

Towards theoretical foundations of mathematics education

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Abstract Although research in mathematics education developed in the last decades as a vivid scientific field, the nature and the place of theories in the field are still under discussion. H.-G. Steiner contributed to this debate in several ways. He not only intervened in the scientific debate but also played an active role in organizing the discipline and the scientific discussion at the international level. This paper attempts to give a synthetic view of the evolution of mathematics education with respect to theory by focusing on the French situation of research in mathematics education and paying a particular attention to the role and contribution of H.-G. Steiner.

1 Theories of mathematics education: two stories

Reflecting on the teaching of mathematics, introducing innovations in teaching, formulating ideas and proposals of change is not new and one can find traces of discussions about learning and teaching of mathematics even in writings done many centuries ago, such as in Clairaut (1765). With the attempt to create research on mathematics education as a scientific field, the importance of developing theories emerged. H.-G. Steiner was certainly a pioneer in working for the establishment of “didactics of mathematics as a scientific

discipline” as celebrated by the book edited by Biehler et al. (1994) on the occasion of Steiner’s 65th birthday. He supported very much the importance of developing theories for the growth of didactics of mathematics as a scientific discipline. One of his enterprises in favour of this idea was the foundation of the international group “Theories in Mathematics Education”. Since the seventies, several theories have been developed and simultaneously a debate has taken place about the nature and role of theories in this field of research, which is still continuing.

This paper intends to discuss the nature and the role of theories in didactics of mathematics by presenting shortly two stories, the story of the evolution of theories in the French research community and the story of research about the use of information and communication technology (in the following: “ICT”) in mathematics teaching and learning. We consider these two stories as case studies allowing us to better identify the nature, the role and the functioning of theories.

The French denomination of “didactique des mathématiques” similar to the German “Mathematikdidaktik” does not have the connotation it has in English (Kilpatrick 1994) and was chosen to express a specific scientific approach to processes related to teaching and learning mathematics: a study of the aspects of those processes specific to mathematics. The French research community in “didactique des mathématiques” is well known for its constant commitment to theories, lasting from the very first year of the international community in research in mathematics education. As such it may give the impression to be like a Gallic village within the international research community in which the role of theories as well as the relationships and tensions between theory and practice

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are still under discussion. The necessity of domain-specific theories and methods of research in mathematics education has been claimed in France for a long time, in particular in the first year of the establishment of a research community. This is why one of the connected enterprises of H.-G. Steiner with his campaign for the international group TME was the development of bilateral exchanges between Germany and France that gave rise to the bilateral French German conference¹ in the centre for mathematics conferences of Luminy² in France in 1986. During the past 20 years, the same recognition of the importance of the role given to theory has been maintained in France while theories evolved and new theories came into being. The aim of this paper is to sketch the nature and the role of theory from a short survey of permanent and changing features of theories. The meaning of current theories cannot be detached from their origins as claimed by de Guzman (1992, p. 147) about mathematics research: “I strongly believe that the crucial insights in research in a particular field tend to come from a deep knowledge of the origins and evolution of the theory one is working with”.

It is well known that ICT is not widely integrated into the mathematics teacher practice because of the deep changes it may introduce in several aspects of this practice. In the same manner, it may illuminate the discussion about the function of theory to analyse how integrating an additional element of the teaching system into a theory may affect theory itself, how it allows the researcher to focus on aspects, which had been invisible until now.

2 Permanent and changing features of French research in didactique des mathématiques

2.1 An invariant: the mathematical content at the core of research

From the very beginning of the story of the French “didactique des mathématiques”, a strong claim was made about the central place of the mathematical content in theories, even in the line of research more centred on students’ conceptualizations. Let us comment this specificity with the example of two theories that appeared at the beginning of the official existence of the French community in “didactique des math-

ématiques”: the theory of conceptual fields (Vergnaud 1990, 1994) and the theory of didactical situations (Brousseau 1997).

