

Discernment of invariants in dynamic geometry environments

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Abstract In this paper, we discuss discernment of invariants in dynamic geometry environments (DGE) based on a combined perspective that puts together the lens of variation and the maintaining dragging strategy developed previously by the authors. We interpret and describe a model of discerning invariants in DGE through types of variation awareness and simultaneity, and sensorimotor perception leading to awareness of dragging control. In this model, level-1 invariants and level-2 invariants are distinguished. We discuss the connection between these two levels of invariants through the concept of *path* that can play an important role during explorations in DGE, leading from discernment of level-1 invariants to discernment of level-2 invariants. The emergence of a path and the usefulness of the model will be illustrated by analysing two students' DGE exploration episodes. We end the paper by discussing a possible pathway between the phenomenal world of DGE and the axiomatic world of Euclidean geometry by introducing a dragging exploration principle.

Keywords Dynamic geometry · Discernment · Variation · Perception · Dragging control

1 Introduction

Dynamic geometry environments (DGE) provide an epistemic domain where movement and variation together with visual and sensorimotor feedback can guide the identification of geometrical properties of figures. Even though DGE are modelled after a theoretical system like Euclidean geometry, the dynamism that characterizes DGE phenomena opens new perspectives not only for geometry as a mathematical discipline but also for geometry

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education (Laborde, 2000; Strässer, 2001; Lopez-Real & Leung, 2006). In particular, dragging in DGE has been studied in pedagogical settings and has been gradually understood as a pedagogical tool that is conducive to mathematical reasoning, especially in the process of generating conjectures in geometry. The epistemic potential of the drag-mode in DGE lies in its relationship with the discernment of invariants.

Identifying invariants is a major activity in mathematical thinking. Invariants concern what remains the same when different aspects of a phenomenon vary, and a sensorial aspect of discerning invariants is to perceive them visually and to *separate them out* during variation. For those who have a keen mathematical sense, especially in the domain of geometry, this may take place through a mental simulation. For example, it is possible to discern the symmetry of a given geometrical object by mentally rotating it or reflecting it. DGE are hinged on visual variation, and through dragging, they introduce the user to a pseudo-reality that helps to facilitate visualization of such mental simulation. In DGE, when a figure is constructed, it varies (via dragging) while keeping all the constructed properties unchanged together with all the properties that are their consequences according to Euclidean geometry. This opens up a context where geometrical properties (interpreted as invariants) can be visually perceived and hence linked to the theoretical control that is behind them. However, discussion is still wide open on how to link phenomena like those in DGE that are experimental in nature, with the Euclidean world that is instead axiomatic and deductive. In other words, how can we make geometrical sense of “dragging phenomena” in DGE? Specifically, how can dragging lead to identifying invariants that potentially correspond to geometrical properties? How is perception involved when particular ways of dragging are used during explorations in DGE, and how are invariants and their relationships identified and geometrically interpreted?

These are questions that contribute to the ongoing discussion on the epistemology and the related pedagogical potential of DGE. In this paper, we propose a cognitive model to address these questions elaborating on the idea of *invariant* and providing a description of the cognitive processes involved in identifying invariants. Such a model stems from the combination of two cognitive lenses. On the one hand, identification and classification of invariants will be described through constructs from the lens of variation (Leung, 2008). On the other hand, processes of geometrical sense making will be described through the notions of *direct and indirect control*, introduced by Baccaglini-Frank and Mariotti (2010; Baccaglini-Frank, 2012). These constructs allow the distinction of two levels of invariants.

We start the development of our model by introducing perception and discernment in Section 2, and then by discussing discernment of invariants under dragging in DGE in Section 3. Section 4 and Section 5 constitute the core of the paper: Here, the means of discernment in our model are described. In Section 6, we use a student exploration episode to illustrate the epistemic value of a specific aspect that can emerge, the *path*. In Section 7, another student exploration episode will be analyzed to illustrate how our model can be used to interpret a case of “failure.” The paper ends with Section 8 in which we propose a dragging exploration principle to discuss a possible pathway between the phenomenal world of DGE and the axiomatic world of Euclidean geometry

2 Perception and discernment

In the introduction we have mentioned that identifying or perceiving invariants is a typical activity in mathematical thinking. The cognitive model we introduce allows us to analyze

this activity in DGE contexts in greater detail than has been done before and can be useful to explain possible difficulties and obstacles in the complex process of identifying different types of invariants. Before introducing the model, we would like to further discuss cognitive aspects involved in ‘identifying’ invariants, referring to the basic notions of ‘perceiving’ and ‘thinking.’ To do this, we will refer to Neisser’s definitions (1989). According to Neisser, interpreting stimuli relies on two complementary processes: perceiving and thinking. In particular, *perception* is carried out on the local environment, and it is “immediate, effortless, and veridical” (p. 11); referring to Gibson, Neisser states that perception is “direct.” On the other hand, Neisser speaks of *thinking* as being “indirect” and anything but effortless. What we perceive are *affordances*, or “the possibility of actions we have not actually undertaken.” Moreover, “an affordance is not a property of an organism taken alone, but a relation between an organism and its environment” (p.12). If we apply such perspective to the context of DGE, a figure can be seen as a set of affordances that the dragger¹ perceives. Through interaction with the figure, based on the perceived affordances, the dragger can discover *invariants* as they are described in Neisser’s theory:

aspects of stimulus information that persist despite movements of the perceiver (or are actually brought into existence by those movements), and that correspond to relatively permanent features of the objective situation. (p.12)

This definition of invariants seems to be quite appropriate in the context of dragging in DGE: The dragger’s movements in this case are the actions carried out on an element of a constructed figure (for example, a point), selected and acted upon through dragging.

However, if we are concerned with the interpretation of such invariants within a particular theory, in our case, the axiomatics of Euclidean geometry, perception as this basic means of interpretation is not enough to explain the whole process leading to geometrical stances emerging from dragging experiences. Thinking, which is complementary to perceiving, acquires a specific connotation: It must bring conceptual aspects, associated to the perceived aspects, to the dragger’s mind. Take, for example, Fischbein’s theory of figural concept (1993). These conceptual aspects must refer to the theory of reference, in this case, Euclidean geometry. To distinguish simple perception from perception accompanied by this form of geometrical interpretation, we use the term *discernment*, echoing Leung’s use of such term (Leung, 2003, 2008) adapted from Marton’s Theory of Variation (Marton, Runesson & Tsui, 2004). In fact, this perspective gives a new theoretical foundation to the use Leung has been making of the term *discernment* and to his adaptation of the Theory of Variation to DGE, which is part of the cognitive model we present. Basically, the Theory of Variation is about using the four patterns of variation (contrast, separation, generalization, fusion), which are different aspects of simultaneous focus, to describe how by varying different aspects of a phenomenon, intentionally, one can unveil critical features of the phenomenon itself. More specifically, the theory focuses on discovering and designing conditions of learning so that learners can discern what is intended in the design of a task through variation experiences. However, the model we develop in this paper, continuing with Leung’s work (2008), adapts and further develops some of the basic constructs in the Theory of Variation to obtain a fine grain analysis of cognitive processes involved in the complex activity of *discerning invariants* in DGE. With this orientation, our model diverges from Marton’s Theory of Variation.

