

THE ROLE OF CONTRADICTION AND UNCERTAINTY IN
PROMOTING THE NEED TO PROVE IN DYNAMIC GEOMETRY
ENVIRONMENTS

ABSTRACT. In many geometrical problems, students can feel that the universality of a conjectured attribute of a figure is validated by their action in a dynamic geometry environment. In contrast, students generally do not feel that deductive explanations strengthen their conviction that a geometrical figure has a given attribute. In order to cope with students' conviction based on empirical experience only and to create a need for deductive explanations, we developed a collection of innovative activities intended to cause surprise and uncertainty. In this paper we describe two activities, that led students to contradictions between conjectures and findings. We analyze the conjectures, working methods, and explanations given by the students when faced with the contradictions that arose.

KEY WORDS: deduction, design processes, explanations' categories, proving in geometry

INTRODUCTION

For generations, proofs have been considered as tools for verifying mathematical statements and showing their universality. Leibniz believed that a mathematical proof is a universal symbolic script, which allows one to distinguish clearly between fact and fiction, truth and falsity (Hanna, 1990).

In consequence the two classical reasons for teaching proofs were, (a) to teach deductive reasoning as part of human culture, and (b) to serve as a vehicle for verifying and showing the universality of mathematical statements. According to this cultural perspective of proof, the addressee for 'showing' is unspecified, and the function of proof is to fulfill the internal 'demands of mathematics', not the psychological needs of any learner. The goal for the teaching of proof is to establish the students' norms of mathematical knowledge: experimenting, visualizing, measuring, inductive reasoning and checking examples are not considered as normative means for this purpose.

Recently there has been a change in this approach for three reasons, the failure to teach proofs, the recognition that the activity of proving must comply with the motives of the learner, and the appearance of dynamic tools.



The failure to teach proofs

The failure to teach mathematical proof appears to be universal. Senk (1985) found that only 30% of the students in full year geometry courses that teach proofs (in the U.S.A.) reached 75% mastery in proving. In addition, even those students that succeeded to function in the proving ritual were not always aware of its meaning. They rarely saw the point of proving, and/or the need to prove, especially when the statement to be proved was given as a ready-made fact without any discovery by the learners. Balacheff (1991) claims that if students do not engage in proving processes, it is not so much because they are not able to do so, but rather because they do not see any reason or feel any need for it. High-school students, even in advanced mathematics and science classes, do not realize that a formal proof confers universal validity to a statement. A large percentage of students believes that, even after they have proved, checking more examples is desirable (Fischbein and Kedem, 1982; Vinner, 1983). Many students do not distinguish between evidence and deductive proof as a way of knowing that a geometrical statement is true (Chazan, 1993). Similarly, Schoenfeld (1986) claimed that after a full year geometry course in college, most students do not see the point of using deductive reasoning in geometric constructions, and that they are still naive empiricists whose approach to constructions is an empirical 'guess and test' loop. They produce proofs because the teacher demands them, not because they recognize that they are necessary in their practice (Balacheff, 1988).

The goals of teaching proofs

As mentioned above, for mathematicians, proofs play an essential role in establishing the *validity* of a statement and in *shedding light on* the reasons that support that statement. Lately mathematics educators have extended considerably the goals that must be fulfilled when teaching proofs. For example de Villiers (1990, 1999 p. 5) lists six functions of proof:

- *Verification* (concerned with the truth of a statement),
- *Explanation* (providing insight into why it is true),
- *Systematization* (the organization of various results into a deductive system of axioms, major concepts and theorems),
- *Discovery* (the discovery or invention of new results),
- *Communication* (the transmission of mathematical knowledge),
- *Intellectual challenge* (the *self-realization/fulfillment* derived from constructing a proof).

It seems that the first two functions mentioned by de Villiers, are related to the main roles of proof in mathematics discussed above: – *verification* directly relates to *validation* and *explanation* is synonymous with *shedding light*. An analysis of teaching materials indicates that these functions often remain hidden in textbooks. The list presents new perspectives on proving, both from the point of view of the professional mathematician (as shown above) and the experienced teacher. De Villiers considers these functions as desirable, although he is aware that they are not fulfilled in class.

Balacheff (1988, 1991) identifies roles of proof, which, in a sense, are similar to the communication function mentioned by de Villiers. Like de Villiers, Balacheff also notes that such a desirable function is not fulfilled in class and suggests that classroom activities in which students become aware of such aspects of proofs be created.

Hersh (1993) and Hanna (1995) assert that the main function of proof in the classroom should be to promote understanding by explaining (which is also in de Villiers' list).

The existence of Dynamic Geometry (DG) environments

The appearance of dynamic geometry environments raised a question concerning the role of proof in the curriculum, since conviction can be obtained quickly and relatively easily by dragging. Dragging a geometrical object enables students to check the invariance of a conjectured attribute on a whole class of objects. Such an operation leads students to be convinced of the truth of the conjectured attribute. Therefore DG environments may prevent students from understanding the need and function of proof. These suggestions are corroborated by Yerushalmy, Chazan and Gordon (1993), who have investigated how students function in open inquiry activities in DG environments, that support experimentation, conjecturing, checking invariant properties of a figure, and thus lead to conviction. De Villiers (1997, 1998) illustrated how one can enrich investigations in DG environments by asking 'what if' questions, to lead to generalizations and discoveries. He claimed that the search for proof then becomes an intellectual challenge, stemming from the need to understand *why* the conclusion is true, and not an epistemological exercise in trying to establish truth. In addition Goldenberg, Cuoco, and Mark (1998) stated that:

A proof, especially for beginners, might need to be motivated by the uncertainties that remain without the proof, or by a need for an explanation of *why* a phenomenon occurs. Proof of the too obvious would likely feel ritualistic and empty (p. 6).