Vergnaud considered that modelling student’s knowing about a mathematical notion makes sense only when taking into account the mathematical problems in which the notion is involved. For Vergnaud, modelling students’ knowledge is modelling the interaction between problem and student: “la solution de problème est la source et le critère de savoir” (Vergnaud 1981, p. 220) (“solving a problem is source and criterion for mathematics knowledge” Translated by CL). Later Vergnaud, probably influenced by the theory of didactical situations, replaced the word “problem” by the word “situation”. For Vergnaud, modelling the state of students’ knowledge and their learning processes could be done only when referring to a subset of mathematical problems related to the mathematical content to be learnt.

One of the main questions in the development of a theory is to delineate an object of study. In this case, it consisted in choosing the adequate size of parts when subdividing the set of all possible mathematical problems, adequate with respect to learning a mathematical content. A widely shared hypothesis of epistemological nature underlying Vergnaud’s question deals with recognizing the existence of strong interrelations among mathematical notions. The learning of a notion, let us say for example addition, cannot be considered independently of children’s conceptions about numbers, but conversely those conceptions are not fixed and are likely to evolve during the learning of addition. Analysing learning processes with the help of too small units would prevent explanations taking into consideration a more coherent view of the construction of mathematics knowledge by students and finally give rise to very local models of students’ knowledge building. Considering a too large subentity of mathematical problems is likely to favour global analyses missing mathematical specificities of the objects.

This question led Vergnaud to define what he calls the conceptual field: “On appelle champ conceptuel un ensemble de situations dont le traitement implique des schèmes, concepts et théorèmes, en étroite connexion, ainsi que des représentations langagières et symboliques susceptibles d’être utilisées pour les représenter” Vergnaud (1994, p. 65) (“We call conceptual field a set of situations for which solving processes imply schemes, concepts and theorems in tight connection as well as language and symbolic representations, likely to be used for representing them” Translated by CL).

Vergnaud himself contributed to investigate the learning of two conceptual fields: the field of additive

¹ Proceedings in French were published in 1988 *Actes du premier colloque franco-allemand de didactique des mathématiques*. C. Laborde (ed) Grenoble, France: La Pensée Sauvage Editions

² The equivalent German centre is in Oberwolfach.

structures and the field of multiplicative structures. Vergnaud (1988) claimed that taking into account mathematics deals even with the choice of terminology. He attempted to describe as much as possible students' competences in mathematical terms as visible in names like theorems in action (*théorèmes-en-acte*, in French) or homomorphism theory (*théorie de l'homomorphisme*, in French).

Whereas, in the theory of conceptual fields, the focus was on the interaction between student and problem as a means of investigating students' conceptualizations, the theory of didactic situations focused on the relationships between the functioning of mathematical knowledge and the situation. The notion of problem present in Vergnaud's work corresponds to the notion of situation in the theory of didactical situations, which was deeply theorized, since the notion of situation itself was used to characterize the theory, as visible in its denomination. As Brousseau explained in a colloquium talk in 1975, the "...underlying hypothesis of Didactique is that a learning process can be characterized by a sequence of reproducible situations that lead to the students' learning of a particular piece of knowledge, or more concretely to a set of modifications of the students' behaviours which characterize the acquisition of that piece of knowledge [...]. The objective of research in Didactique is to study the ways, in which these Situations evolve, to characterize them in terms of the didactical conditions of learning and of the informational content of specific concepts and to find the "laws" governing their evolution" (Brousseau quoted by Warfield 2006, p. 18).