¹ Since dragging is the fundamental activity, we will be referring to within DGE, we refer to the student, or, in general, the person dragging, as dragger.

3 Discernment of invariant under dragging in DGE

Dragging in DGE consists of selecting an element of a dynamic figure (a figure constructed according to a set of properties within a DGE) with a pointing device and moving the pointing device with the result of changing the location of the selected object (and consequently possibly others) on the screen. The changes in the image on the screen obtained by the successive rapid reconstruction of the figure produced by the DGE are perceived in contrast to what simultaneously remains invariant, and this constitutes the base of the perception of “moving image” (Mariotti, 2010). The movement and the identification of invariants are what lie at the heart of activities that aim to exploit the potential of DGE (for example, Laborde & Laborde, 1995; Hölzl, 1996; Healy & Hoyles, 2001; Healy, 2000; Arzarello, Olivero, Paola & Robutti, 2002; Olivero, 2002; Leung, 2008; Baccaglini-Frank & Mariotti, 2010). A dynamic figure may be considered, in Hölzl’s words, as “the materialised representation of an ideal geometric figure only characterised by its internal relationships” (2001, p. 83). Moreover, in DGE, invariants are determined by the geometrical relations defined by the commands used to construct the dynamic-figure, and by the relationship of dependence between the original relations of the construction and those that are derived as a consequence within the theory of Euclidean geometry (Laborde & Strässer, 1990). However, such logical dependencies cannot be seen when the figure is acted upon, because all the invariants appear simultaneously during dragging. To illustrate this, let us consider the following construction (see, Mariotti & Baccaglini-Frank, 2011):

ABCD is a quadrilateral in which D is chosen on the parallel line to AB through C, and the perpendicular bisectors of AB and CD, r and s respectively, are constructed.

Figure 1 shows a dynamic figure derived from such construction in Cabri, a particular dynamic geometry environment. It has a set of *constructed* invariants (the parallelism between AB and DC, the perpendicularity of s to DC and of r to AB, the passing of s and r through the midpoints of DC and AB respectively) and consequently, all the invariants *derived* from these (for example the parallelism between r and s) which are all conserved simultaneously during dragging. In particular, in this Cabri construction, A, B, and C are free base-points which can be moved by dragging them directly; D is restricted to move along a line parallel to AB through C; r and s are dependent objects that move, indirectly, as a consequence of the movement of other base-points.² Hence, as they appear simultaneously during dragging by a dragger, different invariants may have a different status according to the type of control—*direct* or *indirect*—exercised by the dragger on them. In a constructed figure, there is a relationship of dependency between the constructed invariants and any that are derived from these according to Euclidean geometry. It may happen that a *relationship between invariants* is an *invariant* itself, and such a relationship can be discerned through dragging. In our example, there is an invariant relationship between invariants that can be expressed by the following statement.

If two lines are respectively perpendicular to two parallel lines, then the first two lines are parallel.

² We remark that, in DGE like Cabri, dependent elements cannot be moved by dragging them directly but only indirectly by dragging the elements they depend upon. However, in some other DGE for the same construction, r and s could be moved by directly dragging them.

Discerning invariants and discerning invariant properties between invariants are cognitively different tasks. We intend to speak extensively of both, and for this reason, we introduce the following terminology.

- *Level-1 invariants*: aspects of a dynamic figure that are perceived as constant during variation of the figure through dragging. For example, “AB parallel to CD” and “s parallel to r” are level-1 invariants of the dynamic figure constructed in Fig. 1.
- *Level-2 invariants*: invariant relationships between level-1 invariants. For example, “AB parallel to CD causes (or implies) s parallel to r”.

We will later describe a cognitive process that involves shifting between these two levels of invariants.

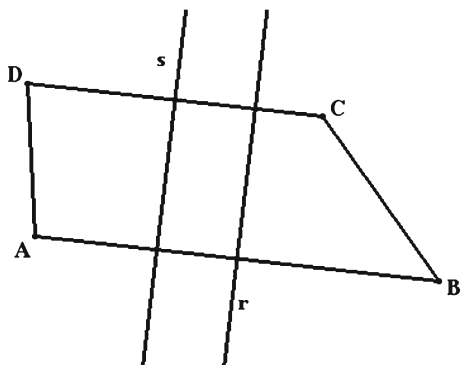
4 Level-1 invariants through the lens of variation (part 1 of the model)

The notions of variation and simultaneity are central in discerning *critical features* of a phenomenon. Critical features are aspects of a phenomenon that, when put together, have the potential to describe, represent, or even define the phenomenon. Variation instigates different levels of contrasting experience which seek to distinguish, separate, and generalize features of a phenomenon under the “eyes” of simultaneity. (cf. Marton et al., 2004, p. 6). We expound now these elements of variation in the context of DGE.

There are two forms of simultaneity that play crucial roles in the thoughtful process of perception that leads to discernment. *Synchronic simultaneity* (spatial type) “is the experience of different co-existing aspects of the same thing at the same time” and *diachronic simultaneity* (temporal type) is “to experience instances that we have encountered at different points in time, at the same time” (ibid., pp. 17–18). Synchronic simultaneity happens in real time while diachronic simultaneity is sequential in time. Leung has used these ideas of variation and simultaneity in what he has defined a “lens of variation” to discuss discernment in dragging explorations in DGE (Leung, 2003, 2008). To stress the intricate relationship between simultaneity and observable invariants, we develop a model describing how the use of variation along with types of simultaneous focus could form basic means to perceive invariants in DGE. The exploration depicted in Fig. 1 will be used as an illustrative example to discuss this means.

Contrast is a type of intentional simultaneous focus that seeks to establish whether something satisfies a certain condition or not. This corresponds to a very common habit in

Fig. 1 ABCD as a result of the construction described in the example above



mathematicians' practice: To fully comprehend a mathematical concept, one often resorts to finding non-examples in order to reveal critical features of the concept. Contrast brings about perception of invariant through awareness of differences when something varies. In DGE, a dragger *drags for contrast* to locate positions where an invariant feature could be perceived. In Fig. 2, B is being dragged in a wandering fashion to seek for possible positions that make the perpendicular bisectors r and s coincide (a level-1 invariant).

