

EXPONENTIAL FUNCTIONS, RATES OF CHANGE, AND THE MULTIPLICATIVE UNIT

ABSTRACT. Conventional treatments of functions start by building a rule of correspondence between x -values and y -values, typically by creating an equation of the form $y = f(x)$. We call this a *correspondence* approach to functions. However, in our work with students we have found that a *covariational* approach is often more powerful, where students working in a problem situation first fill down a table column with x -values, typically by adding 1, then fill down a y -column through an operation they construct within the problem context. Such an approach has the benefit of emphasizing rate-of-change. It also raises the question of what it is that we want to call 'rate' across different functional situations. We make two initial conjectures, first that a rate can be initially understood as a *unit per unit* comparison and second that a unit is the *invariant relationship between a successor and its predecessor*. Based on these conjectures we describe a variety of multiplicative units, then propose three ways of understanding rate of change in relation to exponential functions. Finally we argue that rate is different than ratio and that an integrated understanding of rate is built from multiple concepts.

INTRODUCTION

Constructivism is widely used to support reform efforts in mathematics education. In the community of mathematics educators, significant attention has been paid to its radical implications for student learning. Children, and for that matter, all students, are recognized not simply as inadequate or incomplete adults, but as thinkers and sense-makers in their own frames of reference. Constructivists have effectively documented that student errors are seldom random or capricious – they have a rationality and functionality of their own. In this regard, constructivists have documented that teachers and researchers must pay close attention to how a mathematics problem is conceptualized, worked on and evaluated by students.

What has been given less attention is that the constructivist research program implies much more than revising our views of students and their learning. Our own understanding of the mathematics needs to be challenged and reformulated in light of students' approaches and methods. All too often, the constructivist agenda is reduced to the question of how to get the rich and diverse concepts and strategies of students to be aligned with our own more expert views, and those expert views are assumed as complete, well-structured and non-problematic. One result of such an interpretation of a constructivist program is that reform efforts which attempt to open up and rethink the mathematical content are targeted mostly at the elementary grade levels, while secondary educational reform is more typically limited to pedagogical approaches as the content is assumed to be well-secured in its expert structure. We would claim that we witness too little attention to how

the research on student conceptions demands that we rethink what we claim as mathematical knowledge across *all* grade levels.

In addition, the increasing recognition that a teaching experiment, in which a researcher and a student interact around a set of tasks, is itself a social interaction implies that results need to be interpreted as a particular form of discourse. The interviewer is trying to understand both the student's problematic and what s/he is doing and attempting to do to resolve that problematic. Simultaneously, the student is constructing a problematic in conjunction with what s/he views as the interviewer's expectations and expects that the interviewer will be more expert in the tasks in which s/he then engages. The result is that when the interviewer analyzes the data to build a model of what the student's actions, words, schemes and concepts are like, the interviewer's own understanding of the task influences that model. What has often been neglected in the past is that what the interviewer learns about the student (as s/he constructs a model of the student) evolves together with her/his own changing understanding of the subject matter. Because of this inseparable and simultaneous development, researchers have tended to describe these changes only in terms of the models of the student and in doing so have neglected their own learning.

In previous work, Confrey (in press a) has labeled this as a distinction between voice and perspective. Voice is used to describe the model of the child – and interviewing is an attempt to give voice to a child's ways of approaching a problem. Perspective surrounds voice, as the context from which the interviewer hears, probes and develops and sets new tasks. In analyzing the data, Confrey suggested that researchers need to try to separate out these two different strands and to acknowledge the extent to which the researcher's own perspective changes – one's own mathematical knowledge evolves through interactions with the student and through the analysis of student voice.

This paper provides an example of how researchers' mathematical knowledge needs to be examined in light of research on students' conceptions. We extend the analysis of perspective even further than in Confrey's previous work to argue that analyzing student data can lead to a need to reexamine and extend our mathematical understanding *beyond the immediate reaches of student voice* – into new territory. This has led us to a whole new territory of mathematical meanings for functions, units and rates.

Although the *National Standards for Curriculum and Evaluation* (NCTM, 1989) advocates placing the topic of functions in a central position in the secondary curriculum, it does not explicitly address the epistemological issues raised as the treatment of functions moves away from an emphasis on formal definitions. In our own work, we have found that as students generate functional relationships by acting within contextual situations and by using multiple representations in both creating and representing their solution processes, legitimate and diverse ways of thinking about functions are created. In this process the meaning associated with the function concept is created through the interaction of context, multiple representational forms, and technological tools (Confrey, 1991a, 1991c, 1992; Rizzuti, 1991; Rizzuti and Confrey, 1988; Smith and Confrey, 1992; Confrey

et al., 1991; Afamasaga-Fuata'i, 1991; Borba, 1993; Borba and Confrey, 1992). This research has led us to challenge many attempts to label alternative students' conceptions as either misconceptions or as "prefunctional" (Dubinsky and Harel, 1992). We see this as too often indicative of a narrow view of the function concept, bounded by its placement in a decontextualized and heavily abstract view of mathematical systems. Such a view ignores the richness and viability in the student conceptions and thus lacks serious attention to the epistemic content of the student *voice* and the resulting implications for the researchers' own *perspective*.

In this paper, we have chosen the issue of rate of change as an entry to thinking about functions. Recent research has shown that even young children can use rate of change as a way to explore functional understanding (Nemirovsky and Rubin, 1991; Confrey, 1993; Kaput and West, in press). Arguing against the delaying of the introduction of rate of change issues until calculus, where it becomes studied as a 'property' of functions, we have shown that students exhibit strong intuitive understandings of change and can use this understanding to generate functional relationships. This rate-of-change approach is also closely related to a "covariational" understanding of functions which has both historical roots (Boyer, 1968; Smith and Confrey, in press) and is closely related to student problem-solving actions (Confrey *et al.*, 1991; Rizzuti, 1991). Thus, the balance of this paper will begin with a discussion of covariation then move to examples from linear and exponential functions, discussing our initial discoveries about the exponential function in terms of its underlying operational character and its multiple meanings for rates of change. By analyzing these different meanings, we offer a new conceptualization of the unit and rate concept and articulate the role of the 'split' in rate of change.

Covariation as an Alternative Function Concept

We have described two general approaches to creating and conceptualizing functional relationships, a correspondence approach and a covariation approach (Confrey *et al.*, 1991). Most common in the curriculum is the correspondence approach in which one initially builds a rule that allows one to determine a unique y -value from any given x -value. Thus one builds a correspondence between x and y . This approach is emphasized in conventional algebraic notation in which we write:

$$y = f(x).$$

A covariation approach, on the other hand, entails being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1} . For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table.¹ In working with many applications, students find a covariation approach easier and more intuitive. It can also lead nicely to an algebraic coding of the correspondence rule for a function; however, this skill is learned through a variety of experiences. Supporting the distinctiveness of a covariation vs. correspondence approach, we find that even those well-schooled in

building correspondence relations can face a challenge in moving from covariation to correspondence approaches. For example, in an informal meeting, a group of secondary mathematics teachers were quick to approach to a tuition problem (tuition increases at a rate of 11% per year) as covariation, but hypothesized that the correspondence equation would be a polynomial rather than an exponential.