They concluded that DG environments may provide opportunities for the creation of uncertainties, leading students to seek for explanations.

Movshovitz-Hadar (1988) identified the role of surprise in supporting the teaching of proofs. Similarly, Dreyfus and Hadas (1996) argued that students' appreciation of the possible functions of proof can be achieved by activities in various learning environments (with and without DG software) in which the empirical investigations lead to unexpected, surprising situations. In other studies (Hadas and Hershkowitz, 1998, 1999) we demonstrated how appropriate activities in DG environments can be carefully designed to create situations of contradiction followed by surprise or by uncertainty, to lead students to seek for explanations.

In summary, the findings concerning the failure to teach proofs, the recognition of the multiple aspects of proving, and the existence of DG tools, lead naturally to the design of investigate situations in which DG tools may foster these multiple aspects.

THE STUDY AND ITS FRAMEWORK

In this paper we describe a study involving two activities designed to promote several aspects of proving in a dynamic geometry environment (Geometry Inventor, 1994). Each activity consists of a sequence of tasks, which together create a global coherent activity. We made use of 6 possible mediating functions of the DG environment:

- To serve as learning environment for raising conjectures.
- To refute (or confirm) an initial conjecture (formulated with or without the DG tool), and in cases of refutation, to create a contradiction between expected results and actual findings.
- To 'push' students from one conjecture to a second one in cases where the conjecture is neither refuted nor confirmed (i.e., in situations of uncertainty, see discussion in section 4).
- To lead students to be convinced that a conclusion is right, based on inductive trials.
- To enable the construction of an existence example.
- To provide additional sources of explanation.

The first activity was designed to create a surprise resulting from a dialectic process stemming from contradiction between students' hypotheses and their findings and leading to the need for explanation. The setting from which the surprise arose left the contradiction unresolved. The need to explain thus emerges quite naturally.

In the second activity, contradiction also occurred, but led to uncertainty in addition to surprise, since students were not sure whether a specific geometrical setting can exist. This type of activity has already been mentioned

by Movshovitz-Hadar (1988), and is in correspondence with a situation of creating surprises by the 'existence of the unexpected' (p. 35). In order to overcome the uncertainty, students either find or construct an example, or explain why such an example is unconceivable. In both cases, careful design triggers students to base their actions on deductive considerations. The environment serves as a laboratory for overcoming uncertainty.

Each of the two activities was designed through a process of design-research-design as an outcome of semi-structured interviews (Hadas and Hershkowitz, 1998; Hershkowitz et al., in press). The version presented in this paper is the end product of this process. In each case, we first bring the activity itself, then analyze the role of the various tasks in the activity, and present some global findings on students' actions, and finally we describe both qualitatively and quantitatively the various categories of student explanations.

The categorization of students' explanations is similar to 'proof schemes' of college students made by Harel and Sowder (1998). Harel and Sowder use three main categories (each of which is divided into subcategories): 1. 'External conviction proof schemes' where doubts are removed by ritual proof, by symbolic form arguments without meaning, by authority, or by memorizing statements. 2. 'Empirical proof schemes' where conjectures are validated or refuted by physical experience, perceptual or inductive arguments. 3. 'Analytic proof schemes' where the statement is validated by logical deduction which can be transformational or axiomatic.

Our approach is 'bottom-up', as we aimed to classify students' explanations in cases where contradiction and uncertainty situations are created and investigated by students with the mediation of the DG tool. Hence five categories were defined by which we were able to describe qualitatively as well as quantitatively students responses.

The two activities in this research are part of a course that covers the official syllabus in Israel, according to which students learn geometry for 2 years (from mid Grade 8 to mid Grade 10), two periods per-week, with emphasis on proving. The students in this study were from Grade 8 for the first activity and from Grade 10 for the second activity. Both research populations belong to the upper 60% of the entire population.

Each activity was used first as a research tool for semi-structured interviews with pairs of students, and second for written report of pairs of students in a few experimental classrooms. The interviews enabled us to trace the changes and development in students' conjectures and explanation processes in contradiction and uncertainty situations. All written reports of students working in pairs (from classes and interviews) were collected and analyzed according to students' conjectures and explana-

tions. The results of this analysis provided a quantitative dimension that clarified to what extent students faced contradictions, to obtain a variety of explanations and to establish a categorization.

THE ROLE OF CONTRADICTION IN 'THE SUM OF EXTERIOR ANGLES' ACTIVITY

The 'sum of exterior angles' activity belongs to the beginning of the geometry course. Students had already studied an introductory chapter about parallel lines including the sum of angles in a triangle. They were also familiar with the dynamic geometry software.

In many traditional geometry textbooks students are given a task similar to the following:

- a. Prove that the sum of the interior angles of a polygon is $180^\circ(n-2)$, where n is the number of sides.
- b. Prove that the sum of the exterior angles of a polygon is 360° .