In this case, contrast is based on a time sequence of the varying position of B, hence a varying quadrilateral ABCD, that allows the dragger to perceive special related positions where r and s coincide. This is a diachronic simultaneous focus of attention in the sense that the dragger is paying attention to the variation of a dynamic object in a time sequence as it is being dragged (the changing quadrilateral ABCD).

Separation is a type of simultaneous focus that *brings about awareness* of certain critical aspects of a phenomenon. This awareness comes about when an aspect varies while some other aspects are being kept fixed intentionally. When the varying aspect leads to perception of a critical feature of the phenomenon that becomes an invariant, we call this varying aspect a *dimension of variation*. This perception rests on simultaneous focus on the different appearances of certain critical features that remain constant under variation. In particular, separation arises from exploring part-whole relationships within a phenomenon, pinpointing how to differentiate an invariant part from the whole. In Fig. 3, trace (a function in DGE) is activated for B. A wiggly path reflecting the dragger's feedback responses to dragging while keeping r and s coinciding is being traced (separated out). This path roughly appears to take the form of a circular arc and can lead to the awareness of a possible (yet to be discerned) level-1 invariant under dragging with intention to keep r and s coinciding (Fig. 3)—this dragging technique is referred to in the literature as maintaining dragging (Baccaglini-Frank, 2012). This *drag to separate* intentionality results in a visual *path* appearing on the screen which can be thought of as a visualization of diachronic simultaneous focus since this path is a simultaneous representation of a time sequence (different appearances in time) of positions that have visually satisfied a desired condition. In this sense, the activated trace converts diachronic simultaneities into synchronic simultaneities. Hence, perceptually, as long as point B varies along this path with A, D and C kept fixed, r and s *seem to* coincide. In Healy's term the property " r and s coincide" is a soft property (Healy, 2000), and in our perception framework, this means that an aspect of the dynamic figure becomes, under dragging, a relatively permanent feature of the dragger's expectation (refer to Neisser's definition in Section 2). Thus, a soft property can be considered as an affordance to uncover causal relationships between level-1 invariants.

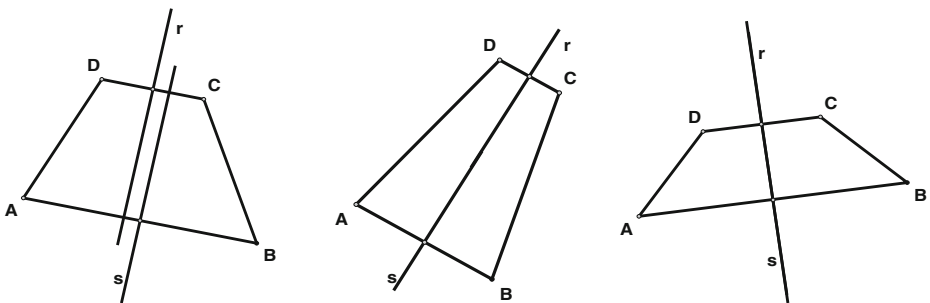
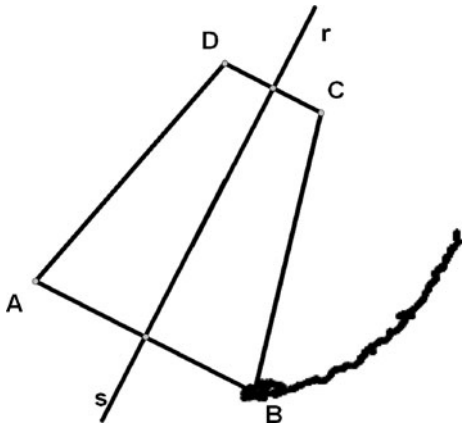


Fig. 2 B is being dragged in a wandering fashion to seek for possible positions that make the perpendicular bisectors r and s coincide

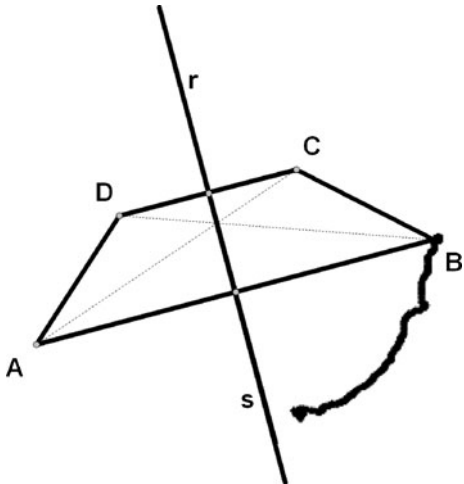
Fig. 3 The trajectory of B is being visually traced as B is dragged to keep r and s coincide



If the exploration goes on with the dragger simultaneously focusing on the diagonals AC, DB, and on the coinciding r and s (regarded as a level-1 invariant), then direct dragging on B by maintaining the shape of this path a circular arc can bring to the awareness that the centre of “the circle” that causes the coincidence of r and s seems to lie on the intersection of AC, DB, and the coinciding r and s (Fig. 4). This is a synchronic simultaneous focus of attention, and it separates out a possible causal relationship. Thus, separation is a simultaneous focus under variation that brings about awareness of the “inner structure” of a phenomenon.

Generalization is a type of simultaneous focus that seeks to verify the (global) invariance of a critical aspect. Continuing with the above exploration, construct a circle with centre in the intersection of AC and BD passing through A. Drag B along the constructed circle to test the validity of the perceived geometric interpretation of the traced path. The fact that A belongs to the circle now becomes a constructed property and consequently a level-1 geometrical invariant. This *drag to generalize* intention is based on a kind of drag-to-fit

Fig. 4 The center of the traced arc seems to lie at the intersection of the diagonals. Thus, an invariant critical feature is separated out



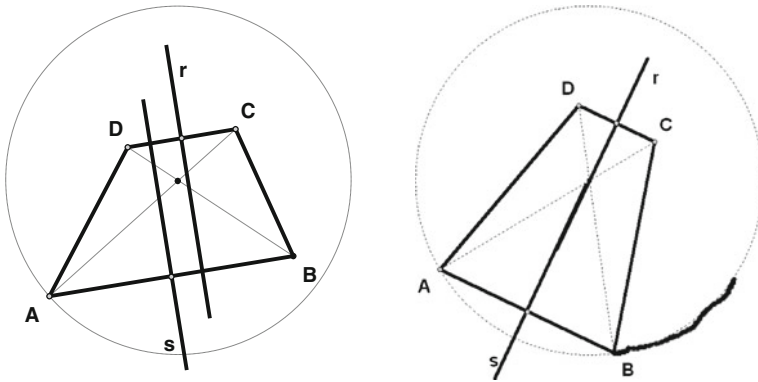


Fig. 5 Drag B along a level-1 invariant to generalize an observed relationship between level-1 invariants

soft dragging modality (Lopez-Real & Leung, 2006) to verify a global invariant relationship of a phenomenon in DGE (Fig. 5). This is discussed further in the next section.