RATE OF CHANGE

A covariation approach to functions makes the rate of change concept more visible and at the same time, more critical. Rizzuti (1991) documents that students find it useful to describe a function in terms of its rate of change, both graphically and numerically. She further illustrates that when students were given arithmetic sequences and geometric sequences, each mapped onto the natural numbers, and asked to describe their graphs, they discussed the differences in terms of rates of change even though they had not been introduced to this language. One student explains the difference in relation to the numeric values:

Umm..... maybe its because with geometric you multiply, so your numbers would – like you would go from, for example, -2 to -8 to -32 . There's so much of a gap between the numbers. Whereas in an arithmetic [sequences] there's gonna be the same amount of gap between the numbers. [p. 126]

Not all students argue as this one did that a geometric sequence has a changing rate. Some students describe the rate of change of the exponential function as constant, consistent with the argument by Confrey (in press b) that a multiplicative rate of change could be regarded as constant. Other students show evidence of multiple views of the rate of change construct as illustrated in the following example. Given tabular data on the growth of bacteria over time, students were asked why the numbers were so large. One student replied:

Because it's just increasing so rapidly. It's increasing by 16 each – it starts out with three. But just every minute, each of them increasing by 16 – it divides itself up. It makes itself into 16 more cells. [p. 134]

This data begins to suggest the multiple meanings of the term 'rate'. We discuss this further in the following section.

Rate of Change in the Exponential Function

Our early work on exponential functions has led us to consider rate of change as a primary point of entry when working with students. This analysis was spurred by our research on students' development of exponential functions wherein we witnessed:

1. A student building a number line which combined primary units constructed multiplicatively as powers of ten with subunits increasing additively between each pair of primary units (Confrey, 1991c). This helped us begin to recognize the viability of a multiplicative unit.

Table	
t	c
time (hrs)	number of cells
0.00	9.00
1.00	81.00
2.00	729.00
3.00	6561.00
4.00	59049.00
5.00	531441.00
6.00	4782969.00
7.00	43046721.00
8.00	387420489.00
9.00	3486784401.00

Fig. 1. Growing cells.

2. A student working on a problem where the number of robots increases exponentially. He described the pattern in a table of time vs. the number of robots in the following manner. "If the time interval is multiplied by six then the number of robots produced would be raised to the sixth power" (Confrey, in press a). This allowed us to recognize his ability to coordinate two 'covarying' worlds, one built additively (where the unit is repeated addition) and one built multiplicatively (where the unit is repeated multiplication).

We point out that our discussion of rate will rely on its use in a variety of contexts. Thus, whereas we do not seek a context-independent interpretation of rate, we do seek a description of rate which can describe its use across various contexts.

We often begin teaching exponential functions with the following situation: The number of cells growing in a lab experiment has been recorded every hour. We show three entries of the table presented in Fig. 1. We ask: 'What would you predict for the number of cells present at the next reading?' We use this example to anticipate the question of what might happen at a half-hour, an issue of interpolation. This problem was designed to be discrete, while being amiable to determining a value for the number of cells at the half-interval. Splitting in biology happens most often in two's but does occur in three's and, hence, if sampled every other split, it happens in nine's.

A conventional analysis of rate of change in this problem would involve a straightforward calculation, taking differences in the 'c' column for each change of one in the 't' column. However, in the context of cell splitting, many students chose different ways to describe rates of change. Three such approaches include:

t	c	Δc	$\Delta(\Delta c)$	$\Delta(\Delta(\Delta c))$
time(hrs)	number of cells			
0.0	9.0			
1.0	81.0	72.0	576.0	4608.0
2.0	729.0	648.0	5184.0	41472.0
3.0	6561.0	5832.0	46656.0	373248.0
4.0	59049.0	52488.0	419904.0	3359232.0
5.0	531441.0	472392.0	3779136.0	30233088.0
6.0	4782969.0	4251528.0	34012224.0	272097792.0
7.0	43046721.0	38263752.0	306110016.0	2448880128.0
8.0	387420489.0	344373768.0	2754990144.0	
9.0	3486784401.0	3099363912.0		

Fig. 2. Differences for an exponential function.

1. *An additive rate of change.* Students will often start by calculating the differences between succeeding values of c . An additive rate of change is defined by finding the differences in succeeding c -values for constant unit-changes in the t -values. Typically they arrange their first column in incremental units of one, avoiding (or possibly neglecting) the unit change issue. By choosing the difference command, students can have Function Probe calculate differences for any column of values. Note that in this problem (Fig. 2), differences increase rapidly as one moves down the table. Also note that taking differences of differences, and differences of differences of differences continues to produce vertically increasing gaps between consecutive values. If, however, one looked at the ratio of these differences (see Fig. 3b), one would find the sequence of the differences to be exponential, a suggestion of what comes when one examines the derivative of an exponential. This pattern in the differences is unlike the situation students have experienced with polynomial functions where for a polynomial of degree n , the n th difference column shows constant values. (See Confrey, 1992, and Afamasaga-Fuata'i, 1991, for a discussion of a rate of change approach to quadratic functions.)

2. *A multiplicative rate of change.* Looking at the first two or three values, students will often suggest that the table is increasing by multiplying by nine, by 'a constant rate of nine'. This suggests a second concept of rate of change which is found by calculating the ratios between succeeding y -values for constant unit-change in the t -values. Students can check such a hypothesis by using the ratio command in Function Probe (see Fig. 3a). In this case, the rate of change is a constant ratio of nine, i.e., nine times as many cells per hour. Because we want to encourage students to develop a strong intuitive sense of ratio change as well as difference change, we have developed a symbol for multiplicative rate of change, $\textcircled{R}$ and encourage students to use both $\textcircled{R}x$ and Δx in their mathematical explorations.

t	c	@c	$\Delta(\Delta c)$	$\textcircled{\Delta}(\Delta(\Delta c))$
time(hrs)	number of cells			
0.0	9.0			
1.0	81.0	> 9.0		
2.0	729.0	> 9.0		
3.0	6561.0	> 9.0		
4.0	59049.0	> 9.0		
5.0	531441.0	> 9.0		
6.0	4782969.0	> 9.0		
7.0	43046721.0	> 9.0		
8.0	387420489.0	> 9.0		
9.0	3486784401.0	> 9.0		
			4608.0	
			41472.0	> 9.0
			373248.0	> 9.0
			3359232.0	> 9.0
			30233088.0	> 9.0
			272097792.0	> 9.0
			2448880128.0	> 9.0

Fig. 3. Ratios for an exponential (a) and for differences (b).

3. A 'proportional new to old' rate of change. We have also had students describe the pattern in this table as an 'increase by eight times' for each change in time by one hour. They describe the amount being added on each time as being eight times the last value. For example in the table of first additive differences (Fig. 2), the amount being added to 9 is 72, which is eight times nine; the amount being added to 81 is 648, eight times 81, etc. This is a way of describing what is new, what is being added, as a proportion of what is old, what was there before.

Many of us might think of this as an awkward way to describe the pattern in this context where the multiplicative rate is an integer. However, its usefulness becomes more apparent in the context of interest rates and compound interest. An interest rate of 5% per year means that the new money will be 5% of the old money and then must be added on to get the total amount in the account. This use of rate sits squarely between an additive and multiplicative treatment of rate, and can cause ambiguities in the use of language. For example, if one says that one's principle has increased 4% per year, one would assume that \$100 would become \$104 after one year. If one's principle has increased 200% per year, then one could mean that \$100 becomes \$200 (multiplicative rate) or \$100 becomes \$300 (proportional new to old) after one year. In compound interest, the formula for the final principal is $P_f = P_i(1 + r)^t$. The rate here is the r in the formula or 4%/year in the first example.²

What is a Rate?

