimming pool in the middle of the house (house 1)*

While working with house 1, Costas created the foundations of the house and he wanted to create the pool in the "middle of the house". In order to do so, he added extra rows of cubes left and right of the existing foundation of the house as shown in Fig. 5(1)

<sup>6</sup> Costas said 1,500 pounds instead of 1,400 in this estimation, however we regard it as his attempt to approximate the calculation.



**Fig. 5** Costas' ways of creating the pool in the middle of the three houses

**Fig. 6** 1st use of the artefact-strategy (house 3)



### **Episode 2** *Making the swimming pool in the middle of the house (house 2)*

Costas created the foundations for the second house and in order to design the swimming pool in the middle of the house he added extra whole rows (Fig. 5(2)). In order to achieve the same result as in the first house, he deleted the “inside” parts of the extra rows, instead of adding extra pieces. Costas said: “OK, it’s 15 cubes all together, so, let’s say 7 cubes for the pool, that’s 4 cubes on the left and 4 cubes on the right. That’s OK I think. So I will delete  $3 \times 7$  cubes from the middle”. Then he deleted cubes from left and right in order to get the same shape of the house as in the first house.

### **Episode 3** *Getting the door in the centre of the main entrance’s wall (house 2)*

Costas wanted the main entrance’s door to be set in the centre of the wall. This has been described above (MP11–MP13).

### **Episode 4** *Getting the house in the middle of the plot (house 3)*

At the beginning of building the third house Costas wanted to create a foundation of  $18 \times 18$  cubes for his house. He also wanted to know where the middle of the foundations would be in advance so he created two cubes (which he set at the back side of the house, see Fig. 6) which he put near the middle of the empty plot,<sup>7</sup> and he said: “the middle is the line between those two

<sup>7</sup> Costas thought that he had put the two cubes in the middle of the plot but he had not. He lost track counting and misjudged. However, he based his building upon the “false” idea of those two cubes set in the middle of the plot.

*cubes*". Later he used those two cubes as a central "point of reference" to build the foundations; he counted out a row of 8 cubes starting from the left of the two-cubed "point of reference" and another row of 8 cubes starting from the right, so that he could get  $8 + 2 + 8 = 18$  cubes overall, which was the length of the foundations that he wanted. In this way he had marked where the middle of the back side of the house was. This two-cube-point-of-reference (artefact) together with the strategy of counting an equal number of cubes from the left and from the right is what we mean by an "artefact and an accompanying strategy" and we will henceforth refer to it as the 'artefact-strategy'.

### Episode 5 *Making the swimming pool in the middle of the house (house 3)*

Costas had already created the  $18 \times 18$  foundation for the house and he decided to design the swimming pool on the left side of the house and not at the back as it was his original plan. Costas said:

Since the other houses were too big when I added extra rows for the pool, I am thinking of cutting the (unwanted) cubes differently this time. I think I will draw a line in the middle like I did with the cubes (he meant the two-cubed point of reference) before, and then start cutting from left and right.

He counted the cubes starting from left to right until he reached the 9th cube and said: "the middle is the 9th and 10th cube together, because it's 18". He then painted them as black, to see what to cut. He used the black cubes as an outline of what he would cut, in order to get the swimming pool in the middle of the foundation (see Figs. 7, 5(3) for greater detail).

## 6 Discussion

As we said at the start of this paper, this work began as exploration of what mathematics is involved in playing *The Sims 2* but the interpretation of data analysis led to some difficult questions about: what we regard as mathematics; the nature of abstraction; types of mediation; the distinction between scientific and everyday concepts; educational design; and 'target knowledge'. Our presentation above has also raised issues concerned with learning and emotions and the influence of technology in Costas' activity. We do not have definitive answers to all of these interrelated matters but they merit a discussion in the light of related literature. We structure the discussion in four sub-sections: mathematics, context,

**Fig. 7** 2nd use of the artefact-strategy (house 3)



task ownership and knowledge; the construction and use of the two cube artefact is a mathematical abstraction; instructional intent and design; and game play, non-traditional mathematical abstractions and mediation.

## 6.1 Mathematics, Context, Task Ownership and Knowledge

In this subsection we argue that Costas' goals, decisions, meanings and actions were shaped by the fact that he 'appropriated' (in the sense used by Wertsch 1998) this activity as his own and by the interplay of mathematical and everyday knowledge and artefacts in the context of video game play.