In summary

Discernment of level-1 invariants in DGE is awareness of invariant aspects of a dynamic figure perceived under dragging through contrast, separation, and generalization.

5 Level-2 invariants through dragging controls (part 2 of the model)

The variation of a dynamic figure depends on the degree of freedom of its draggable base-points, in particular, the figure's possible movements depend on the steps of the construction that induce corresponding invariant properties of the figure. This constitutes an essential aspect of the 'being dynamic' of such a figure. In the example that we have been exploring, A, B, and C are base-points constructed with two degrees of freedom while D was constructed as belonging to the line passing through C and parallel to AB. Therefore, A, B, and C can be dragged to any place on the screen while D can be dragged only along the line to which it belongs. As we have discussed, in DGE like Cabri dependent elements of a construction cannot be directly acted upon,³ they can only be moved *indirectly* by the movement of their dependent base-points.

Making sense of the sensorimotor perception that constitutes the feedback of any dragging activity is entirely up to the dragger who will need to interpret the 'construction steps' as the origin of specific (perceived) invariants, relate them to other invariants, discover new ones, and logically link the perceived geometrical properties and relationships to one another. As said, linking level-1 invariants and perceiving the invariance of such a link corresponds to perceiving a level-2 invariant. Such as in the previous example, "AB parallel to CD causes (or implies) s parallel to r". In the preceding section, we have discussed how the lens of variation could facilitate such sense-making in perceiving level-1 invariants. To perceive level-2 invariants, the dragger needs to cognitively distinguish between direct and indirect movement of elements of a dynamic figure, relate (or learn to relate) such feedback

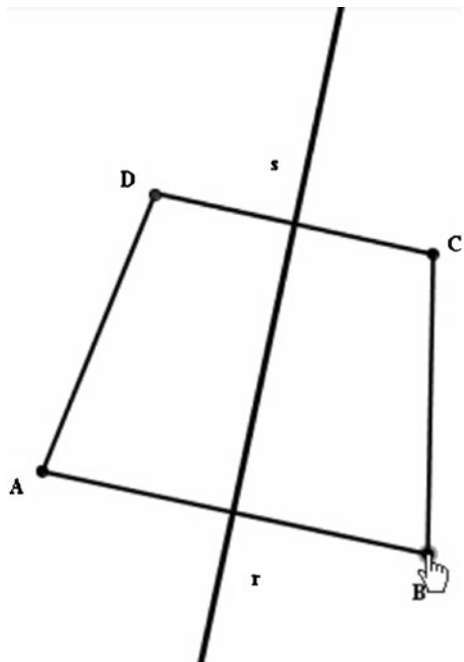
³ In some other DGE, they can be acted upon directly but in this case they lose their defining properties.

to the “construction steps,” and hence elaborate a dependency hierarchy among the elements of the dynamic figure. A leap at the cognitive level is to become aware of the *hierarchy induced* not only on the elements of the figure but *on their properties*, that is, to become aware of the fact that such a hierarchy corresponds to a logical relationship between specific properties of the dynamic figure, and that corresponds to a level-2 invariant. Not only can the dragger experience different types of control over elements, but different types of control may also be experienced over invariants.

We claim that there is a particular dragging modality that may make possible the experience of the different types of control. In the exploration that we have been analyzing, we discussed how a dragger tries to maintain an interesting property “coinciding perpendicular bisectors” by dragging a base point, and we have referred to this type of dragging modality as *maintaining dragging* (MD). Figure 6 shows a snapshot of how such property can become a *soft invariant* when the base point B is dragged. A soft invariant for a dynamic figure is an invariance perceived and maintained when a base point of the figure is dragged

The movement of B cannot be random as in the case of robust invariants (those related to the commands used in the construction and to their geometric consequences), but in order for the desired property to remain invariant, B must follow some pattern (a circular path, in this case). As far as the dragging experience is concerned, r and s move as an effect caused by the intentional and controlled movement of B, thus the *control* exercised over the movement of B may be experienced by the dragger as *direct* while that over the invariance of our desired property (r and s coincident) may be experienced as *indirect*. Awareness of these two different kinds of control can be developed through the experience of and reflection on sensorimotor perception, and it becomes fundamental when maintaining dragging is used as

Fig. 6 The figure shows a dragger in the act of trying to maintain the coincidence between the perpendicular bisectors, by dragging point B



a strategy of exploration (Baccaglini-Frank, 2010; Mariotti & Baccaglini-Frank, 2011; Baccaglini-Frank, 2012).

This kind of discernment, concerning level-2 invariants, involves *fusion*, that is, a simultaneous focus on different dimensions of variation while specific components of a dynamic figure vary under different dragging controls. In terms of the lens of variation, *dragging to fuse* involves the intention of a dragger to causally relate different invariants, inducing co-variation of different critical aspects through dragging control. With this proclivity, dragging to contrast, separate, or generalize can each be seen as components of dragging to fuse. A simultaneous focus to fuse different level-1 invariants under variation is central to perception and awareness of level-2 invariants. This involves perceiving a logical relationship between a property (level-1 invariant) being maintained by dragging (say by MD) and, consequently, an emerged pattern which may lead to another level-1 invariant. Let us revisit the situation where dragging to generalize was discussed in the previous section (Fig. 7).

Under MD with direct control on B while keeping r and s coincident, an emergent pattern taking the shape of a traced path resembling a circular arc may be separated out. The center of the anticipated circle may be further separated out and constructed. Thus, when B is being dragged along this circle directly, r and s consequently always coincide. In this way, direct control over B moving along the circle causes an indirect control over the coincidence of r and s , a level-1 invariant. This causal relationship transforms the status of “point B moving on the circle” to become a robust level-1 invariant that implies “the coincidence of r and s ,” another level-1 invariant. Thus, dragging to fuse and awareness of direct/indirect control result in a conditional relationship between two level-1 invariants:

IF (premise) B lies on the circle with the intersection of the diagonals as the centre, and A is the end point of a radius, THEN (conclusion) the two perpendicular bisectors r and s coincide.

