

Proof and proving in the classroom: Dynamic Geometry Systems as tools of semiotic mediation

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The objective of this paper¹ is to discuss the didactic potential offered by the use of a Dynamic Geometry System (DGS) in introducing students to theoretical thinking and specifically to the practice of proof. Starting from a discussion about what constitutes the general objective in developing students' sense of proof, the notion of Theorem is introduced as a system consisting of a statement, a proof, and a theory within which such a proof make sense. The paper discusses how the use of Dynamic Geometry Systems may support the teacher's intent to achieve her educational goal. The Theory of Semiotic Mediation provides the main theoretical reference for describing and explaining the role of Dynamic Geometry Systems in fostering the development of students' sense of theorems. Examples will be presented drawn from teaching experiments involving 9th and 10th grade students, illustrating how semiotic potential may unfold in the solution of specific tasks. The didactical focus consists in exploiting the semiotic potential of a DGS, especially the use of Cabri tools and particular dragging modalities that may function as tools of semiotic mediation.

Keywords: semiotic potential; Dynamic Geometry System (DGS); Theory of Semiotic Mediation (TMS); dragging; conditional statement

Introduction

For a long time, proof and proving have been themes for debate not only in the mathematics education community, but also in the mathematics community (see, for instance, Thurston 1994; Hanna 1989a). In this respect, proof represents a very special case among school mathematics topics, like Geometry or Algebra. The objective of this paper is to discuss the didactic potential offered by the use of a Dynamic Geometry System (DGS) to introduce students to theoretical thinking, and in particular to the practice of proof. I believe there are some epistemological issues that must be clarified before addressing the issue of proving and proof from a didactical perspective. In particular, I will introduce the notion of Theorem, as a system of three interrelated components: a statement, a proof and a theory within which such a proof makes sense. Before presenting a possible pedagogical approach developed in a long term teaching experiment carried out in the classrooms and aimed at introducing students to a 'practice of Theorems', I will introduce the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti 2008). This theory provides the theoretical framework within which the teaching experiment was conceived, and it will provide the main elements for the analysis of the examples.

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Epistemological considerations on proof and proving

The debate about proof and proving among researchers has been very passionate, revealing sometimes great divergences. This makes the theme fascinating, but also demonstrates its complexity, and the difficulty of having a productive discussion. Dangerous misunderstandings are possible in the use of specific terms: first of all the term *proof*, for which we think that we have a shared understanding of the meaning much more than is the actual case. As clearly stated by Balacheff:

Currently the situation of our field of research is quite confusing, with profound differences in the ways to understand what is a mathematical proof within a teaching-learning *problématique* but differences which remain unstated. (Balacheff 2008, 501)

We are becoming aware of the need to make our epistemological assumptions (stance) explicit in our research work. Actually, this point is crucial in order to foster mutual understanding, and is needed to find a way of overcoming unfruitful divergence. However, the need for clarification becomes of the utmost importance in elaborating on research results in order to outline didactical implications for school practice and make them suitable for an extended scientific community, including teachers and curriculum developers.

At the risk of oversimplification, I claim that when approaching the issue of proof there are two opposing positions that may be referred to, two different epistemological, and perhaps also cultural, perspectives.

On the one hand, proof is meant as idiosyncratic, strictly dependent on the individual who produces it. The core of its meaning concerns the epistemic value of a statement (its truth or falsity) and the potential change in this value that may occur after a proof. A paradigmatic example of a study consistent with this perspective is offered by the research work developed by Harel and Sowder (1998), and the classification (taxonomy) provided by these authors is an exemplar of the results coming from this type of studies.

The authors provide an explicit definition of a proving process:

Proving is the process employed by an individual [...] to remove or create doubts about the truth of an assertion. The process of proving includes two sub-processes: ascertaining and persuading. (Harel and Sowder 1998, 241)

As the authors explain, ascertaining is the process an individual employs to remove her or his own doubts, whilst persuading is the process employed to remove others' doubts. The meaning of the term proof stems from that of proving: proof is the product of the proving process.

According to this definition, the term *proof* is largely referred to any kind of argument, and "need not connote mathematical proof" (Harel and Sowder 1996, 60).