These three alternative approaches to change in the exponential led us to claim that the treatment of rate in many mathematics textbooks and classrooms is ambiguous and overly narrow. Our epistemological investigation led us to argue that the dominance of calculus in the mathematics curriculum encourages instructors to emphasize an additive view of rate while neglecting the legitimacy of the multiplicative approach. Our intention is to remedy this situation by building a broader analytic approach to rate of change as follows: *rate is a unit per unit comparison*. This analytic approach will be embedded in a qualitative approach to rate and in

an epistemology of multiple representations. Before doing so, we need to discuss our approach to the concept of a unit.

THE CONSTRUCTION OF A UNIT

A unit is “the invariant relationship between a successor and its predecessor; the unit is [created as the result of] the repeated action” involved in numeration (Confrey, in press b). For example, in order to create a *counting* unit of one, a child must first recognize a multitude composed of objects sharing a particular quality. For young children the objects are typically identical, varying only in location or color, like three fishes. The child then points a finger at each object in turn and utters a sequence of number words. The unit which s/he creates is the result of the operation of carrying out mentally the repeated action (which may or may not be done physically by gestures) and forming it into a unit. Thus, adding one, the primitive that later is modified to become addition of $n > 1$, emerges concurrently with the unitizing activity, the count. This is a claim that the genesis of numbers and basic unitizing operations are inextricably intertwined.

As Steffe has demonstrated, creating units out of non-one groups involves a complex developmental sequence of making the units, treating the composite as a unit and iterating it, and finally creating an anticipatory scheme that allows one to imagine mental actions of carrying out the unitizing operations (Steffe, 1990, in press). In our work, we seek to broaden the definition of unit to include other underlying repeated actions, particularly the action of splitting (Confrey, in press b). However, before we examine the splitting construct, we wish to discuss the implications of defining a unit in terms of the mental operation of internalizing a repeatable action, rather than in terms of standardized units such as inches, pounds, etc. This approach to *unit* has a number of radical implications:

1. It stresses that units are not ‘things’ that can be disassociated from the mental operations involved in their construction. That which one identifies as the *same* (*unit-izing*) in these repeated actions evolves into the unit. As a mental operation, unitizing frequently involves situations into which a person enters with an intention or goal. The goal might be to answer the question ‘How many?’ or ‘How much is there?’, in which case the answer is intended to be a number. However, units can also relate to non-numeric goals as designing and cutting out swatches to make a quilt (see Fig. 4). The unit encapsulates what the actor perceives as repeated action. This involves a mental segmentation of a potential or real physical action. Our approach does not ignore standard unit-names (inches, lbs, etc.), but recognizes that these are social conventions. As these are standardized onto measurement tools, rulers, clocks, calendars, we tend to separate the units from their construction; however, watching children learn to use a ruler will quickly convince one of the process of the development of mental operations involved. Recall that feet were originally connected to the footstep and meters to the distance between the nose and the hand. As these standard unit-names become integrated into our construction

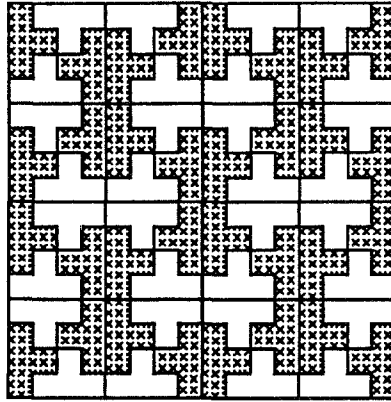


Fig. 4. A quilting pattern.



Fig. 5. Possible quilting units.

of units, they come to play a role in the units we construct.

2. It means that units involve mental operations for *comprehending change or pattern*. That is, we come to perceive pattern and/or change by answering two questions simultaneously: 'What is different?' and 'What is the same?' As we strive to answer these questions, we create the pattern or unit via *segmentation*. Sometimes an obvious choice of unit exists, such as in discrete cases when we count, for example, the children in a classroom. In others, we might override the discrete choice in favor of a unit that is perceived to be more relevant to our goal, such as when we buy potatoes by the pound rather than by the potato (to assure price uniformity over potatoes of varying sizes). Other times, such as in a rainbow, the segmentation is arbitrary and conventional. Does a rainbow always have eight colors? Consider the pattern in Fig. 4 (from Rubin, 1992). Depending on how one views it, it can be described as having multiple candidates for the unit, and each candidate leads towards different actions in creating the quilt. These units could include those presented in Fig. 5.
3. It implies that initially it is the 'unitizing' operations that create numbers. That is, it is the goal of finding out how many or how much and the operation of creating units that must proceed before numbers are available as an answer. That answer, represented as a number, is *made* from the units. Two errors are frequently made if one does not recognize this claim:

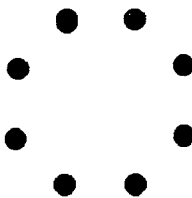


Fig. 6. Counts.

- a. First, treating numbers as independent from the unitizing operations used to create them can lead one to act as if all numbers are the same kind of entity. For instance, Greer (1988) has argued that we should encourage students to develop “conservation of operation” so that the operation is seen as independent of the number type. We would claim that numbers have “biographies” and that their character is inherently tied to how they are constructed. Thus we would view a claim for an interaction between a number type (ratio, decimal, fraction, integral) and an operation as a reasonable expectation on the part of students.
- b. Second, treating numbers as logically prior to the operations leads us to assume that all numbers are characteristically the same, regardless of how they are made. Thus the number distinctions we do make (counting, integer, rational, real) are only loosely related to the operational bases for the numbers and are, to a large extent, a posteriori distinctions. For example, we neglect the geometric origins of square and cube roots treating them simply as unending decimal strings like all other irrationals. Likewise, we call 0.2, $1/5$, and $\sqrt{1/25}$ the *same*, despite the differences one might expect in their constructions implied by the notations.
4. It can lead one to make certain assumptions about numbers while failing to recognize that assumptions cannot be separated from the operations used to make those numbers. For example, one counts to create the natural numbers and then the natural numbers have a unit of one, symbolizing the underlying repeated one-count. When we see a display such as the one shown in Fig. 6, we are likely to say immediately that there are eight objects. However, if someone then tells us that the figure is a schematic for planning the steps in a square dance, we look again and suddenly see four objects (couples). It is that which we make the same as we count around the circle (or square) that makes the unit and simultaneously creates that number.

Additive Units

To illustrate our treatment of units, we will work through an example. Suppose a house is located at a certain distance away from a cliff which is eroding towards the house. For five years, the person measures the distance to the cliff and the table is shown in Fig. 7. Since she wants to predict when the house will fall over the cliff, she analyzes her table for the amount of change in the distance each year.

t	d	Δd
time (yrs)	distance (inches)	units
0.00	100.00	> -3.50
1.00	96.50	> -3.50
2.00	93.00	> -3.50
3.00	89.50	> -3.50
4.00	86.00	> -3.50
5.00	82.50	

Fig. 7. An additive unit.

t	d	Δd
time (yrs)	distance (rope lengths)	units
0.00	28.50	> -1.00
1.00	27.50	> -1.00
2.00	26.50	> -1.00
3.00	25.50	> -1.00
4.00	24.50	> -1.00
5.00	23.50	

The unit as rope length

Fig. 8. An additive unit.