A more open approach using a dynamic geometry environment gives students the opportunity to discover these attributes before being invited to prove (Yerushalmy and Chazan, 1990). We designed an activity intended to create a contradiction between plausible student conjectures about the sum of the exterior angles as the number of polygon sides increases, and the results obtained by the measuring operation done in the DG environment. We aimed to investigate the effect of the activity on students' explanations, and on their feeling for the necessity to prove. The wording of the activity is as follows:

Task A: Measure (with the software) the sum of the interior angles in polygons as the number of sides increases. Generalize, and explain your conclusion.

Task B: Measure (with the software) the sum of the exterior angles of a quadrilateral.

Hypothesize the sum of the exterior angles for polygons as the number of sides increases.

Check your hypothesis by measuring and explain what you found.

Comments: 1. In both tasks, students were first led to check (by dragging) the sums of the interior/exterior angles for polygons with the same number of sides, and only then to change the number of sides. 2. The DG software measures only angles smaller than 180° . Therefore we confine this activity to convex polygons.

Both tasks were deliberately posed in open form to increase the contradiction between the conjectured and the invariant property of the sum of the exterior angles, and thus the need to explain the results.

The goal of Task A was to familiarize students with the situation and to provide an opportunity to generalize the increase in the sum of the interior angles as the number of sides increases, without necessarily relying on algebraic considerations. After generalizing, students were asked to explain their generalization. The generalization and explanations in Task A may encourage students to conjecture that the sum of the external angles is also increasing. Consequently, the contradiction that students may face in Task B, is likely to become stronger.

The activity was first given in the form of a worksheet to 4 pairs of students (from grade 8) in a semi-structured interview. The interviewer encouraged students to make conjectures before measuring, to check their hypotheses by measuring, and then to explain verbally their universality. In addition, each pair of students reported their hypotheses and explanations on the worksheet. The interviews were videotaped, analyzed and interpreted.

In a second stage, the activity was given to 82 students in three 8 grade classes, with students working in pairs. As a result we collected and analyzed 45 reports (4 from the interviews and 41 from classrooms). In the following we analyze the tasks, reports and interviews, and describe findings and the types of explanations given by students in interviews and in class.

Activity analysis and findings

Task A:

(i) Students in 32 reports generalized from working with the DG software that the sum of the interior angles increases by 180° when the number of sides increases by 1. (ii) In 18 reports students expressed their generalization algebraically: $180^\circ (n-2)$, $180^\circ (n-4) + 360^\circ$, or $180^\circ n - 360^\circ$. (iii) 5 pairs gave both kinds of generalization.

Following the request to substantiate their generalizations, students produced various explanations. Most based their explanations by generalizing their measurements, or, by adding a triangle when the number of sides increases by one (Figure 1).

Task B:

In 37 of the 49 responses (the 8 students in the 4 interviewed pairs who hypothesized individually, and the 41 written responses in the classrooms), students hypothesized that the sum of the exterior angles increases with the

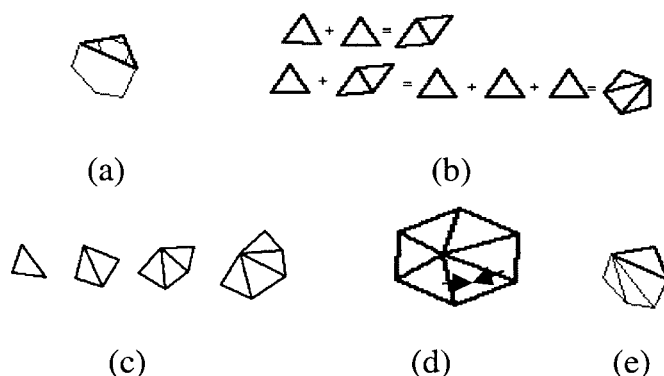


Figure 1. Students' drawings in various explanations for Task A, Activity 1.

number of sides. Another 10, that were all from the same class, hypothesized that the sum is fixed. From the information we had from the teacher of this class, students knew this fact from previous work. Two girls, in an interview, did not make any hypothesis.

Task A pushed many students to generate the hypothesis that the sum of the exterior angles increases. The degree of confidence that the hypothesis is true became quite strong as reflected in the following episode.

In the course of a discussion, Jonathan claimed that the sum would always remain 360° . Galia asked him if he thought that when the number of sides increases beyond 100, the sum would still remain 360° . Jonathan hesitated and then agreed that the sum would increase. This episode indicates that the intuitive belief that the sum increases with increasing number of sides, is quite strong even when the student has some knowledge about the true statement. This belief may produce contradiction and surprise while checking this conjecture, and the contradiction may trigger a need for explanation.

Types of explanations

Fifty explanations relating to Task B were given by the students (in the interviews students gave more than one explanation), to refute (or confirm) the initial conjecture. The explanations were classified by the authors, into the following five categories, and two other experienced teachers validated this classification (as Task A does not lead to contradictions, we did not include the explanations given by students to this task in the analysis.)

1. No explanation category

Seventeen of the 50 responses (34%) belonged to this category. It included responses without any argument, responses that were mainly tautological (i.e., in which students rephrased the result of their measuring and added some ‘supporting’ words as in Example 1), and responses whose arguments relied on external authority.