Some of the mathematics that resulted from Costas' game play was 'visible' (Noss and Hoyles 1996b), e.g. the category 'calculations' in Table 1,  $144 \div 4 + 4 = 40$ . This is worthy of note and shows that *The Sims 2* can provide a platform for school mathematics. However, much of the mathematics was what Noss and Hoyles might call 'invisible' but we prefer to call 'blended' as it was, in a complex way that we shall attempt to explain, blended with everyday knowledge and artefacts used in the course of game play that Costas appropriated as his own. We address appropriation first because the depth of his involvement was so strong and this, we feel, is important and relates to Roth's comments, referred to in Sect. 2, about "learning *generally*, including emotions". We also comment on the physical context of the game play (his bedroom) and the lack of instruction because these, we feel, are related to Costas' total appropriation of the task.

Although it was the researcher's intention to ask Costas to build a house, at the end of the familiarisation session Costas requested this without the researcher asking him to do this. The researcher said that he could build a first house without financial constraints prior to building a house to a given budget. Costas readily accepted this and referred to the final house as his "dream house". Costas abandoned (left incomplete) two houses, not just one, in order to get his dream house perfect. We regard the above as evidence that Costas appropriated the task of building his dream house to a budget as his own. The researcher introduced Costas to *The Sims 2* and occasionally suggested technical help but this, we feel, merely accelerated game play as *The Sims 2* is an internationally popular game that many children play with only virtual assistance. The researcher provided no assistance on the direction of game play or on any mathematics—and Costas was not aware, during game play, that the researcher was interested in his mathematical actions. Costas appropriation of the task was an intense emotional and intellectual appropriation.

The game was played in his house not in a classroom. We contrast this with Monaghan (2007) in a study in which students worked on an authentic task that *was set* by a company director and carried out *in a classroom*. Monaghan argues that students did not actually address the company director's task but transformed the company director's authentic task into a school-like task. Monaghan claims that the fact that students were working in a mathematics class mattered, as students stated that they expected to do school mathematics in such a classroom. So we feel that the physical context was important, that the task may have been appropriated differently in, say, a school mathematics class physical context. This, unfortunately, has negative implications for the potential for learning through game play in a school context.

We now consider what we have called 'blended mathematics'. We look at mini-problem 41 (see Table 2 and the accompanying quote from Costas). Costas' goal was to choose doors for the living room and the master bedroom's veranda. But this was not a free choice; like all choices in computer-mediated activity the software was a player in the 'distribution of agency' (Gresalfi et al. 2009). Costas had to choose from what was on offer, an

expensive glass door (he would need four of them) or less expensive but also less personally desirable doors. In making this ‘choice’ he used everyday and mathematical knowledge: everyday knowledge—a glass door is more suitable for the family to be able to see the swimming pool; mathematical knowledge—four doors at 350 pounds is “up to 1,500 pounds” and that was expensive for a family with an overall budget of 40,000 ‘pounds’, approximately. Nevertheless, Costas decided to buy those doors, even though they were expensive, because that suited his family’s needs, which was the object of his activity: to build a house for a specific family within a specified budget. Costas interpreted the *Sims* money as pounds, presumably Cypriot pounds. 350 Cypriot pounds is not relatively expensive for a door of an average Cypriot house which costs up to 250,000 pounds. However, in this game the family did not have a budget of 250,000 pounds but 40,000 ‘*Sims* money’. The fact that Costas interpreted the currency of the game as Cypriot money is additional evidence of the way Costas appropriated the activity. He interpreted the currency in a way that it made sense to him, using a currency system that he already knew and understood. However, at the same time, he was well aware of the particular situation and the context of the game.

The single goal linked to mini-problem 41 was the only goal that was initiated, processed and terminated by a combination of everyday knowledge, mathematical understanding and computer artefacts. Costas reached a (bounded) decision that, if taken within a classroom setting under the influence of “suspension of sense making” (De Corte et al. 2000) might have been very different. Costas blended his everyday and mathematical knowledge in a way that was meaningful to him in a task he had appropriated and in a familiar physical context.