According to our units analysis, the repeated action in the example is the annual loss of 3.5 inches. Thus, we claim, in this example, the unit is -3.5 inches. Many would be inclined to identify inches as the unit. Inches do describe the repeated unit used in measuring a single 3.5, as the ruler is slid along to mark each inch, and as such they are units, but they are not units to be used in describing the experiential invariance in the erosion of the cliff. The decision to segment one's experience into annual intervals sets the stage for a unitizing operation to create a unit of -3.5 inches from the initial one inch measure. Thus, a unit is constructed in relation to an experiential situation which involves both segmentation and an invariance across segments. In this case, this invariance could be described as -0.29 feet or as -8.9 cm. Likewise, the owner could have constructed an equally viable unit by stretching a rope from the house to the cliff edge and marking off the annual changes in the rope length. This action would create the unit shown in Fig. 8. If she used that unit to measure the original distance and made a table equivalent to the one above, she could have created the table shown in Fig. 8.

Our selection of a standard unit name (inches) to describe the unit of change in a situation is one of convenience, efficiency and the form of measurement tool. The lesson here is that a unit is defined in relation to the situation and to the recognition of a repetition. In relation to these situations, we use unitizing operations similar to those described by Steffe to create a unit from standard unit names. However, the sense of invariance across units is not necessarily tied to the selection of the unit name. Communicating to others is simply made easier by the use of standard units.

What is particularly pleasing about such an approach is that if a student looks at this example in terms of its unit (-3.5 inches), is aware that this unit is created by segmenting the change over a year, and wants to create a correspondence relationship between the number of years and the the distance to the cliff, s/he may recognize that it will be:

{the initial distance to the cliff} plus {the number of years} * {the number of units}

and write:

$$d = 100 + t * (-3.5).$$

That is, we suggest that we might revise the traditional form of $y = mx + b$ to read $y = b + x(m)$ where m is the unit of change. That is, take an initial value and increment (or decrement) it by x units of change. We have found evidence that students often do, in fact, approach linear problems in this way, particularly when using the calculator on Function Probe (Confrey *et al.*, 1991b).

Multiplicative Units

This treatment of the unit creates an interesting possibility. Suppose one begins the process of quantification with a different action than counting. Could another equally legitimate unit emerge that would open up a world of quantitative possibility that remains hidden without it? This idea is the basis of the 'splitting conjecture':

Among the actions we observe in children and which can be interpreted multiplicatively are some that are related to addition (affixing, joining, annexing, removing, etc.) and others that seem relatively independent of addition (sharing, folding, dividing symmetrically, and magnifying). It is in these latter actions that we find the basis for splitting. In its most primitive form, splitting is defined as an action of creating simultaneously multiple versions of an original, an action which is often represented by a tree diagram. As opposed to additive situations, where the change is determined through identification of a unit then counting consecutively copies of that unit, the focus in splitting is on the one-to-many action.... Closely related to this primitive concept are actions of sharing, where verifying the outcome can be based on looking for congruence of parts, rather than counting... Splitting can also be differentiated from counting (and repeated addition) by its geometric connections to similarity. From this it can be seen that similarity forms the basis for our depth perception as we maintain the identity of objects as they move towards (magnification) or away (shrinkage) from us (Confrey, 1990, pp. 1–2).

Thinking of splitting in terms of the above description of units, a repeated splitting action should lead to the creation of a *multiplicative unit*. Consider, for example, the following situation in which a child shares a pie: a child splits a pie in half, and then splits the halves in half. In carrying out the action again,

would she do one additional cut (resulting in two quarters and four eighths) or two additional cuts resulting in eight eighths? In our experiments with children, they typically make two additional cuts. Their reasoning is that they want to have equal pieces, and to do so, they must cut each piece in half. The repeated action cannot be simply the physical action of cutting, but must be the action of cutting each piece into two equal parts. We would describe this as a three 2-splits, or three multiplicative units of two.³

In the cell example given earlier, there are nine times as many cells every hour. The invariance between predecessor and successor is most immediately described as 'nine times as many'. This is the unit. Notice we did not include the word 'cells' in the description. 'Cell' is not a description of the repeated action which created the unit in this situation. As opposed to the cliff problem, where the additive unit, -3.5 inches, could always be made from iterating 1 inch 3.5 times (viewed as taking away), one cannot create the splitting unit, '9 times as many' from 'cells' for there is no invariant relationship between '9 times as many' and 'cells'.⁴ In fact, if the number of cells was changed, for example, to the number of 'pairs' of cells (or number of 'hundreds', or number of 'millions', etc.), the invariance would still be 'nine times as many' each time. That is because the change between number of cells and number of pairs of cells involves counting in an additive world. Thus the standard unit names from the counting world will *not* be relevant to our discussion of units in the splitting world.

There are, however, standard units in the splitting world which will make a difference in how we describe multiplicative units. They do not look like our conventional additive standard units *at first glance*. However, by creating a language to describe them, they create an appropriate measuring device for converting among units in a splitting world. Let's begin by creating a 'state' type name for the unit nine times as many.⁵ Call it a nine-split. Now, suppose that in the cells example, the researcher observes the cells through a microscope and sees that every time an individual cell splits, it actually splits into three new cells. This observation may change the perspective she chooses to take. If she becomes interested in the 3-split from a biological perspective, she may choose to identify this as a unit and perhaps to create a table such as the one in Fig. 9. Although she may recognize that this repeated action will occur (on the average) at half hour intervals (and thus that the first column could be labeled 'Number of half hours'), her focus in creating the unit of a 3-split was a change from segmenting a unit in time (every hour) to segmenting a unit in reference to a biological process.

If the researcher maintains her interest in describing the repeated action each hour, the nine-split is still the unit of change. She may choose, however, to describe it as two 3-splits. The unit is created through her intention to seek out and organize her experience in terms of a repeatable action.

Within this multiplicative framework, it would be relatively basic to assert that the amount of time required for a quantity to double would be fixed over the life of the biological process. This is because the action of doubling creates a multiplicative unit which is then paired with an additive unit in time. Doubling (time) and halving (half-life) are standard multiplicative units in their own right

n	c	u = $\otimes c$
number of 3-splits	number of units	unit
0.00	9.00	
1.00	27.00	> 3.00
2.00	81.00	> 3.00
3.00	243.00	> 3.00
4.00	729.00	> 3.00
5.00	2187.00	> 3.00
6.00	6561.00	> 3.00

Fig. 9. A 3-split unit.

and become part of the fabric of the exponential function within such a perspective.

To convince one that this use of multiplicative units is not as strange as it might seem, consider the following situation. A child is asked how many biological great-great-grandparents she has. She answers '16' and says 'I doubled four times': 2 parents, 4 grandparents, 8 great-grandparents and 16 great-great-grandparents. The number of great-great-grandparents is 4 generations back of *doublings* for the little girl. A 16-split is four 2-splits. We propose that doubling and halving are the most primitive of units in a splitting world. We further propose that there are relatively few familiar primitive units in the sense of doubling and tripling. As argued before, 2-splits, 3-splits, 5-splits, 7-splits, 10-splits (often created as a 5-split followed by a 2-split in harmony with our fingers or toes), cover most splits we use regularly, i.e. most of what we would call standard multiplicative units (Confrey, 1990, in press b).