Example 1:

The sum of exterior angles in every polygon, either pentagon or 7 sided polygon, is always 360° .

2. Inductive explanation category

In two of the explanations (4%), inferences, not based on general deductive arguments, were drawn from features of one or several examples. For example, in one of the interviews, Noa and Inbar explained that the sum of the exterior angles is always 360° by checking numerically as follows:

Example 2:

Noa: We will multiply 180° by the number of sides and we will subtract the interior [sum of angles] we found [in Task A].

I: And what we will get from it? It will increase or decrease? By how much?

Noa+Inbar: We don't know.

They started to write down a table:

Interiors + Exteriors	Interiors	Exteriors
180×4	$-$	$360 = 360^\circ$
180×5	$-$	$540 = 360^\circ$
180×6	$-$	$720 = 360^\circ$

Noa: The sum is always 360° , and we don't have to continue to check. But, we still don't know *why* the sum doesn't change.

Although Noa and Inbar elaborated a global process for computing the exterior sum and were convinced of its universality, they did not consider their inductive process as an explanation of *why* it is true.

3. Partial deductive explanation category

There were six explanations (12%) in this category.

Example 3:

A pair of classroom students who concluded in Task A that the sum of the interior angles is $180^\circ(n-2)$, explained in their written response to Task B: One should take off the sum of the interior angles from the sum of the adjacent [interior and exterior], and then one gets 360° .

This is a deductive type of explanation in which the 2 computations, $180^\circ n$ and $180^\circ n - 180^\circ(n-2)$ are missing.

4. Visual-variations explanation category

In this category we included explanations that make use of visual variation, which might be a result of imagery or the dragging action. As such, this category is particular to explanations in DG learning environments. The sixteen explanations (32%) in this category include variations based on visual imagery, described verbally and by drawings.

Example 4:

A pair of students noted a visual attribute of the sum of the exterior angles: There is a whole turn around the polygon, therefore the sum is 360° .

And they drew Figure 2.

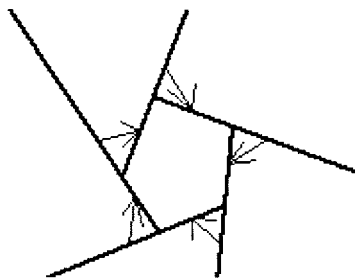


Figure 2. A 'whole turn around a polygon' as an explanation for Task B, Activity 1.

Their use of imagery for expressing a global visual property of polygons, which is independent of the number of sides, is clear. Harel and Sowder (1998) call such arguments *transformational analytic proof schemes*.

Note: This example is included in category 3 also, because the students based their explanation on the statement that a complete turn is 360° .

Example 5:

A typical response in this category is the following:

When the number of sides increases, we have more angles. But they are smaller, because the internal angles become bigger.

The Inventor DG software, enables students to choose a polygon with a given number of sides. The polygon appears in its regular form which can be 'messed' by dragging. Hence, students who choose polygons (created by rays) sequentially with increasing number of sides, see the added angle simultaneously with the decreasing measure of all the exterior angles.

5. Deductive explanation category

In this category there were nine explanations (18%). They consist of a complete chain of logical arguments as illustrated by the following examples.

Example 6:

In their interview, Tomer and Adir, hypothesized that the sum of the exterior angles will be always equal to the sum of the interior angles (as in quadrilaterals). After measuring with the DG tool, they concluded that the sum of the exterior angles is constant and equal to 360° , relying on compensation between the sums of the interior and exterior angles. They then checked numerically a sequence of polygons with increasing number of sides, like the two girls in Example 2. They calculated the sum for polygons of 4, 5, and 6 sides and then 'jumped' to 12 sided polygons. They felt then that they had reached a dead end:

Adir: 12 sides it is 12 multiplied by 180, minus the sum inside which is . . .

They did not have the formula nor numerical information about the sum of the interior angles of 12-sided polygons.

Adir: I can say that the sum of all pairs of the adjacent angles [the interior and exterior angles together] minus the entire interiors, equals the sum of the entire exteriors.

I: O.K. But how can you be sure that when you subtract the sum of the interior angles, that is changing all the time, the sum of the exterior angles will stay 360° ?

Adir: I can explain this to you: to the sum of all the interiors 180° is added, and 180° is added to the sum of all the adjacent. And if I subtract the sum of the interior. . .

Tomer: Then this is coming back. . .

Adir: It compensates the 180° back, it remains the same. Let us write it down
Tomer.

(Adir was writing what Tomer summarizes):

Tomer: When we add one side to a polygon, the sum of the interior angles increases by 180° , and the sum of the adjacent angles increases by 180° . Therefore when we subtract the sum of the interior from the adjacent, we get the same number.

Adir: And it starts at 360° because it is like that in a triangle, and this is the smallest polygon one can get.

In addition to explaining why the sum of the exterior angles in polygons is constant, Adir felt the need to complete the explanation why this constant sum is equal to 360° . We can conclude that the students felt that their explanation is adequate, from their need to summarize it by writing. It is worth noting that because this example is the record of an interview, we are able to see the process of construction of the justification.

Additional pairs of students used the *explanation element* they produced in Task A – the triangle they built (See Figure 1.a) – to explain why the sum of the exterior angles is fixed. For example: Inbal and Limor in the following episode.