The focus is not on students' conceptualizations in relation to the problems students are faced with, but on the situations as leading to learning a particular piece of knowledge. One of the aims of the theory is to identify conditions on situations, with which to face the students and in which they develop a scientific activity that can be the source of the construction of knowledge, i.e. of learning. Learning is considered as resulting from an adaptation to a new situation in which the student's prior knowledge does not allow him/her to develop an efficient solving strategy. She/he must develop new means of solution that are the source of new mathematical knowledge. In the theory of didactic situations, knowledge takes its meaning from the economy it brings in the situation and the prior choices it rejects: making groups is an efficient way for counting the number of elements of a large collection that prevents from counting errors, decimal fractions offer economical means for comparing fractions with contrast to other fractions. The economy brought by a concept in mathematics with respect to

problems is the first driving force behind the notion of situation developed by Brousseau, even if the notion of situation cannot be restricted to this aspect. Numerous examples of such situations can be found in Brousseau (1997) that were experimented in school.

Both theories, the theory of conceptual fields and the theory of didactical situations are based on learning hypotheses of the cognitive nature or on two non-domain specific "grand theories" (as called by di Sessa and Cobb 2004), constructivism and socio-constructivism. Both theories aimed at creating conditions for improving mathematical learning and teaching.

Another theory developed in France by Chevallard (not based on those grand theories) was rather meant for analysing the functioning of the teaching system in mathematics as it is: the "Anthropological Theory of Didactic" (Chevallard 1999) presented for the first time at the bilateral French German conference. A germ of this theory was the concept of didactic transposition (Chevallard 1985), extensively discussed in Germany (Seeger et al. 1989), probably because of the German tradition of elementarization of mathematics (Klein 1925), mathematical background theories of presentations of the taught mathematical content (Vollrath 1985) and of "Stoffdidaktik". This theory and in particular the concept of didactic transposition is not only focused on mathematical knowledge, it questions mathematical knowledge and does not consider it as defined for ever and everywhere. Knowledge taught in school results from a process transforming knowledge constructed in institutions outside school in order to adapt it to the constraints of the teaching institution such as

- the societal requirements for the mathematical curriculum,
- the views and aims of the "noosphere" when defining the programme of studies and curricula and,
- eventually the sequential way of introducing new notions in the teaching, the requirement to define new knowledge in function of already introduced pieces of knowledge, the algorithmization of knowledge.

Several steps in the process of didactic transposition were distinguished: scholarly knowledge, knowledge to be taught, actual taught knowledge, learned available knowledge.

The notion of scholarly knowledge gave rise to several discussions, in particular between France and Germany. Sträßer (1992, 1994 at the conference for the 20 years of didactics of mathematics in France) by means of the example of elementary geometry argued how geometry taught in school did not originate only

from the discipline of mathematics but also from spatial knowledge, he also quoted Bourbaki claiming that elementary geometry is no longer a research field but a mere dictionary. In the same way, the didactical transposition of activities involving mathematics (often invisible mathematics) in vocational education does not necessarily start from a scholarly knowledge but from the target work situations that are transposed into training oriented situations (Samurçay and Rogalski 1998).

The anthropological theory evolved over time for modelling not only transformations of mathematical knowledge but mathematical practice itself since mathematical bodies of knowledge are inseparable from this practice. It gave rise to the notion of “praxeology” modelling any human action at large (Chevallard 1999). Didactics of mathematics is conceived as the study of mathematical praxeologies and didactic praxeologies in various institutions (institutions producing, disseminating, using or teaching knowledge). A praxeology is described by Chevallard as a set of tasks defining the field of human activity in question, of techniques developed to solve the tasks, of technologies, which are the justifications for the techniques and of theories founding the technologies. The evolution and changes of the teaching of a mathematical notion over time are made visible through the description of the various praxeologies in use. A good example is provided by the teaching of proportionality that was object of numerous changes in the history of mathematical teaching (see below).