The different units in the splitting world may appear to be dull in comparison with the counting world with its linear measurements: feet, inches, centimeters, miles, etc.; its volume measures: cups, cubic cm; its area measures: square inches, acres; and its weights: tons, pounds, grams, and so on. Because of the more common usage of additive units, the vocabulary for describing them is more robust and widely recognized; however, we would also suggest that as we learn to be more discriminating in our visualization and articulation of multiplicative units, we will begin to make more distinctions and create more diversity within multiplicative worlds. The standard additive units we use can often be classified in terms of the situational action we take in making them. Likewise, in multiplicative situations, we have situational actions which serve to classify multiplicative units. Parallel to counting, in discrete situations, we have used splitting-units. We also might wish to distinguish sharing-units, similarity-units, percentage-units, magnification-units, order-of-magnitude-units, and probability-units.⁶ Some examples are given here.

1. *Splitting*. The split (most easily viewed in discrete cases) is easily represented by the tree diagram and involves replication (see Fig. 10). The repeated action is one of an object being replaced by some fixed number of copies as a repeatable process. Thus we measure it in copies, and answer the question, 'so many times

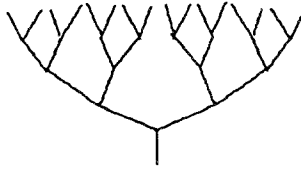


Fig. 10. Splitting.

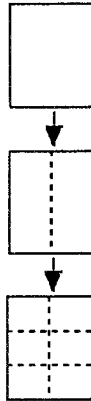


Fig. 11. Sharing.

as many'. Both cell splitting and the creation of family trees represent the use of splitting units.

2. *Sharing.* A sharing-unit, such as the one illustrated in Fig. 11, is a unit created with the intention of creating a certain number of equal shares, an action common in young children and, for which, they develop a variety of strategies (Pothier and Sawada, 1983; Confrey, 1992). Confrey found that elementary students were able to develop a variety of strategies for sharing cookies with friends which included both basic shares (or splits) into two or three pieces and compound shares such as sharing with 4 by two consecutive 2-shares or with six by a 2-share followed by a 3-share. Since sharing is undertaken with the intention of creating a certain number of shares, we would call a 2-share an intention to share with two such as dividing a cookie in half or creating 2 equal size piles of candy from one original pile.

3. *Similarity.* In situations where we identify growth or change through the identification of similarity in shape, and are further able to segment this change in such a way that a constant proportionality is maintained across segments, we identify a similarity unit. Although many organisms grow in such a way that they maintain a similar shape, we do not necessarily identify a similarity unit because we have

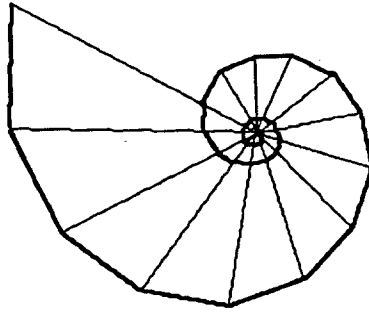


Fig. 12. Proportional similarity.

no apparent segmentable action that maintains a constant proportion across units. However, in cases like the nautilus shell, which grow by periodically creating new chambers which are similar and maintain a continuing constant proportion, we can identify a similarity unit. The growing triangles in Fig. 12 illustrate this pattern.

4. *Percent.* The multiplicative unit with which we are probably most familiar is percent. In its simplest form, percent is a standardized way of indicating an action based on a proportionality: 'percent is to 100' is the same as 'new is to old'. Perhaps because of its common use, percent has developed several context-specific idiosyncrasies, especially when used as a unit in a rate relationship. Davis (1988) posed the question: 'Is percent a number?' We would answer that percent is a multiplicative unit.

5. *Magnification.* Although magnification as a multiplicative unit is similar to proportion/similarity, magnification is generally used to indicate segmentation relative to different perspectives on the same object whereas proportional similarity is used to indicate a relationship between separate objects, such as the chambers of a nautilus, or of changes in a single object over time (see Fig. 13). We have previously argued that magnification is basic to our visual construction of our environment for it provides our perceptual basis for object identity as objects move closer or further away (Confrey and Smith, 1989; Smith and Confrey, 1989). Magnification often involves combining multiplicative units to obtain a new unit. Thus a microscope with $4\times$ eyepiece and a $300\times$ lens produces a $1200\times$ magnification.

6. *Order of Magnitude.* Order of magnitude is commonly used in scientific fields as a multiplicative unit, a multiplication by 10. For example, astronomers describe the distances between stars in terms of orders of magnitudes and biologists use the same units in describing, for example, populations of microscopic organisms.

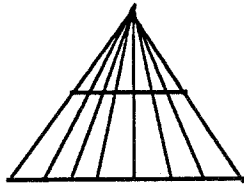


Fig. 13. Magnification.

For example a plant pathologist described comparisons among his laboratory bacteria cultures in terms of orders of magnitude, describing orders of magnitude as a primary indicator of differing population levels (Alan Maloney, personal communication). For him an order of magnitude is an appropriate segmentable unit for describing change. Other commonly used units based on multiplications by ten include the Richter and pH scales.

7. *Probability*. Probability can be viewed as a multiplicative unit in the sense that it provides a repeatable expectability between two events. Just as the constant probability that the nuclei of any atom will decay gives rise to our exponential models of radioactive decay, the expected return of anyone going to Las Vegas with a fixed sum of money who plays repeatedly in the same game exponentially approaches zero.

8. *Half-life and doubles*. Half-life is common in descriptions of residuals of a depleting quantity over time. For example, we talk about the half-life of a substance undergoing radioactive decay or the half-life of a toxic substance in the environment. In scientific contexts, doubling is often used to the change in a population. Research has shown that children from an early age use doubling as a description of change, and as a way of creating 'coordinated counts' (Confrey, 1993).

We do not claim that this list is complete or that the examples are necessarily exclusive. We have presented it for two reasons: first to describe the types of situations in which we see multiplicative units as relevant, and second to begin to develop a language to talk about the actions we take in creating multiplicative units.

Issues of Commensurability of Conversions of Units

Returning to the example of growing cells, we could imagine that the biologist might choose to focus on the issue of doubling, that is, her intent might be to determine when the amount (or number) of bacteria will double (see Fig. 14). If so, we could imagine that she would make a table such as the one shown in Fig. 15. However, if she were interested in using doubles (2-splits) as a unit-name for describing her hourly unit (a 9-split), she would have more difficulty. As

d	c
number of doubles	number of cells
0.00	9.00
1.00	18.00
2.00	36.00
3.00	72.00
4.00	144.00
5.00	288.00

Fig. 14. Doubling units.

t	Δt	c	@c
time	unit	cells	unit
0.0		9.0	
1.0	1.0	81.0	9.0
2.0	1.0	729.0	9.0
3.0	1.0	6561.0	9.0
4.0	1.0	59049.0	9.0
5.0	1.0	531441.0	9.0
6.0	1.0	4782969.0	9.0

Fig. 15. Covariation and rate.

previously mentioned, using a 3-split as a unit name allowed her to think of a 9-split as two 3-splits. In this sense a 3-split is *one half* of a 9-split. The question now before her is how many doubles make a 9-split. From the table in Fig. 15, one can see that it will be between 3 and 4 (the first 9-split makes a total of 81 cells). However, there is no rational solution to this problem, for the number of 2-splits needed to make a 9-split is irrational (and could be calculated as $\log_2(9)$). In line with Nicole Oresme who discussed this issue in the 14th century (Smith and Confrey, 1989, in press; Oresme, 1966), we will describe two multiplicative units as *commensurable* if one unit can be created by repeating the second unit a rational number of times.⁷ We will call two multiplicative units *incommensurable* when this is not the case. In this sense, $\sqrt{3}$ and 3 are commensurable multiplicative units, for repeating $\sqrt{3}$ two times creates the 3-unit. However 2 and 3 are incommensurable for neither can be created from the other through a rationally iterative process. Because of the incommensurability of common multiplicative units, particular attention needs to be paid to the issue of unit conversion when working with students in multiplicative settings.