Example 7:

Inbal: If we add a triangle, it is like the sum of the angles is not. . . I am not sure about it. . .

Inbal then drew Figure 3 to support her process of explanation.

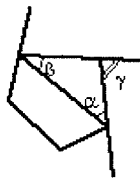


Figure 3. A drawing for explanation based on adding a triangle for Task B, Activity 1.

Inbal: To continue?

I: Yes, Yes. . .

Inbal: This is what was added (γ), and this is what was reduced (Inbal is pointing at α & β).

Limor: This was outside before (pointing at α & β). [α is opposite to a part from an external angle].

Inbal: Because this is ahh. . . I know why. . . the sum of the exterior angles. . . This (pointing at γ) is equal to these two (pointing at α & β) it is equal for both of them.

Limor and Inbal: Yes, yes it is true.

Both girls constructed their explanation while talking and drawing cooperatively. From their last common sentence above, it seems that they became confident that their explanation was correct. It is worth noting that although the girls used a special drawing to exemplify their explanation, in Task A they talked about adding a triangle as general way of adding a side. Thus we consider their explanation as also general.

Summarizing, as intended in the design of the activity: (i) Many students in Task B faced the contradiction between their hypothesis and the results; (ii) In spite of being at the beginning of the course on Euclidean geometry, nine of the fifty explanations were complete deductive explanations and eight more used partial deductive or inductive explanations; (iii) The generalizations and the explanations students produced in Task A, served as artifacts in their explanations in Task B (see examples 6 and 7).

ROLE OF UNCERTAINTY, IN THE ‘CONGRUENT TRIANGLES’ ACTIVITY

The ‘congruent triangles’ activity was designed for the last months of the geometry course (Grades 9 and 10). Students were asked to investigate sufficient conditions (equality of sides and angles) for congruence in a situation of uncertainty. The uncertainty emerged from situations in which the construction was possible, but was opposed to students’ intuitive conjectures or alternatively in situations in which the construction was impossible and some students conjectured that it was possible. Students at that stage were already familiar with congruence and similarity and made use of the relevant theorems.

The activity

In this activity, we will investigate if and when, two triangles having several equal elements, are congruent.

Part 1: Task 1a. Given a dynamic $\triangle ABC$, construct another $\triangle DEF$ having one angle and one adjacent side equal to one angle and adjacent side of $\triangle ABC$. Drag the vertices of both triangles and investigate the variance of the triangles and how they relate one to another.

Task 1b. Given a dynamic $\triangle ABC$, construct another $\triangle DEF$ having two angles and the included side equal to two angles and the included side of $\triangle ABC$. Drag the vertices of both triangles and investigate the variance of the triangles and how they relate one to another.

Task 2. Is it possible to construct a triangle with one side and two angles equal to those of a dynamic $\triangle ABC$, but *not* congruent to $\triangle ABC$? If it is possible, construct such a triangle; otherwise explain why.

Part 2: Task 3a. How many equal sides and angles are there in the non-congruent triangles constructed in task 2?

Task 3b. Is it possible to construct two non-congruent triangles, with *five* equal elements (sides and angles)? Generate an hypothesis.

Task 4. Is it possible to construct two non-congruent triangles with *six* equal elements? If yes, construct two such triangles, otherwise explain why.

Part 3: Task 5a. Let's suppose that we already have 2 non-congruent triangles with *five* equal elements, specify the equal elements.

Task 5b. Try to construct two non-congruent triangles, with *five* equal elements.

This activity was first given to 2 pairs of Grade 10 students in semi-structured interviews, which were videotaped and analyzed (for more details and analysis of the interviews, see Hadas and Hershkowitz, 1999). In the second stage the activity was given in a Grade 10 classroom, where students worked in pairs. As a result 12 written reports (10 from the class and 2 from the interviews) were collected and analyzed.

In the following we first analyze the various tasks in the light of students' actions and conjectures and then discuss the various types of explanation qualitatively and quantitatively.

Activity analysis and findings

- Task 1 was aimed to invite students to grasp the concept of congruence as a possible invariant under dragging. In Task 1a, both triangles can be changed by dragging their vertices and they may or may not be congruent, because they have only one side and one angle equal. On the other hand, in Task 1b, where there is an additional equal angle, the second triangle ($\triangle DEF$) cannot be changed by dragging its vertices, and when one changes $\triangle ABC$ by dragging, $\triangle DEF$ changes accordingly and remains congruent to it. This task prepares students for Task 2 when they realize that congruence may be an invariant attribute under dragging.
- In Task 2, four pairs of students constructed two non-congruent triangles satisfying the required conditions. The other eight pairs claimed that this construction was impossible because of the congruence theorem of one side and two angles. These eight pairs then received *Task 2**.

In Figure 4, side $AC = DE$ and $\angle A = \angle D$. Examine all possible ways of 'copying' an additional angle of $\triangle ABC$, in order to complete the second triangle. Is the completed triangle congruent to $\triangle ABC$ for all possibilities?