As visible in this short survey, various theories over the past 30 years have developed in the French community but all give a prominent place to the mathematical content. This is not surprising as it is the main specificity of research in “didactique des mathématiques”, as claimed by the French research community. However, what emerges from the various approaches is a new perspective on mathematical content, a perspective that differs deeply from the widespread view of mathematics as a universal science independent of the context of its uses. A common feature of all models underlying the approaches presented above is that, what is modelled is the use of mathematical content: the use by the learner while solving problems (theory of conceptual field), the relationships between the features of a situation and the economy of uses of a given mathematical content (theory of didactical situations), the relationships between the institution and the techniques for solving tasks within this institution (praxeological approach). In all these approaches, mathematics is not reduced to a text of axioms, definitions and theorems: problems or tasks, and solving

strategies or techniques are the first focus of attention. All approaches are based on a “grand epistemological assumption” according to which the meaning of mathematical concepts is twofold, on the one hand it deals with more or less formalized theories, on the other hand it is related to practice, it consists of the ways of using the concepts in various situations.

2.2 The multifaceted complexity of mathematical knowledge

The focus on mathematics in use revealed that the mathematical content is not independent of the individuals using it and of the conditions in which it is used. Mathematical knowledge itself is not transparent (Chevallard 1994, p. 314), it poses problems. Whereas, research in mathematics is oriented towards generalizing and extending theories, research in mathematics education often deals with the variations of mathematical concepts and investigates how these variations depend on conditions of a various nature, cognitive, social and societal or institutional. Research in didactics of mathematics is thus not without relationships with other disciplines, like epistemology, psychology or sociology as stressed by Steiner (1985). Some examples of mathematical concepts are very relevant for illustrating the large scope of uses and practices attached to them, whether it is in the history of mathematics, in the history of the teaching of mathematics, or apart from school. Proportionality and functions are two of them.

The variety of strategies that appeared in the teaching of mathematics for finding the fourth term in a mathematical proportion between two magnitudes is huge: from the “rule of three” which prevailed for a long time in many countries, through the use of operators of the modern mathematics to the equality of product of extremes and means, currently in use in French school. The praxeological approach illuminates very well the variety of techniques and corresponding technologies attached to each praxeological organization (Bolea et al. 2001). However, the students’ preferred strategy identified by Vergnaud (1983) differs from strategies presented in teaching. The students do not spontaneously try to find the operator even when several values of the two proportional magnitudes are given. They prefer a “scalar” strategy, which consists in expressing the value of the magnitude as a linear combination of values for which the value of the proportional magnitude is known. If they have to find the price p for a given weight w , they express w as a function of a weight or several weights for which the price is known.

- as $w1 + w2$ and use the rule that $p = p1 + p2$
- or as $nw1$ and use the rule that $p = n w1...$

Students do not operate just on numbers, they operate on magnitudes and they take into consideration the dimension of the magnitude. Operating among values of same magnitude makes sense for them whilst the dimension (often complex) of the operator may not. Students conceive proportionality as the transfer of the same relationships from one set of values of a magnitude to another set. This example shows how, from the investigation of the practices and uses of a mathematical notion, research infers the meaning attached to these uses, as in the above case of proportionality. This was exactly the programme of Biehler (2005) when he reconstructed the meaning of function in the context of reform attempts in mathematics education. Function is again an excellent choice for a study of this kind as it is a complex notion resorting to several systems of representations, used in several domains of mathematics and extensively used outside mathematics, just as proportionality. It is clear that some concepts are especially well adapted for such studies: as in the theory of conceptual fields, the size of the concept in terms of relationships with other concepts should be large enough for involving several systems of representations and several techniques coming from various parts of mathematics; the concept should be used outside the academic mathematics. Biehler considers that reconstructing the “meaning landscape” of mathematical concepts (p. 62) is a critical task for researchers in mathematics education to provide a base for curricular developments and to enrich teacher education. He argues that it is important for teachers to know not only the academic uses of a mathematical notion but also its uses outside mathematics. Let us note the convergence of this approach taking into account mathematics outside school with the praxeological approach of Chevallard meant for modelling the use of mathematics not only in teaching institutions but also in institutions using mathematics, even in an invisible way.