RATE OF CHANGE REVISITED

Having established the viability of a multiplicative unit, we can now return to our discussion of rate of change. In the exponential example above, we identified three meanings of the term rate. Are these all legitimately called rate? To answer this,

we propose the following approach to rate: *A rate is a unit per unit comparison.* The focus in this definition is on the word ‘per’. By it we mean:

1. At a primitive level, ‘per’ means one unit for one unit, ‘one of this for one of that’. This is an extension of a thesis by Schorn (1989) who documented young children’s command of a closely related idea, ‘so much of this for so much of that’, however, we insist that the child recognize each block as a composite unit.
2. This primitive understanding of ‘per’ can be seen as evolving from building down two columns of a table, a ‘covariational approach’. We believe that the covariation approach is central to the rate concept. One recognizes that constructing rate involves recognizing the repeated action in each column (+1 in the *t*-column; *9 in the *c*-column) as a ‘unit’, understanding that one wants to compare these units as one $\textcircled{R}$ -unit of cells for one Δ -unit of time, and seeing that as one moves down the *unit* columns (Δt and $\textcircled{R}c$) one is always comparing one unit to one unit.⁸
3. One builds on this primitive by combining units. The ‘per’ relationship is maintained whenever an action which joins units together in one unit-column is simultaneously mirrored joining units in the other unit-column. For example three Δ -units of time correspond with three $\textcircled{R}$ -units of cells – or a change in time of ‘plus 3’ corresponds with a change in cells of ‘times 729’. One can combine units through a variety of actions, for example *splitting* (combining units by doubling, tripling, halving, etc.), squaring, etc. This leads initially to the certainty that any *n:n* relation of units maintains equivalent rates where *n* is a positive integer.
4. These combining actions can, in general, be extended to create combinations of rational number of units, extending the equivalence understanding to *p/q*: *p/q* combinations, provided one can build fractional units. In the example above, we will be able to describe a comparison of $\frac{1}{2}\Delta$ -unit to $\frac{1}{2}\textcircled{R}$ -unit and see it as equivalent to a unit per unit comparison provided we have an action that can create half a unit in both dimensions.

This definition of rate has similarities and differences with the treatment of rate within the research community. Kaput *et al.* (1986), Behr *et al.* (1983), and Schwartz (1988) have described rate as a quantity per one unit comparison, an intensive quantity. We have modified this description in two ways: Firstly, their unit reference seems to be focused on standard additive units for both quantities and lacks a discussion of the construction of that unit. As a result, the multiplicative rate is eliminated from the frame of reference. Secondly, for us, the numerator of the rate is no less a unit than the denominator in that it too expresses a repeated action. We focus on the mental construction of the unit, rather than on what is typically known as ‘units analysis’ involving standard units.

Although Thompson (1990, in press) describes a ratio as a comparison of two quantities multiplicatively, his description seems to eliminate or neglect the possibility for rates composed of multiplicative (or non-additive) units. Thus he claims that: “a specific conceived rate is... a linear function that can be instantiated with the value of an appropriately conceived structure” (1990, p. 13)

and restricts his discussion of rate to additive situations (speed as distance per unit time). Although he stresses that we can create multiplicative comparisons and multiplicative combinations of quantities, his quantities themselves are never built of multiplicative units. We do not necessarily believe that Thompson himself excludes the possibility of multiplicative units, but do suspect that his use of the term 'quantity' has an unintentional outcome; readers infer that a quantity is a kind of difference (typically a difference from zero) and is, thereby additive.⁹ We believe that an understanding of multiple ways of constructing units provides the basis for rate as a unit per unit comparison by establishing the role of units in both parts of the 'per' comparison.

Kaput and West (in press) differentiate between a particular ratio and a rate-ratio. They suggest that an "ingredient of a genuine rate conception is understanding of homogeneity" (p. 5). Homogeneity, they argue, is "the idea that all purchases [for example] are governed by the same price, no matter how that price may be stated, or all samples of homogeneous mixture have the same ratio of ingredients. Equivalence, numerical and semantic, applies only when the property of the situation being modeled is homogeneous in that situation" (p. 5). Homogeneity serves the purpose of establishing the invariance "in the situation and not the description" and securing numeric and semantic equivalence.¹⁰

We suggest that the covariation approach is a central part of how we construct homogeneity in rate. For example, if one is traveling at 30 miles per hour and asks what happens over a four-hour period, we typically will understand that we have four times as many hour-units and thus must also have four times as many distance units. Since four 30-units is 120, we state that 120 miles per four hours is the same as 30 miles per hour. Thus the unit per unit rate construct does provide for the homogeneity which Kaput and West emphasize as central to the rate construct.

However, we do differ from both Kaput and West and from Thompson in the distinction between ratio and rate. A ratio for them is described as a particular example of a rate-ratio relation. A rate concept requires homogeneity, shown as the invariance across the apparent variability as illustrated across the top of the table in Fig. 16. A ratio is any particular entry in a given cell interpreted within a situation. A student who thinks a mixture selling for eight dollars per pound can only be bought by the single pound would fail to have a rate concept, but would have a ratio concept for these researchers.

For us, the *ratio* concept is defined as the invariance across a set of equivalent proportions, a position we take in concert with the Greeks. In an earlier paper, we documented how through a process called anthyphairesis the Greeks created a representational symbol system in which the notation for any pair of segments in a given ratio, $a:b$ would be notated as 'the same' (Smith and Confrey, 1989). Thus for us, the homogeneity concept (going across the table in Fig. 16) is essential to *both* rate and ratio. However, what is central to an understanding of rate for us is the sense that it can vary (moving vertically in Fig. 16) as a quantity itself.

We believe that the ability to recognize *variation* in a rate of change is essential for the transition to calculus, and is the point where ratio and rate concepts depart ways. In calculus, all non-linear functions exhibit varying rates of change.¹¹

Rate Variation ↓	Rate Homogeneity →			
	7.04 / 1 lb	14.08 / 2 lb	10.56 / 1.5 lb	5.28 / 3/4 lb
	7.72 / 1 lb	15.44 / 2 lb	11.58 / 1.5 lb	5.79 / 3/4 lb
	8.00 / 1 lb	16.00 / 2 lb	12.00 / 1.5 lb	6.00 / 3/4 lb
	8.32 / 1 lb	16.64 / 2 lb	12.48 / 1.5 lb	6.24 / 3/4 lb

Fig. 16. Price rates.

x	Δx	$y=x^2$	Δy
1.00		1.00	
1.50	0.50	2.25	1.25
2.00	0.50	4.00	1.75
2.50	0.50	6.25	2.25
3.00	0.50	9.00	2.75
3.50	0.50	12.25	3.25
4.00	0.50	16.00	3.75



(a)

(b)



(c)



(d)

Fig. 17. Variable rates: (a) increasing, (b) decreasing, (c) varying, and (d) varying constant.