To verify this logical relationship (a conjecture consists of a premise and a conclusion), we can devise construction steps respecting the logical relationship to robustly construct the two invariants (the circle and the coincided perpendicular bisectors) that reflect the direct and indirect controls exercised (Fig. 8). In this case, B and D each lose one degree of freedom. B can only be dragged along the circle, and D cannot be moved directly while A and C remain free points. In Fig. 8 are snapshots of a robust dynamic figure that could be interpreted as embedding a level-2 invariant (a set of two robust level-1 invariants are related within it).

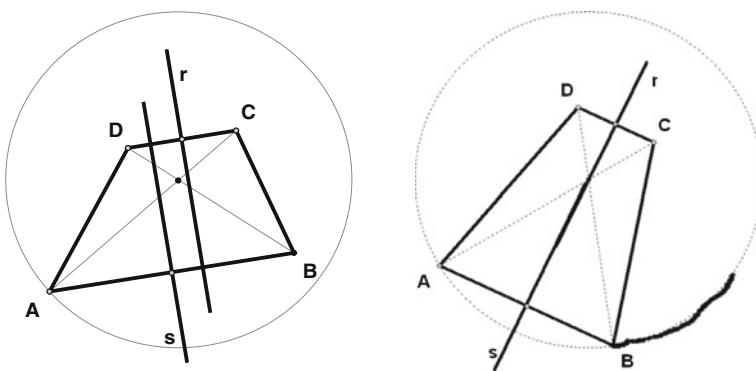
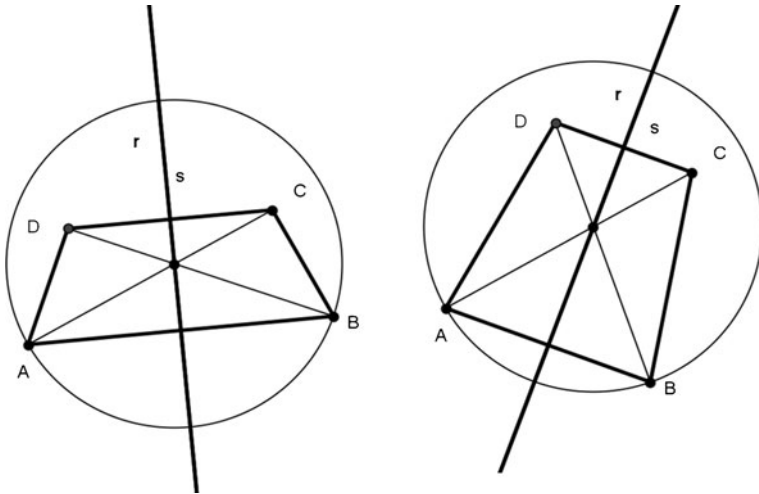


Fig. 7 Drag B directly along a level-1 invariant (the *circle*) indirectly and simultaneously causes the occurrence of another level-1 invariant (the coincidence r and s). Drag to fuse and awareness of direct/indirect control result in a logical relationship between two level-1 invariants



Our model⁴ describes the transition from discerning level-1 invariants and discerning level-2 invariants in summary as follows:

6 Discerning paths

⁴ It is beyond the scope of this paper to compare our model with other models. In particular, it should not be thought of as “developmental”: The transition it describes is given in terms of a set of means of discernment that we do not relate to any particular stage of cognitive development of the individual. Being able to discern a level-2 invariant and interpret it geometrically, per se, with respect to a model like Van Hiele’s certainly implies being able to operate at least at level 2 (Van Hiele, 1986, p.53); however, what we are concerned with is describing, from a cognitive point of view, what might be involved in discerning such an invariant during an (even single) experience in a dynamic geometry software.

a key element that emerges is the *path* (Baccaglini-Frank & Mariotti, 2009)⁵: It is the objectification of the constraint that will make property A be maintained, it is not an invariant itself, but it may provide a bridge to discerning a possible level-1 invariant that will become the premise of the conditional relationship that constitutes the level-2 invariant. We now present an example from a student exploration, showing how the path and its evolution may create a bridge between a level-1 soft invariant, maintained during dragging, and the level-1 invariant the dragger is searching that allows the student to reach a level-2 invariant, the conjecture.

A path could be conceived as a visual record of a *controlled variation*, and at the same time, it could be a record of simultaneous invariance: the *maintained invariant* and a possible new invariant causing it.⁶ As the exploration proceeds, the representation of the path (both mental and within the software used) passes through a prototypical sequence of transformations: First a path is envisaged (envisaged path); then a path can be roughly traced (traced path); the path may be constructed, and along such path, a base-point can be dragged (drag-along path), and finally, a generalized robust path can be constructed (generalized robust path). We will use this sequence to structure our discussion of the discernment of a *path* experienced by students working with Cabri. The episodes were collected during Baccaglini-Frank and Mariotti's study on MD (Baccaglini-Frank & Mariotti, 2009; Baccaglini-Frank, 2010, 2011). For such study, the draggers (who belonged to tenth Grade in Italy) had been preliminarily introduced to MD as a possible dragging modality. The exploration problem was:

Construct three points A, B, and C on the screen, the line through A and B, and the line through A and C. Then construct the parallel line *l* to AC through B, and the perpendicular line to *l* through C. Call the point of intersection of these last two lines D. Consider the quadrilateral ABCD. Write conjectures on the kinds of quadrilateral it can become, trying to describe all the ways it can become a particular kind of quadrilateral.

The problem asked students to “write conjectures” and in to doing so to “describe all the ways” in which certain configurations could occur. In the excerpts below, the bold letters indicate the student who was dragging.

6.1 Envisaged path: conception of a possible path

The students decided to use maintaining dragging.

- 1 Gio: and when... do like maintaining dragging when it is a rectangle [Fig. 9].
- 2 **Fab**: Never... I mean one point and that's it.
- 3 Gio: really? If you move... moving A... let's write moving ... [G starts to write the conjecture]
- 4 **Fab**: Moving A ...
- 5 Gio: Moving A... there is only one point... but are you sure, even going over there? Can't you go over there?
- 6 Gio: There... Already two...
- 7 **Fab**: two...[as F seems to find a new good position for A]
- [...]
- 13 **Fab**: two... I mean, one...[as he moves A back on these]
- 14 Gio: one...two...three, four...twenty thousand!
- 15 **Fab**: yes, there are really many of them [laughing]... let's do trace... we made a mistake. There are really too many.