2.3 Changes in research focus

During this evolution of theories, two kinds of changes appeared:

- The establishment of relationships between a microdidactic level and a macro didactic level,
- the move from the design of teaching experiments to the study of ordinary classrooms.

The theory of conceptual fields led to investigate students’ conceptions in specific classes of problems.

Although Brousseau’s teaching process on decimal numbers is a long-term process, the theory of didactic situations in a first phase gave rise mainly to study the microgenesis of new conceptions in carefully designed situations. The first examples of so-called “didactical engineering” (for the presentation of this method, see Artigue 1994) stressed more the fine microdidactic choices of each situation than the global choices of the organization of the sequence of situations which often were justified by an epistemological study and not by a didactical study. The notion of institution allowed research to introduce different scales in analyses and to relate micro and macrolevels: “The different scales for institutions allow the researcher to take into account and to study the different expectations concerning the same piece of knowledge and the ways to address the same problem through school levels or in different parts of the school system, i.e. changes in the institutional relationship to knowledge. This framework also allows one to observe the agreement (or not) between the personal and institutional relationships to knowledge and to make a connection between the notion of didactic contract at a microdidactic level and the notion of didactic transposition at a macrodidactic level” (Laborde and Perrin-Glorian 2005, p. 6).

Whereas, empirical data came mainly from the observation of designed teaching experiments (didactical engineering) in the eighties, the studies currently deal with a larger type of data, as ordinary classrooms give rise to analyses based on the same theories. Concepts coming from the theory of situations are used to analyse microprocesses used by teachers for managing the classroom (Sensevy et al. 2005): how teachers modify the milieu (mesogenesis process) or how they manage the organization of time (chronogenesis process). Simultaneously mathematical organizations and didactic organizations of long-term teaching are analysed in a praxeological approach (Barbe et al. 2005). The nature of empirical data has changed, they are no longer provided by ad hoc experiments, the classroom-reality itself and textbooks provide empirical data for an analysis based on theories. In this sense, didactic research methods come closer to methods of human and social sciences.

3 Research about the use of technology in mathematics teaching

The use of technology in mathematics teaching gave rise to many studies across the world that were devel-

oped within various theoretical frameworks. One can also identify different trends in the progress of research, may be more explicit than in the general case of research in didactics of mathematics.

At the beginning, the focus was essentially on the interactions taking place between the learner and the machine and how they impacted learning. A constructivist perspective in a broad sense was generally adopted. It is only in a second phase that growing attention was paid to the epistemology of the mathematical content whose learning is at stake. In the first year of technology in school, learning was mainly considered as emerging only from the interactions between the student and the machine, rather than between the students and appropriate tasks to be done with the machine. The focus then moved to the design of adequate tasks and from there to the teaching environment, and in particular to the role of the teacher and the social interactions she/he could organize in the classroom as well as the social norms developing in the context of the classroom.

A twofold process took place in the development of research with regard to theories:

- Already existing theories of didactics of mathematics were used to investigate the learning and teaching processes based on the use of technologies,
- but phenomena probably more amplified than in paper and pencil environment were observed and either new theoretical concepts were proposed to take them into account or adaptation of theories was achieved.

For example, the notion of “milieu” coming from the theory of didactical situations is well adapted for analysing how students’ strategies are affected by the features of technology and feedback coming from the technological environment, generally richer than in paper and pencil environment. The drag mode in dynamic geometry environment brings mathematical information on a geometrical construction, because it follows a geometrical model on which the environment is based.

An example of a new concept created in the context of research about the use of technology is provided by the concept of “situated abstraction” introduced by Noss and Hoyles (1996) in their research on conceptualization processes of the learner with digital technologies. When analysing strategies of students faced with tasks, Noss and Hoyles could observe how students’ actions are shaped by technology. Students start by using the technology and progressively discriminate the mathematical relations and concepts underpinning the behaviour of the tools.