We are concerned that placing the emphasis in defining rate as a description of the invariance across equal proportions may create an obstacle to understanding varying rates of change for it tends to neglect the importance of one's experiential understanding that a rate can vary over time (for example, an understanding of the changing speed for a freely falling body). We seek to establish a rate construct that allows one to explain both the uniformity of unit to unit comparisons (homogeneity) and the variation in rates over time (non-homogeneity). This is essential to us because our work has concentrated on the development of rate within a variety of families of functions, most notably, linear, quadratic and exponential. Within these environments, one must explain rates which vary, such as those Fig. 17.

For example, we find students examining differences in the y -values of a quadratic function, and, in comparison to a linear function, describing it as an increasing rate (Fig. 17a). We would argue that the students do not see a homogeneity in the situation in terms of a constancy in the rate, yet use the word 'rate' in the situation appropriately to imply a description of a potential repeatability. Thus the unit 1.25 (in the Δy column) per unit 0.50 (in the Δx column) is conceived of as repeatable. The recognition that it does not repeat, that this unit per unit is not large enough to describe the next unit change leads them to describe the situation as having an increasing rate. We have not encountered any student spontaneously describing this as having an increasing ratio, suggesting that ratio and rate may mean different things in this context. In the other figures, students can explain that the rate of the graph is decreasing (Fig. 17b), or the rate varies (Fig. 17c). In Fig. 17d, one can describe it as having either an increasing rate or increasing ratio as one moves from the lowest spoke to the highest one. We suggest that the variation across rates of change should be introduced earlier in order to establish a more robust rate construct. We further believe that they pose a significant epistemological challenge to our understanding of the rate construct.

It appears that there is evidence of a primitive, experiential understanding of varying rates of change. Children can distinguish more, less and perhaps the same (to some degree of accuracy) rates of change, even though they may not be able to discuss them in terms of units, or of unit per unit. Thus, they possess an experiential dimension of rate as extensive, but need not have made the connection to the ratio/rate, unit per unit, approach. This experiential dimension allows them to compare rates non-numerically as more, less or the same, which is essential to experiencing something as a rate at all. A child recognizes a change of speed in an automobile; a gust of wind is heard as 'blowing harder', both implicit rate concepts.

Volume turned up, running hard to end a race, breathing slowing down after a rest, are all rate concepts to children. Discrete cases also are common, although their comparisons may be more difficult for children to articulate. When it rains or snows harder, they perceive more flakes falling in a given visual frame; they prefer the cookie brand known to have more chips per cookie, etc. Following the distinctions of Davydov (in Steffe, 1991) we use the term 'comparative'; thus, we refer to this as a 'comparative dimension'.

One may question whether such an understanding is pre-mathematical or mathematical. Basic to our argument for the importance of multiple representations is the claim that it is through the ability to coordinate representations that one engages in mathematics. Graphs provide an environment in which children demonstrate their ability to treat this experiential dimension as mathematically situated. Tierney and Nemirovsky (1991) have demonstrated elementary students' competence in displaying and justifying a series of changes graphically (the addition and subtraction of blocks into and out of a paper bag, the accumulation of rain water and the description of plant growth).

Furthermore, in work with secondary students, Nemirovsky (1991) has suggested that the transition to calculus could be facilitated if students were to become

facile at identifying and working with six prototypic shapes of curves, each demonstrating a different pattern of variation in rate of change. These analyses suggest that a second understanding of varying rate lies in the coordination of one's experiential knowledge of slope and changing slope (as steepness) with its visualization on a graph. We suspect that students form a direct connection between slope and rate of change which is not mediated by numeric analysis. This second rate concept is more holistic than the analytic 'unit per unit', and connects the experiential basis of slope with rate variations in contextual problems.

An interesting question is how to extend the kind of careful analysis done by Thompson and Kaput and West to describe/produce an analysis of unit per unit change in a situation in which the rate is experienced as variable, either as a comparative dimension or visually through slopes of graphs.¹² This returns us to a consideration of varying rate in the covariation, tabular approach discussed earlier.

In extending the unit per unit analysis to varying rates, one can consider how a rate structure, or price, of a mixture can be varied by increasing/decreasing the unit in one column (while holding the other unit fixed), or vice-versa. These are analytic/numeric strategies, and our students find them intuitively easy to express within the table window on Function Probe *if a constant unit change* can be created in one of the two variables (x or y). We have found that students can frequently visually link these curves with their descriptions of situations and discuss rate in this context. For example, Rizzuti (1991) writes of "how several students spontaneously generated the construct of rate of change of a function (both additive and multiplicative) in their explanations of why graphs behaved as they did" (p. 123). She cites an example where a student predicted the shape of a geometric sequence:

All right, this would be linear, because you're multiplying by the same – No, it would not be. No, take that back. Because 512 to 3072 is smaller than from 3072 to 18000 so you're gonna have this slope again. It's gonna go to a real steep slope. [p. 126]

and later,

I believe that we get the curve on the other one because its multiplication and when you multiply numbers they get bigger faster. And then when you add numbers or subtract numbers its always constant. It's always like minus 103, so the slope is actually just a little bit smaller. The slope is smaller here than it is here. I mean, it starts out small, the slope on the first one [the geometric sequence] and then it goes real big. And then the slope on the right one here [the arithmetic sequence] – the straight line – it just remains constant the whole time. [p. 129]

In this example, one sees the student appealing to both a unit per unit comparison and then a visual perception of slope.

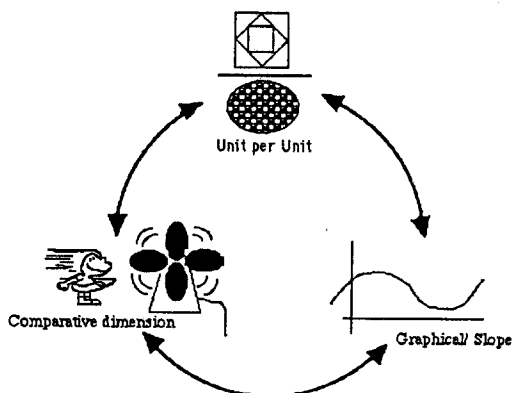


Fig. 18. Rate as a coordination of multiple concepts.

An Integrated Understanding of Rate

We are suggesting that the kind of rich understanding of rate that will serve students well, particularly as they enter calculus, involves all three components we have discussed (Fig. 18). We argue that:

1. The unit per unit comparison is central to a construction of rate and that understanding a unit as an intentional segmenting of experience in a potentially repeatable manner is a necessary first step in making this comparison. Thus, we express the centrality of homogeneity to rate, particularly in situations where rate is constant (additive *or* multiplicative contexts), but through a units-based approach.
2. Children have a 'comparative dimension' for varying rates, which allows them to assert greater, lesser and constancy of rate.
3. Graphs which allow for an experiential interpretation of slope provide an alternative expression for varying rates at an early age.
4. A coordination of these multiple representations of rate will be necessary for a more robust concept of rate.

Rates of Change in Exponentials Revisited

Early in the paper, we described three separate views of rate within the biological cells example. We can now revisit them using the theoretical approach to units and rates.