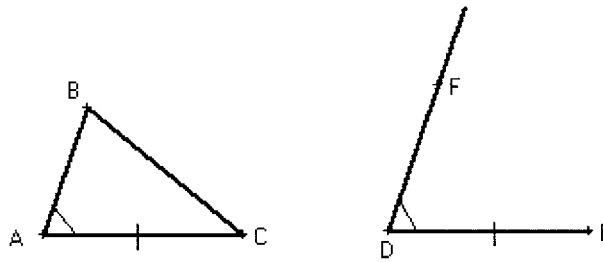


Figure 4. The drawing for Task 2*, Activity 2.

This task ‘pushed’ these eight pairs of students to construct non-congruent triangles with one side and two angles equal. The DG software enables students to try various ways of ‘copying’ $\triangle ABC$ angles to different vertices in $\triangle DEF$, and to get feedback. They then realize that when the ‘copying’ does not preserve the correspondence, the two triangles are not necessarily congruent. This means that even though dragging the vertices of $\triangle ABC$ changes $\triangle DEF$, the two triangles are generally not congruent.

- When in Task 3a, students were asked to identify the equal elements, eight pairs were surprised to discover four equal elements in the two non-congruent triangles. The remaining four pairs claimed that the three elements used in the construction are the only equal elements. It seems that the belief that three equal elements are sufficient to guarantee congruence, causes students to be surprised or prevents them from concluding the equality of the fourth element in the second case.
- While solving Task 3b, the pair of students in the first interview considered the possibility of constructing a triangle non-congruent to $\triangle ABC$, with six elements equal to elements in $\triangle ABC$. We consequently decided to insert Task 4, and thus obtained three parts: a conjecture (Task 3b), a discussion of the case of six equal elements (Task 4), and finally the construction of a triangle with five elements equal to elements of $\triangle ABC$, and yet non-congruent to it (Task 5b).
- The two pairs in the interviews conjectured in Tasks 3b and 4 that it is possible to construct non-congruent triangles with five or six equal elements. The discourse in the interview is described in Hadas and Hershkowitz (1999). The ten pairs from the classroom conjectured first that it is impossible to construct non-congruent triangles with five or six equal elements, based on their belief that three equal ele-

ments guarantee congruence although they had already constructed non-congruent triangles with four equal elements. Following a whole class discussion, they also felt a need for an explanation. Hence we can conclude that all 12 pairs faced a situation in which they had to explain why it is impossible to construct non-congruent triangles with six equal elements.

- Task 5 was designed to challenge students' conjecture that triangles with five equal elements must be congruent. This task comes after Task 2 where students were surprised to discover that they can construct two non-congruent triangles with four equal elements, and after Task 4, in which the students explained why it is impossible to construct non-congruent triangles with six equal elements. This sequence made students uncertain about the problem of five equal elements. This uncertainty encouraged students to link considerations to one of the two alternative reasons they had discovered previously namely relying on a congruence theorem or on lack of correspondence. Only one pair in the interview started directly with a construction (described in Hadas and Hershkowitz, 1999). All the additional pairs received *Task 5**, which was intended to trigger students to use similarity consideration.

In $\triangle ABC$ (see Figure 5) one side is 6 units long and a second is 9 units long. Design a construction of $\triangle DEF$ with sides of 6 and 9 units also and angles equal to the angles in $\triangle ABC$, but which is not congruent to $\triangle ABC$ (Clue: Calculate first the length of the third side in each triangle).

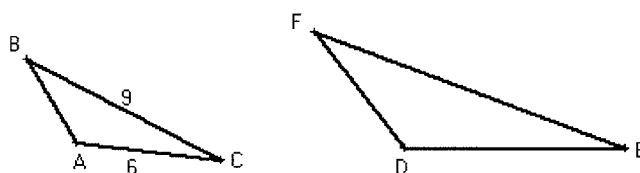


Figure 5. The drawing for Task 5*, Activity 2.

Activity characteristics

First, in this problem the students do not receive any confirmation or refutation of their initial conjectures. Thus, they are uncertain about the existence or the non-existence of the required construction. They therefore were engaged in a situation of uncertainty. Secondly, as in all other construction problems in such an environment, the computerized tool elicited empirical actions followed by prompt feedback, but deductive considerations

were needed for complete success. More generally, students' construction processes depend here on their ability to analyze the problem deductively, leading to the elaboration of an existence proof. In this case the construction itself is a kind of deductive proof – 'proving by doing'.

Types of students' explanations

There were a total of 112 responses in the 12 written reports on the all five tasks of this activity. Eighty-five of them related to tasks 2 to 5 in which students worked in situations of uncertainty. The analysis below relates to these 85 explanations (the preparatory Task 1 does not lead to contradiction and was not included in the categorization of explanations.)

In spite of the differences between the tasks in this activity and the previous one (the sum of exterior angles in a polygon), and hence the differences in the explanations, we found that we could use the same categories.

1. No explanation category

There were 23 (27%) responses in this category. For example:

Two triangles with 5 equal elements must be congruent.

2. Inductive explanation category

Three explanations (4%) fell in this category.

Example 8:

One pair, while explaining why there are no congruent triangles with five equal elements, drew Figure 6 on their worksheet and wrote:

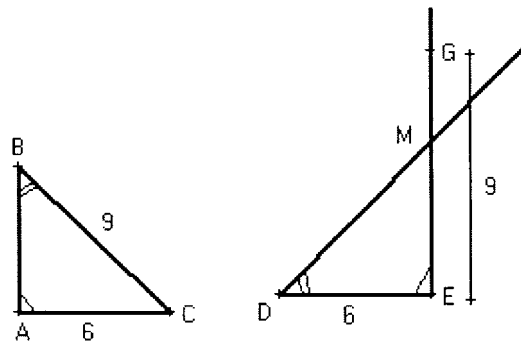


Figure 6. A single drawing as an explanation for the impossibility to accomplish the construction in Task 5*, Activity 2.