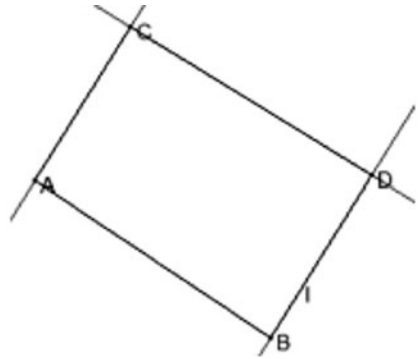
⁵ The notion is consistent with Leung and Lopez-Real's (2002) notion of *locus of validity*.

⁶ This is why a path can become the key element in a process of conjecture-generation. So analyzing how students discern paths and elaborate them can provide insights on their processes of conjecture-generation.

6.2 Traced path

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Fig. 10 The students dragged A until they obtained a “nice rectangle”



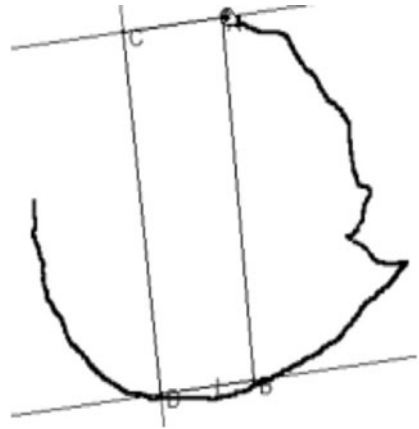
- 16 Gio: circle with...
 17 **Fab:** no
 18 Gio: eh, no.
 19 **Fab:** look at C. C doesn't move.
 20 Gio: I see a kind of circle with...
 21 **Fab:** ... with radius CB, and centre...
 22 Gio: No, with diameter AD, I see.
 [...]
 27 **Fab:** I would say... I would trace CB and its... [Fig. 11]
 28 **F&G:** midpoint
 29 Gio: for the radius

A was varying while B and C were fixed. D was dependent on A, B, and C, and, in particular, as A varied, D varied accordingly, appearing to vary diametrically with respect to the traced “circle”. Gio “saw” AD as a diameter (line 22) of the traced circle, in contrast with Fab’s focus on the constancy of C. It seems that this circle was in the students’ mind rather than on the screen. This constitutes an interesting (and frequent in students’ explorations) *separation* experience associated with maintaining dragging: a varying object—a dragged point moving along “something”—is being separated out as an invariant object under maintaining dragging. When the trace mark was activated on the point being dragged, Fab and Gio seemed to perceive C and B as more than just constructed free points; they seemed to be perceiving them, through *synchronic simultaneity*, as parts of a dynamic whole, the varying quadrilateral ABDC. The prominent level-2 invariant that starts to emerge during this episode is the relationship between dragging along the recognized circle and the invariance of ABDC rectangle. The students’ simultaneous focus on the two soft level-1 invariants allowed them to perceive a regularity, which they eventually interpreted as a level-1 invariant observed during dragging that “causes” the initially perceived level-1 invariant (ABDC rectangle) to be visually verified.

6.3 Drag-along path

Once the students recognized that BC was a diameter for the circle they sought after, they constructed a *drag-along path* (a circle with BC as a diameter) in order to drag the base point A along such path to ascertain (by eye) that in this case the initially perceived level-1 invariant was visually verified.

Fig. 11 Trace mark left during the performance of maintaining dragging as the students tried to maintain $ABDC$ a rectangle



30 Gio: Circle, do circle.

31 Gio: eh, let's choose...

32 **Fab:** well, I would say B and C because they are the two points that don't move... here...yes, because actually now we take A.

33 Gio: eh, we did it...cute! [Fig. 12]

34 **Fab:** yes, definitely.

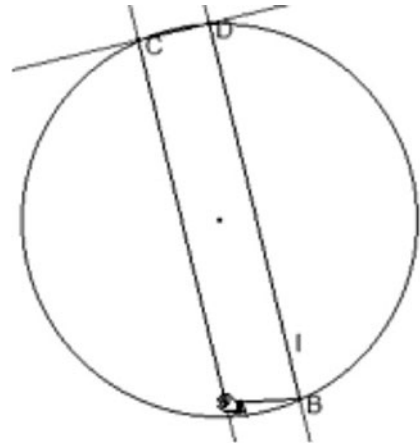
Here, two level-1 invariants are experienced simultaneously, with a difference of control: direct over “base point A on the circle” and indirect over “ $ABDC$ rectangle.” Such experience of simultaneity and awareness of the type of control exercised is an experience of *synchronic simultaneity* leading not only to *fusion* of two level-1 invariants but also to a conditional link between them, which constitutes a level-2 invariant.

6.4 Generalized robust path

The students linked the base point A to the circle they constructed (using the tool “redefinition of object” in Cabri) and dragged point A around the circle to verify that $ABCD$ is always a rectangle. The dragging was not by “eye control” anymore as in the drag-along path, as point A was robustly fixed on the path. The students expressed their discernment of a level-2 invariant in the following conjecture: “If A belongs to the circle with diameter BC , $ABCD$ is a rectangle.” Once A was linked to the circle (the drag-along path), the circle became a *generalized robust path* (Fig. 13). Students could then experience *fusion* through “perfect” *synchronic simultaneity* while dragging the base point along the generalized robust path. We finally remark that the final conjecture was expressed in “static” logical terms.

In the analyses above, we showed how the means of discernment we presented contributed to discerning a level-2 invariant through something conceptually new, a *path*, during an exploration in which students used maintaining dragging. Since the episodes we analyzed are prototypical with respect to explorations that we have observed other students carrying out, we advance the hypothesis that the use of MD can be a way to discern different types of level-1 invariants, and consequently, level-2 invariants.

Fig. 12 The students constructed a robust path to drag along



7 The model as a lens for identifying students' difficulties

We now use the model we have developed to analyze an exploration of two students who “got stuck” and were not able to produce a conjecture. The model will allow us to interpret their difficulties. The task is the same as the one introduced in the preceding section; the two students exploring the construction have used MD, trying to maintain the quadrilateral ABCD a rectangle. Performing MD, with the trace activated on the dragged point, the shape of a circle could be discerned, and the students constructed a circle which they thought might “fit the trace mark.” The circle they drew had center in the intersection point of the diagonals of ABCD, and it passed through the point that they were dragging (Fig. 14).