We now consider the interplay between computer artefacts and conventions, both built-in tools of *The Sims 2* and those developed by Costas, and knowledge. We use his initial attempts to get a door centrally placed as an example to illustrate this interplay. This problem was initiated through the goal of getting the door in the middle of the wall. While he was actually *moving* the door (using the mouse) over the wall he *saw* that it was not in the middle, i.e. the goal emerged as he was interacting with computer artefacts (his own and built-in tools) and the problem was confirmed using mathematical knowledge, “Ah! The door takes two cubes and the house (wall) is 15! I can’t put it in the middle”. He then used his mathematical understanding of the situation to resolve the problem and successfully accomplish his goal (for house 2). Enlisting the support of mathematics to resolve a problem was quite common in Costas’ game play and appears in ‘breakdown situations’ in the use of mathematics in the workplace (see Pozzi et al. 1998). But again there was an interplay between mathematical and everyday knowledge, in this problem Costas’ brought his prior understandings of the aesthetics of an ‘ideal house’ into the resolution. Our interpretation of the data has not resulted in viewing either everyday or mathematical knowledge or the use of computer artefacts as primary, they are ‘blended’.

## 6.2 The Construction and Use of the Two Cube Artefact is a Mathematical Abstraction

*Abstraction in context*, as mentioned above, posits three stages: (1) a need for new mathematics; (2) construction of new (vertically reorganised) mathematics; (3) consolidation (use in further mathematical activity) of the new mathematics. We attend to all three stages in this subsection.

Costas was obsessed with the idea of making his house symmetric but this is not absurd as the practice of building houses often involves symmetric shapes. His desire to create the swimming pool in line with the middle of the house and the door in the middle of the wall

reveal aspects of his understandings of middle and of symmetry. Costas could find the middle cube of an odd number of cubes (the middle of a five-cubed wall was the third cube) but when he had to find the middle two cubes and the wall was a large number of cubes long he encountered a problem and had to modify his strategy of finding the middle; he needed (stage 1 of AiC's three stages) to find a cost efficient strategy for his dream house. When the problem of placing a central door occurred in the second house, Costas deleted a column of cubes in order to have an even number of cube wall and he said that the door should be put after the seventh cube (presumably dividing 14 by 2). This was a small but important expansion of knowledge about the middle.<sup>8</sup> It was computer-game-play-mathematics knowledge expansion, he was dealing with cubes, not numbers: the middle of 14 is 7 but the middle of the 14 cubes was "the line between those two cubes", "the 7th and 8th cube together". This *visual middle*, the 7th and 8th cubes is a representation "...specific information is contained in representations ... It is specific information that allows subjects to control for the meaning and reasonableness of their answers in problem situations" (Nunes et al. 1993, p. 147)—Costas' understanding of specific information was held in the representations, which in this case were the cubes. Costas constructed his two cube artefact, together with a counting cubes mode of using the artefact (his strategy) from these representations. Prior to using the artefact-strategy Costas did count cubes from left and right but, as noted, he experienced problems. Costas' constructed artefact-strategy was a vertical reorganisation of prior knowledge (stage 2 of AiC's three stages). As episode 5 in the Results section shows, he went on to use this artefact-strategy to make the swimming pool in line with the middle of the house (stage 3 of AiC's three stages). Whether or not Costas' artefact-strategy is an abstraction with regard to other theoretical accounts of abstraction, it is an abstraction with regard to AiC.

Commentators on a draft of this paper have said, in so many words, "OK, this is a kind of mathematical abstraction, but is it really 'theoretical abstraction'?" This is a good question; our answer is 'yes' and we now attempt to justify this answer. We first note that one reason that we have focused on AiC in this paper is that Costas' artefact-strategy is an abstraction with regard to AiC; it satisfies all three stages of AiC, as argued in the paragraph above. We are not aware that any one has argued that AiC is not in the theoretical abstraction camp. But this is an argument based on authority and we would like to go beyond this argument. If we go to 'past masters', Vygotsky and Davydov, then our 'yes' answer initially runs into problems. By the Vygotskian 'disciplinary knowledge of a scientific community' criterion Costas' artefact-strategy is not a theoretical abstraction but we have, in Sect. 2 above, laid out our thinking that this criterion is not sufficient in the case of mathematics. We move on to Davydov. On the face of it, it may appear, through Davydov's focus on the 'essence', that Costas' artefact-strategy is not an abstraction because there is no preformed construct from a scientific community for which to find the essence. This is founded on the scientific-everyday divide mirroring established-emergent knowledge and a theme of this paper, clearly, is that we do not accept this view. Another view of Davydov's essence is "Grasping the essential relation in a situation is getting the big idea behind it ... [which] requires analysis, establishing relationships between what is known and the big idea" (Dreyfus and Monaghan 2009, p. 104). In terms of Mason's delicate shift of attention this involves foregrounding the big idea to work with it to do mathematics. This is what Costas did with his artefact-strategy; this arose from what he

<sup>8</sup> In a lecture on Costas' work a participant said "but I use strategies like Costas' when I do *do-it-yourself* around the home, it is pretty trivial". To this we repeat, from Sect. 2, "new mathematics will depend on the person(s)—what is new to one person may be familiar to another".

knew by trial and error in houses 1 and 2, he established the relationship between this artefact-strategy and calculations and then he foregrounded and used this artefact-strategy to do mathematics to build a cost-efficient house (number 3).