We tried, and something like that (Figure 6) came out. We can see that if one wants the triangles not to be congruent, one of the two sides we choose will never be 9. [They meant, that EM is never 9].

The two students relied on manipulations on one example to explain their wrong conclusion.

3. Partial deductive explanation category

In nine (11%) explanations students constructed a partial deductive chain of arguments, which mostly led to a wrong conclusion.

Example 9:

One pair from the whole class population, explained in Task 3b:

If the 3 sides and 2 angles are equal it is impossible [to construct non-congruent triangles]. If 2 sides and 3 angles are equal it is also impossible, because then the triangles are similar, and it does not exist without correspondence.

This pair of students did not take into consideration that similar triangles could have two equal (not corresponding) sides and be non-congruent. In spite of the fact that their conclusion was wrong the arguments which tie together the different claims were deductive. The wrong conclusion seems to indicate their difficulty in accepting a situation, which contradicts the intuition based on previous knowledge in which three elements are sufficient for congruence.

Example 10:

In Task 4, Nadav and Gili (in an interview) discussed the issue of six equal elements in non-congruent triangles. Nadav who was surprised to succeed (in Task 2) to construct non-congruent triangles with four equal elements, argued that it was possible to construct non-congruent triangles with six equal elements and tried to apply the same strategy again – drawing equal elements but not in correspondence. He drew Figure 7 on a piece of paper, and asked the interviewer whether to start with the construction in the DG environment. It is interesting to note that yet he sketched congruent triangles.

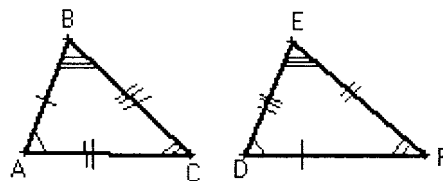


Figure 7. Nadav sketch for Task 4, Activity 2.

Later the two students concluded deductively that this construction was impossible.

4. Visual-variations explanation category

Working in a DG environment creates opportunities for explanations arising from visual variation induced by dragging. This kind of visual variation was absent in the explanations of the first activity. We classified two explanations in this category (2%); one of them is also included in the deductive category. It is worthwhile noting that in Task 1, seven pairs used this kind of explanation to explain why one side and one angle are not sufficient for congruence, but one side and two corresponding angles are sufficient.

Example 11:

Following their success in constructing non-congruent triangles with four equal elements and their correct inference that one cannot construct two non-congruent triangles with six equal elements, (see example 10), Nadav and Gili, tried to solve the uncertainty in the case of five equal elements. They erased the 'equal signs' on BC and DE in their paper and pencil drawing (see Figure 7) in order to plan their construction. Then they started to drag the vertices of $\triangle ABC$ (with four elements equal to four elements in $\triangle DEF$) already constructed in Task 2, and tried to get an additional equal side through a sequence of dragging and measuring actions.

Example 12:

In Task 2, a pair of students in an interview constructed the second triangle (see Figure 8) where: $DE=AC$; $\angle D=\angle A$; and $\angle F=\angle B$, and then dragged vertices in order to try to obtain non-congruent triangles, until they realized that it was impossible. They then explained:

As a matter of fact, It was not possible to drag vertices B or F, in order to make side DF unequal to AB.

5. Deductive explanation category

Forty-eight explanations (56%) fell in this category. All the 12 pairs constructed non-congruent triangles with one side and two angles equal, either as a response to Task 2 or to Task 2*. The deduction here is expressed by the *doing* itself, because without the deductive considerations of correspondence they would not be able to complete the task. An additional six constructions, to solve Task 5 or 5*, are examples of *deduction by doing*. The students deduced that the triangles (see Figure 5), are similar because of the angles' equality, but not congruent, thus FD should be six and DE

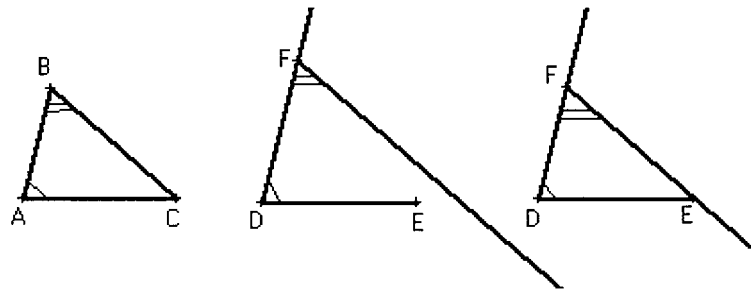


Figure 8. Dragging vertices as an effort to get non-congruent triangles in Task 2, Activity 2.

should be nine. Then they were able to calculate that the ratio between corresponding sides in the similar triangles DEF and ABC must be 1.5, and hence AB must be 4, and FE 13.5. Then they constructed the two triangles with the software.