1. *An additive rate of change.* Looking at the exponential function from the perspective of an additive rate of change, one concludes that the rate is varying. In fact, for increasing exponentials, the drama of the increasing magnitude of the rate of change is an essential component to being mathematically literate about the exponential. Several scientists have described their qualitative understanding of an exponential function as involving a situation in which a population decreases

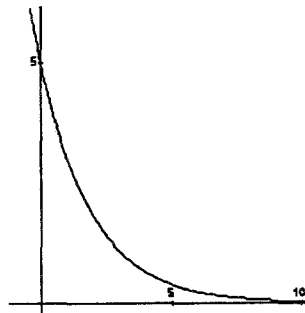


Fig. 19. Approaching an asymptote.

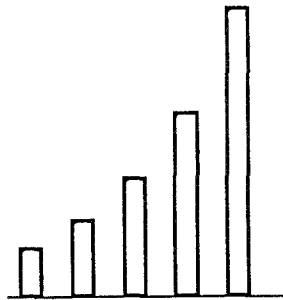


Fig. 20. Multiplicative bars.

rapidly at first, then tends to level off and asymptotically approach a lower limit (Fig. 19). Empirical evidence has confirmed that students can, in fact, coordinate their recognition of an increasing (or decreasing) additive unit and the curvature of a graph placed on an additive scale.

2. *A multiplicative rate of change.* The analysis has legitimized a multiplicative rate of change approach by recognizing the multiplicative unit. Visually, on an additive scale, it is difficult to see the exponential curve's defining quality; although Carlstrom (1992) has demonstrated that students are quick to recognize a visual regularity when working in a computer environment in which exponentially growing bars are generated (Fig. 20). Building concepts of multiplicative rates constructed from multiplicative units should play a central role as students work on understanding how multiplicative worlds generate constant doubling times and constant half-lives.

3. Although the 'proportional new-to-old' description of rate of change described previously seems, in a way, to be neither here nor there, that is it does not quite fit into either an additive or a multiplicative understanding of change, we believe that it is key in assisting students' move into calculus. In Fig. 21, we have redrawn

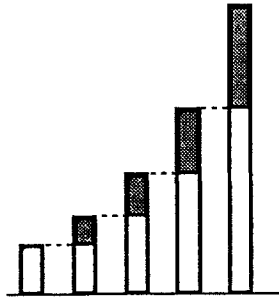


Fig. 21. Proportional new-to-old.

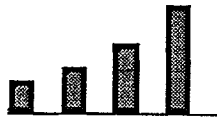


Fig. 22. Proportional new.

Fig. 20 to illustrate the proportional new-to-old rate. As one moves from one bar to the next, what is new, is the shaded area of the next bar. This also represents the additive change (difference) from bar to bar. However, one can also see in Fig. 22 that the additive increments are also increasing. As one investigates further and finds that these added increments are increasing by the same constant ratio as the original bars (also see Fig. 3b), it is only a short step to argue that the additive change (the shaded areas) will always be the same proportion of the whole bar. This principle becomes a fundamental property of exponential functions in calculus, that the derivative of any exponential function is directly proportional to the function itself. In formal terms, this can be expressed as:

1. If $f(x) = a^x$, then $f'(x) = k * a^x$, where k is constant for fixed a .

We would suggest that building this understanding should precede the introduction of e , for as we have already discussed, e cannot be constructed as a segmentable multiplicative unit. Thus, we expect e to be constructed out of its relationship to equation 1, since it has the unique property that when $a=e$, then $k = 1$ or:

2. If $f(x) = e^x$, then $f'(x) = 1 * e^x$.

Within this setting, multiple approaches can be developed for constructing e and the natural log function. We plan to explore this issue further in a future paper.

CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we have presented an epistemological discussion for understanding rate and unit that builds on multiple approaches. The description of a multiplicative unit is based on language and actions we have observed in our work with

students and scientists. We believe this description has a potential to bring a rich set of concepts and language into our mathematical discussions that do not fit into standard additive/quantitative models. Likewise, although we do not have serious disagreements with others who have described rate concepts, we believe that the tendency to limit rate to situations comparing quantity to (additive) units unnecessary restricts rate and does not capture the variety of situations and mental actions which people need and use in building a rate concept. Furthermore, we argue that an understanding of varying rates of change must accompany the development of the homogeneity of the rate concept. The multi-conceptual approach to rate which we have proposed holds particular promise in preparing students for calculus.

ACKNOWLEDGEMENTS

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NOTES

¹ The potential role of technology in our changing concepts of function cannot be ignored. For example, with Function Probe[©] (Confrey, 1991) a student can use the method of co-filling two columns to generate a functional relationship. The ready access that the software provides for students to create and explore various patterns in tables broadens the opportunities for them to develop a rich covariational concept of function that would be much more time-consuming and difficult with paper and pencil.

² This is not intended to be an exhaustive description of concepts of rate of change. In particular the r in equations such as $P_f = P_i e^{rt}$, often used in population, radioactive decay and interest problems encourages us to explore additional concepts of rate-of-change. We do not have the space to discuss this further in this paper, but will address this issue and the development of e in a follow-up paper.

³ Many people suggest that this is division and not multiplication. We are arguing that the operations of multiplication and division are paired, and that splitting is a description of the overriding primitive that can be used in both cases. However, we identify the units as multiplicative units in the general case.

⁴ Although we describe the outcome in terms of number of 'cells', it is used as a way of describing what one gets after the split in a language that allows for a single naming system.

⁵ Dienes (1967) makes the distinction between a state and an operator. We operate to create states. State language is easy to use to describe the outcome of a unitizing action. However, as already discussed, creating the unit as 'state' simultaneously involves creating the unit as the repeated action, the successor operation.

⁶ We refer to this world of activities as 'splitting world'. We use this overriding description because we like to contrast the primitive 'split' with the primitive 'count'. However, we will also use split as a particular kind of multiplicative unit (see #1).

⁷ Commensurable magnitude was used by the early Greeks to mean 'have a common measure', a common unit that could be used to evenly measure each magnitude. The issue of incommensurability first came up for the Greeks in the investigation of the diagonal and side of the square by the Pythagoreans (Smith and Confrey, 1989).

⁸ We will use Δ -unit and $\mathbb{R}$ -unit to denote additive and multiplicative units respectively.

⁹ Carlstrom (1992) has suggested that a way to avoid the additive inference about quantity is to switch to using the term dimension where a dimension is constructed as the progressive integration of units where the units can be constructed using the multitude of ways discussed in this paper.

¹⁰ A difficulty we have with Kaput and West's description is their attribution of homogeneity to the situation, leaving out the active and intentional construction of the situation by the individual. In this sense, we agree with Thompson when he states: "When we shift our focus to the operations by which people constitute 'rate' and 'ratio' situations, it becomes clear that situations are neither one nor the other. Instead how one might classify a situation depends upon the operations by which one comprehends it" (in press, p. 11–12). Although this may be what Kaput and West are suggesting by the idea of semantic equivalence, we believe that a semantic analysis is insufficient if it does not explicitly allow the situation to be viewed in terms of the actions and intentions of the individual.

¹¹ This is partly because in conventional calculus instruction, rate of change is built *only* on additive units. Whereas examining an exponential rate as a multiplicative unit per additive unit leads to a constant rate of change, analyzing an exponential rate as an additive unit per additive unit leads to a varying rate of change.

¹² A transition towards instantaneous velocity and the derivative is implied as a tangent or as an algebraic derivative is implied.

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