### 6.3 Instructional Intent and Design

We consider a potential difficulty with Costas' artefact-strategy as an abstraction in the sense of AiC and vertical mathematization, that there is no target knowledge in Costas' activity. Vertical (and horizontal) mathematising was originally a construct of Treffers but was soon taken up by others, including Freudenthal (1991). AiC is inextricably linked in the literature with vertical mathematisation:

Vertical mathematization relates to reorganization within mathematics, for instance, by finding shortcuts and discovering connections between concepts and strategies and then applying these discoveries. (Schwarz et al. 2009, p. 16)

We think this describes precisely what Costas did but a senior academic in the Freudenthal Institute in a private communication on Avraamidou and Monaghan (2009) said “How is Costas' work vertical mathematisation? Vertical mathematisation links a learner's constructed maths to a target mathematical idea.” We have not found literature on vertical mathematisation to support this claim that there must be an instructional target mathematical idea but there may be. If there is support for this, however, then we would simply like to take a different stance on vertical mathematisation. We do feel that vertical mathematisation *almost always* links a learner's constructed mathematics to a target mathematical idea simply because most developed mathematical abstractions are so far removed from nature that we would not expect anyone to make them without mediated direction to a target mathematical idea. But there are exceptions and, with continued increase in use of technological game play, we feel exceptions will continue to arise; this was one of the reasons for writing this paper. This relates to our qualified objection (see Sect. 2) to Tiberghien and Malkoun's (2009) statement on “the mediation of a teacher, or a more informed peer”. Costas' abstraction did arise without mediated direction to a target mathematical idea but it was ‘artefact mediated’.

Similar comments can be directed towards design—any educational design: teaching sequences, tasks, software/tools and combinations of these. An explicit focus on design has become ‘trendy’ in North American influenced mathematics education research in the last 10 years since the special issue of *Educational Researcher*<sup>9</sup> on design experiments and Cobb et al.'s (2003) call for a greater focus on design. But design, as Ruthven et al. (2009) note, has a long tradition in European mathematics (and science) education research. In the realms of abstraction Pratt and Noss (2002) see the fundamental design problem of mathematics education as gaining “a handle on how to provide tools and activity structure with optimal opportunities for supporting the construction of new knowledge ... [and] It is here that we believe that digital technologies hold particular promise.” (p. 460) Although we do not accept that all educational design is good (in some value system), we applaud developments in educational design but mathematical learning can go on without explicit educational design as Costas' abstraction illustrates.

<sup>9</sup> *Educational Researcher*, 32(1), 2003.

## 6.4 Game Play, Non-traditional Mathematical Abstractions and Mediation

Costas' mathematical activity was greatly influenced by *The Sims 2*. The video game's features facilitated the interrelation of his mathematical and everyday knowledge and the majority of Costas' goals were initiated and terminated, at least in part, by features of *The Sims 2*. The computer game was a means by which Costas' mathematical ideas and meanings became visible to him. We do not make general claims from this case and, indeed, believe that Costas may not have formed his abstraction (with regard to AiC) if the researcher present had said "it does not matter if you exceed your budget" or if Costas had not been so intent on building his dream house.

The mathematical abstraction that Costas constructed and used in the course of game play is not a part of scholastic mathematics—it is not mathematical knowledge 'privileged' (Wertsch 1991) in mathematical curriculum documents. It is, however, to us, certainly mathematics in that he engaged with relationships between objects (even if these objects were, to him, cubes and not numbers). Further to this we view that Costas engaged in vertical mathematisation in the sense of Freudenthal (1991) even if a leading Freudenthal Institute researcher believes that vertical mathematisation must have a target mathematical idea. The value judgement that vertical mathematisation must have a target mathematical idea is related to values associated with privileged mathematical knowledge. Costas' case shows that non-privileged mathematical knowledge can be constructed through computer game play.

With regard to claims that *learning will not happen without the teacher*, we feel these may be best kept to learning privileged knowledge. But Costas' abstraction would not have come about without artefact (*The Sims 2*) mediation, so we are happy with a claim that *learning will not happen without mediation by person or artefact*.

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