In addition to the 18 deductive explanations by doing, there were further 30 typical explanations in the deduction category. For example;

Example 13:

The following discussion took place after Nadav and Gili in the interview (see examples 10 and 11) calculated the ratio and the lengths of the sides in Task 5*.

Nadav: But the angles should be equal as well, (pause) but it is O.K.

I: Why are you sure that the angles should be equal?

Nadav: Because they [the triangles] are similar.

The overall goal of this activity, to push students towards deductive considerations in order to overcome uncertainty in geometrical situations, is exemplified in the following episode.

Example 14:

Udy and Amir, a pair in the classroom, wrote in Task 2 that it is *impossible* to construct non-congruent triangles with one side and 2 angles equal, because of the angle, side, angle theorem. Later in Task 2* (Figure 4) they wrote:

Angle B *can be* copied in two different ways, at vertex E or F. It must be copied at E otherwise the triangles will be congruent.

In Task 3 they wrote again that it is *impossible* to construct non-congruent triangles with five elements equal, and added:

In this way we will get congruency *for sure*, because one of the congruence theorems *should hold*.

In Task 5a they wrote that the equal elements must be three angles and two sides:

because otherwise the side, side, side theorem would hold automatically . . . But, we think that even in this case the triangles will be congruent, because some other theorem *must hold*.

When in Task 5*, the students succeeded in calculating the third side in each of the two non-congruent triangles, they wrote:

In the presence of the new data that were revealed to us, we *must admit* that two non-congruent triangles with 5 identical properties, *do exist*.

This example demonstrates how *deduction by doing* may solve uncertainty.

CONCLUSION

In this paper we have described two activities. In the first, students faced contradiction and in the second they also faced uncertainty. We studied how these activities led many students to articulate explanations. As researchers, our approach is naturalistic, meaning that we classify students' explanations without relating to their correctness. As mathematics educators, and as people involved in the development of mathematical thinking, we are obviously interested in fostering deductive explanations (the preferred norms in the mathematics milieu). Thus we intentionally sought to identify the frequency of deductive explanations and the conditions which encouraged them. However, the appearance of other explanations, especially the visual-variational explanation, often based on the (dynamic) displays or stemming from students' imagery (which in turn may originate from students' prior experience with the DG tool), open an exciting issue. The question is whether these explanations can be considered as a normative way to prove, which obviously depends on the explanations themselves. This issue which deserves a special debate, stresses that norms are not only a function of what is imposed by professionals, but may be influenced by practices induced by new cultural tools and contexts.

As to the findings concerning the context in which deductive explanations arose, we saw that in the first activity, in spite of the contradiction, students were certain that the result they obtained from the tool was correct. In contrast, in the second activity, we capitalized on uncertainty in addition to contradiction, uncertainty that was maintained as long as the tasks were not solved. The sequence of tasks led students to oscillate between two different hypotheses (the impossibility or possibility of constructing

non-congruent triangles). Students' actions relied on deductive considerations, or on the inclination (based on experience from previous stages of learning) to think that three equal elements guarantee congruence. In this second activity some explanations had to be expressed by *doing*, which requires the use of deductive considerations. It seems that these actions pushed them towards deductive explanations in other tasks in this activity. This may be one of the reasons for the fact that the proportion of deductive explanations (56%) in the second activity is much greater than in the first (18%).

The two activities exemplify a design in which learning in a DG environment opens opportunities for feeling the need to prove, rather than considering proving as superfluous. In addition, the design of each activity (in particular the second one) as a chain of tasks, enabled students to be involved and apprehend a comprehensive topic in geometry. In a sense, the second activity seemed to push students to feel the systematization (de Villiers, 1999) of the topic of congruence, – a function of proof that is seldom reached in classrooms.

In both activities, we capitalized on surprise. Surprise occurs when there are expectations which, do not fulfilled. The students had made a *conjecture* concerning the solution of the task and found a reason why this *conjecture* was true. The reason was often rooted in common sense or based on previous learning. In the first activity the refutation was given by students' findings in the DG environment, and thus lead students to more than accept the correctness of the new conjecture, it lead them to construct a new argument. In the second activity we documented and analyzed refutations that evolved when the *proof by doing* was completed, and thus a surprise took place. We believe that in both cases students were led to use deductive arguments, that is, to construct reasons to support the new conjecture or reasons that enabled them to solve the problem. It seems that in several cases, the construction of a new argument on deductive grounds was accompanied by an evaluation of the previous argument. Students related to the previous reasons to show *why* those reasons were erroneous, and why the new conjecture was true. For example, students initially thought that the sum of the external angles increases as the number of sides increases. When they realized that the sum is fixed, they repaired the argument by introducing the idea of *compensation* (Examples 3, 5, 6 and 7).

The various explanations seem to indicate that students started to be aware to the role of explanations in general and to the role of deductive explanations in particular, in the construction of meaning and in increasing their confidence in the geometrical conclusions.

We conclude then that the design in a DG learning environment brought proofs into the realm of student activity and argument; that is, they engaged naturally in true mathematical activity. Laborde (1995) claimed that it is important to design tasks for DG environments, which can only be solved using geometrical knowledge. And indeed in these activities, the students ceased to be recipients of formal proofs, but were engaged in an activity of construction and evaluation of arguments in which certainty and understanding were at stake, and they had to use their geometrical knowledge to explain contradictions and overcome uncertainty.

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