Unfortunately, this circle depended on the dragged point, so it moved with the dragged point, and therefore, it could not generate an effective constraint for the

Fig. 13 Once A was linked to the circle (the drag-along path), the circle became a *generalized robust path*

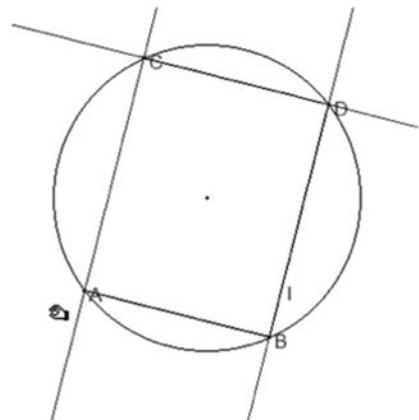
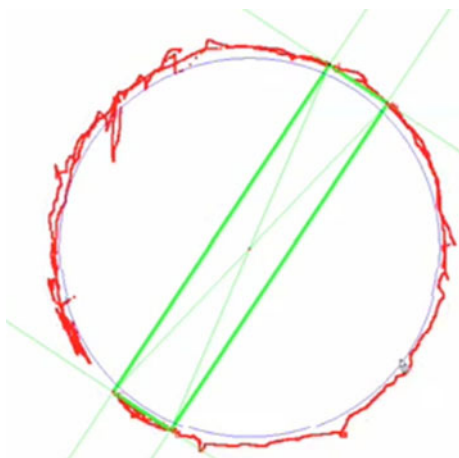


Fig. 14 Screenshot of the trace mark left during MD. The students used this trace to try to “fit” a circle they constructed



dragged point. Below is the excerpt that shows what happened next interlaced with brief analyses.

- 1 Ila: passing through...through...oh my goodness! [They constructed a circle with center in the intersection of the diagonals of ABCD and passing through the dragged point.]
- 2 Val: no.
- 3 Ila: Yes, but make it go through, eh, it isn't...
- 4 Ila: I mean you have to ...
- 5 Val: Do “control Z” [the trace mark disappeared]
- 6 Ila: Nooo!
- 7 Val: But ok, it doesn't matter!
- 8 Ila: Circle...
- 9 Ila: Good! We have seen that it follows.
- 10 Val: Yes, this is the trace...in brief.
- 11 Ila: But...wait, because there ... there are points.
- 12 Val: what points do you have to make?
- 13 Ila: Well,...oh dear! No.
- 14 Val: Wait.
- 15 Ila: I am tracing now...
- 16 Val: Yes

Brief analysis 1 to 16: The solvers constructed a circle that is dependent on the point being dragged and seemed to be trying to compare it to the trace mark that they had obtained through MD. The solvers did not seem to be convinced of what they had found.

- 17 Val: Move A on the circle.
- 18 Ila: Eh!
- 19 Ila: You look to check that it stays...
- 20 Val: There, it remains, it remains a parallelogram.
- 21 Val: Yes, I mean a parallelo...it remains a rectangle.
- 22 Ila: a rectangle.
- 23 Val: Yes, more or less.
- 24 Ila: Yes, ok.

Brief analysis 17 to 24: Val proposed to try to drag A “on the circle” even though the circle jiggled as A was being dragged. They seemed to notice that “A on circle” and “ABCD rectangle” occurred simultaneously.

- 25 **Ila:** But...
 26 **Val:** Ok....why?
 27 **Ila:** Because...
 28 **Val:** Why?
 29 **Val:** So...I know that, uh, so
 30 **Ila:** But B has to always be in that point there.
 31 **Val:** Where?
 32 **Val:** So I think...this remains a rectangle
 33 **Val:** ...when AB is perpendicular to DB, ok but in this case it would also be BA is equ, perpendicular to CA.

Brief analysis 25 to 33: However the solvers did not seem to be able to make sense of this.

- 34 **Ila:** Basically, uh, it's like we said before
 35 **Val:** and...
 36 **Ila:** No that basically uh DB has to always be parallel to CA, and, uh the segments AB, also AB, AB, no we had the points...
 37 **Ila:** Wait...this was fixed...these two were, right! I mean that CA and DB have to always be parallel, uh perpendicular to, uh...
 38 **Ila:** to the line, uh, parallel to DC.

Brief analysis 34 to 38: Ila tried to make sense of the behaviour of the figure, but she did not seem to be able to. Val then suggested different properties that she seemed to interpret as occurring simultaneously with respect to “ABCD rectangle”. Ila seemed to agree with Val. The solvers ended up “explaining” the exploration through a conjecture that linked one of these properties to “ABCD rectangle.”

The students seemed to be unable to discern “ABCD rectangle” and “A on circle” (17–28), as two level-1 invariants and to relate them through *fusion* and *recognition of the invariants as direct and indirect*, which would have led to the conception of a conditional relationship between them (a level-2 invariant). Instead, they finally proposed the following conjecture: “If CA and DB are parallel to DC, ABCD a rectangle.” We remark that the students did not seem to be certain about whether CA and BD should be parallel or perpendicular to DC (lines 37–38). What students could have observed was “CA and DB are always parallel and perpendicular to DC.” According to our model, different interpretations are possible to explain why the students failed to produce a conjecture based on the exploration with MD.

7.1 Interpretation one (unable to discern level-1 invariants)

Although the circle had been *separated out* and even constructed by fitting a traced path, possibly the students did not discern “A on the circle” as a level-1 invariant because the circle they constructed was not independent from the dragged point A and therefore it *moved* as they dragged A. Hence the circle the students constructed was not a drag-along path. The fact that critical aspects appear to co-vary may not have allowed the students to discern the level-1 invariant through *contrast* and *separation*. This made it impossible for the students to discern the expected level-2 invariant relating the level-1 invariants “A on the circle” and

“ABCD rectangle”, since one of them had not been discerned. The only properties the students were able to relate to “ABCD rectangle” are properties that they knew a quadrilateral must have in order to be a rectangle. In this sense, the students proposed the properties “AB perpendicular to DB”, “BA perpendicular to CA”, “CA and DB have to always be parallel and perpendicular/parallel to DC” because of their prior known conceptual understanding (or misunderstanding) with the interesting invariant “ABCD rectangle” but not because they discerned any of them as a level-1 invariant. In this respect, the conjecture produced by the students cannot be considered as generated by a level-2 invariant that could be discerned from exploration of the figure: Based on the exploration, the two properties “CA and DB have to always be parallel and perpendicular/parallel to DC,” and “ABCD rectangle” could have possibly only be put together but with no conditional relationship because they appeared and disappeared simultaneously since they were both indirect invariants. In other words, even if the students had been aware of the type of control on the two invariants, this could have led only to the discernment of two simultaneous indirect invariants, and therefore, no conditional relationship could be established between them (hence, no level-2 invariant could be discerned).