This is followed by generalizations, which are local to the situations from which they emerged. Finally they move on to synthesizing their generalizations with different contexts and representational registers outside the specific technology used. Noss and Hoyles introduced the concept of “situated abstraction” to account for the constructions of the learner. Situated abstractions are invariants that are shaped by the specific situation in which the learner forges them. Although these invariants are situated, they simultaneously contain the seed of the general that could be valid in other contexts.

Two examples of adaptation of theories external to didactics are presented below.

The instrumental perspective, developed by psychologists in the mid-nineties (Vérillon and Rabardel 1995) about the role of the tool on the constructs of the user was recently adopted by researchers in mathematics education to understand the strategies used by students when beginning to use technology for solving mathematical tasks. A tool is not transparent. It affects the way the user solves the tasks and thinks. The instrument, according to the terms of Vérillon and Rabardel, denotes this psychological construct of the user: the instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for him/herself and has integrated it with her/his activity. The subject develops procedures and rules of actions when using the artefact and so constructs instrumentation schemes and simultaneously a representation of the properties of the tool (according to what Vérillon and Rabardel call instrumentalization schemes). The first studies about instrumentation processes addressed the use of computer algebra systems (“CAS”) by students (Guin and Trouche 1999). They were mainly focusing on the difficulties of students using technology and on the detours they sometimes follow in order to perform an activity with technology (for an overview: Artigue 2002). Today, research pays more attention to the mathematical knowledge involved in instrumental knowledge. Because technology used in mathematics embarks mathematics, mathematical knowledge is intrinsically linked to the knowledge about how to use the tool. Developing instrumental knowledge may also involve developing mathematical knowledge (Artigue 2002; Lagrange 1999). The instrumentation theory was first adopted by didacticians working on the use of technology focusing on the instrumentation schemes and-in a second step,

- instrumentation was integrated when planning the design of didactic engineering based on the theory of didactical situations,

- or attention was paid to instrumented techniques when analysing the praxeology of tasks solved in a technological environment (Lagrange 2001).

The second example deals with the Vygotskian perspective on internalization process of an individual when moving from an external use of a tool to its internal use. This approach is adopted for designing teaching sequences using technology to mediate mathematical knowledge to be taught. Operations carried out with technology may be subject to an internalization process with the guidance of the teacher and interpersonal exchanges within the class in the form of collective discussions (Bartolini Bussi 1998; Mariotti and Bartolini Bussi 1998). The interventions of the teacher are essential for making possible the construction of a correspondence between mathematical knowledge and knowledge constructed from the interactions with the computer environment. According to the instrumentation theory, the meaning constructed by the student when using the artefact may differ from what is intended by the teacher. Consequently, the interventions of the teacher are critical to let the meanings evolve towards culturally shared meanings of mathematical knowledge.

The story of theoretical approaches in research about the integration of ICT into teaching shows how it made visible the technical part of the mathematical work and the role of the tools. Again, hidden aspects of mathematics were brought to light: mathematical meanings are affected by the tools used to perform mathematical tasks, doing mathematics requires instrumentation of technology or of any system of representation or symbolism, and finally technical and conceptual parts are intrinsically intertwined in a mathematical activity.

4 Conclusion

From the two stories told in this paper, a dialectical link between mathematics and theories emerges: theories put mathematics at the centre of their focus and conversely analyses of empirical data based on these theories shed light on hidden or invisible aspects of mathematics and revealed its complexity. Steiner (1985) certainly considered mathematics as central in the efforts of developing theories for didactics of mathematics as a scientific discipline. He also stressed the transdisciplinary role research in mathematics education could play: “rather than being restricted to home-grown theories mathematics education should develop more professionalism in formulating home

grown demands to the cooperating sciences and in taking over a transdisciplinary regulative function as a kind of indispensable meta competency” (p. 8). This is a call for the future of research in mathematics education.

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