7.2 Interpretation two (unable to discern the type of control)

The students could have discerned “A on the circle” as a level-1 invariant, but they may not have developed awareness of the type of control on invariants in DGE and therefore not have been able to discern “A on circle implies ABCD rectangle” as a level-2 invariant. The students did not seem to interpret the level-1 invariant “A on the circle” as a *cause* for the maintaining of the other discerned level-1 invariant “ABCD rectangle.” This interpretation may explain the students’ insistence on asking “why” (lines 26, 28). Even though this question might have arisen out of surprise as to “why” a circle could be discerned through dragging, it could also refer to the meta-level of “why dragging along a path” would guarantee the maintaining of a level-1 invariant. Moreover, the students’ inability to construct a circle that remained fixed while A was being dragged denotes that they were not aware of the hierarchy of dependencies of elements from one another in dynamic figures. This, most likely, is indicative of the students’ inability to distinguish between direct and indirect invariants and thus to discern through perception the level-2 invariant.

The impasse shown in this example seems to be related not only to a failure in discernment but also to a partial or missing *geometric interpretation* of what is discerned during the exploration. In other words, a fundamental component of the exploring–conjecturing process is the relationship that needs to be established between geometry and what is experienced and perceived in DGE. Such a relationship is established not all at once, but it can be achieved through a cyclic process where students’ geometrical knowledge leads them to interpret what is perceived during the dragging and drives their attention to specific aspects of what occurs on the screen and that is then discerned. The complexity of this process is evident in our second example; the students were never able to make a conceptual leap using geometrical knowledge to interpret what they perceived and seemed to discern; for example, they were not able to interpret “A on the circle” as a geometrical property that linked A to the other base points of the figure. This could have led them to a robust construction of the circle that described the path geometrically. Such a construction is possible only if the students are able to use their geometrical knowledge to relate, at a theoretical level, elements and properties of the constructed figure. At that point, a successive redefinition of the dragged point on the circle would have led to a robust level-2 invariant, which may have been easier to perceive.

In summary, the model we propose introduces two intertwined components in making geometrical sense of an exploration: (1) a two-level discernment process, describing perception of moving figures, combined with (2) the geometrical interpretation of what is perceived which in turn influences what is attended to and discerned.

8 Concluding discussion

As Lopez-Real and Leung (2006) suggested, if dragging in DGE is accepted as a tool that can bring about structures and patterns, then "...we have new 'rules of the game', or even a new game, for experiencing geometry." (p. 676) In spite of the relationship between the realm of the software and the realm of Euclidean geometry—a relationship that is intrinsic in the design of DGE (Laborde & Laborde, 1995)—a fruitful geometric exploration must take into account experiential aspects that do not have immediate conceptual counterparts in the realm of Euclidean geometry. Previous discussion highlighted the complexity of geometrical explorations finalized to produce a conjecture, as the complexity of discerning phenomena within DGE and making sense of such discernment within the axiomatic world of Euclidean geometry. A fundamental design feature of most DGE is that all elements of a dynamic figure that depend on a given base point move when such base point is dragged, and they move in a way such that the properties defined by the construction are maintained. This basic design feature makes dragging become a powerful epistemic tool supporting geometrical reasoning and, specifically, a tool able to produce geometrical conjectures. Such an epistemic power is based on an assumption, usually implicit and almost unconscious for experts, that relates the realm of DGE and the axiomatic world of Euclidean geometry. Trying to make explicit such assumption we state it as the following *dragging exploration principle*:

During dragging, a figure maintains all the properties according to which it was constructed and all the consequences that the construction properties entail within the axiomatic world of Euclidean geometry.

This principle lies at the heart of design of DGE for which our model is suitable. It implicitly embraces variation, invariants (soft and robust), and potential awareness of sensorimotor perception, and it explicitly relates the dynamics of figures in DGE to geometrical properties in the theoretical world of geometry. In our model, level-1 invariants may be discerned via variation strategies which are susceptible to the different types of control described, they may be put in a relationship of logical dependency (a level-2 invariant) leading to a conditional statement relating geometrical properties (a conjecture).

Thanks to the model that we developed, we were able to gain insight into explorations within DGE, analyzing in fine detail how discernment unfolds, leading a dragger to transform acting, and perceiving into their conceptual counterparts in Euclidean geometry. In particular, it takes into account the issue of time that is not explicit in the logical world of Euclidean geometry, but that is present in explorations in DGE and that plays a crucial role in discerning conditionality. Our model can describe the transition from a temporal sequence of phenomena in DGE to recognizing conditionality in Euclidean geometry. We showed how this could occur through the perception of causality given by awareness of the types of control (e.g., direct/indirect) plus simultaneity. Such perception led to overcoming the temptation of stating the invariants discovered in the order given by the temporal sequence.

The analysis of the episodes presented in the previous section shows how the establishment of conditional links seems to guide the transition from a dynamic to static view of the exploration. In producing a conditional statement with a premise and a conclusion (a

conjecture), previous research refers to a transition from a dynamic exploration to its “crystallization” in a statement (Boero, Garuti & Lemut, 1999). The focus is on a “temporal section” of an exploration (Boero et al., 1999; Boero, Garuti & Lemut, 2007):

a time section in a dynamic exploration of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the condition A, hence ‘if A, then B’. (Boero et al., 1999, p. 140)

As far as dragging explorations are concerned, identifying invariants may determine a temporal section in the exploration process, characterized by sensorimotor perception that can induce discernment of conditionality between them. In particular, when discussing the conditional link that is developed between the level-1 invariants, we introduced the hypothesis that the sensorimotor nature of the exploration may foster a sense of direct control over one of the two invariants that occur simultaneously on the screen. We suggested that the combination of simultaneity and a form of control over a certain invariant can help the dragger attribute to an invariant the status of ‘premise’ or ‘conclusion’ of a conjecture. The dragger may become aware of these types of discernment and use such awareness to discern level-2 invariants. We remark here that this sense of control should be genuine in all types of DGE for which the dragging principle introduced above holds. This means that specific experiences in DGE may guide the dragger’s transition from dynamic to static interpretation of the experienced phenomena, moving from a phenomenal world of experience ordered through time to the crystallized world of axiomatic Euclidean geometry ordered by logic.

In conclusion, the dragging exploration principle imbues DGE with an epistemic quality that is process-oriented and user-centered. In particular, it opens up a type of geometrical reasoning that could be distinct from deduction and induction and that possibly suggests a different type of pedagogical process (Baccaglini-Frank & Mariotti, 2011). Such process should bestow on the teaching and learning of geometry an explorative and experimental nature that is complementary to deductive and inductive approaches. Equipped with the dragging exploration principle, learners can search, via dragging, for reasonable explanations of dragging phenomena occurring in DGE that are consistent with the axiomatic world of Euclidean geometry.

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