Argumentation versus proof

From a different point of view, other researchers use a different and more restricted meaning for the term proof: proof meant as the product of the process of theoretical validation. Moreover, this characteristic makes proof a key element, a specific and not renounceable component of Mathematics as a theoretical discipline. Taking such

85 a stance leads to a clear distinction and opposition between proof and argumentation, and to considering a proof independent on any interpretation and factual verification of the statement /statements that are involved. This stance is held, among others, by Duval (1992–3; 2007), who claims that an epistemological and cognitive gap exists between these two processes, and stresses the pedagogical issue involved in overcoming such a gap. The core of Duval’s argument concerns the distance between semantic and theoretical level, stressing the independence of theoretical validity from the interpretation and the truth value that can be attributed to any statement involved.

90 The epistemological and cognitive analysis carried out by Duval leads him to warn us that, from an educational perspective, the main difficulty lies in the proximity of the two processes and in the danger of mistaking one for the other; thus there is need for a careful distinction between them, disvaluing argumentation as a possible obstacle to the developing of a sense of proof.

95 Of course there is a wide range of intermediate possibilities and variations in conceiving proof between the two extreme positions. However, it seems to me that it is difficult to disregard the theoretical perspective without losing a fundamental component of mathematical knowledge, which would be a serious loss for mathematics education (Mariotti 2006).

100 Returning to the analysis carried out by Duval, we find that the distinction between argumentation and proof is based on the interpretation of the different statements and on their semantic relationship, argumentation is strictly connected to the issue of understanding “why” (Dreyfus and Hadas 1996) a certain statement is true, usually having an explanatory function. However, this same explanatory function belongs also to proof, though not always such a function is fulfilled: Hanna (1989b) speaks of proof that proves and proof that explains. Thus, the complex relationship between argumentation and proof raises the following question:

105 What is the role of theoretical derivation in respect to understanding? If we take a theoretical perspective, what is the place of the explanatory function of proof?

110 Different answers are possible to these questions, however it seems reasonable to agree on the fact that the sense of mathematical proof cannot cut the links with the semantic plane where understanding comes from the interpretation of the statements involved. This becomes more evident if we consider the process of proving in relation to the practice of mathematicians: and specifically, to the whole process that leads one to *produce a new theorem*. The production of a new theorem consists of two main elements, producing a conjecture and producing its proof. It seems difficult to conceive a conjecture that produces a link of logical implication between two statements, without referring to the meanings involved in the formulations of such statements and in the relationships between them:

115 to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge. (Lolli 1999, 6)

120 Meanings and epistemic values, given by the interlocutors to the statements involved, have no theoretical status, nevertheless they play a key role in the conjecturing process,

so that it seems impossible to think of a practice of theorems without any reference to meanings.

In this respect, conjecturing constitutes a fundamental activity where argumentation and proof may be related, resolving the potential conflict between the two main functions of proof, validating within a theory and explaining with respect to interlocutors' conceptions.

Thus we can summarise the pedagogical issue of proof as follows: to find a way, a pedagogical/ didactical way, for developing the complex relationship between argumentation, with its goal of convincing and explaining, and proof, with its goal of theoretical validation. Eventually, this means exploiting the deep link between argumentation and proof, without forgetting to take account of their differences.

Dealing with the complexity of this issue sometimes requires the researcher to assume a position that may appear too rigid or oversimplified. Such simplifications might be reasonable for methodological reasons, but it is important that the researcher (and the teacher) does not miss the complexity of the problematique. In order to better clarify this point, I will present a theoretical construct that is consistent with the previous statement, and that in my view can help to frame the design and analysis of research studies on proof; the construct is the notion of Theorem introduced some years ago (Mariotti et al. 1997). Originally there was a need to provide a shared framework to describe commonalities between different research approaches to proof among some Italian research groups. Later, this notion was used and further elaborated in other studies (see for instance its use in the analysis of the relationship between conjecturing and proving and in the analysis of the proof by contradiction (Mariotti 2006; Pedemonte 2008; Antonini and Mariotti 2008; Mariotti and Antonini 2009).

A definition of Theorem

The definition of Theorem that we are going to introduce is motivated by the following remark. Current literature commonly refers to the issue of *proof*. As a matter of fact, as briefly discussed above, when we want to differentiate it from argumentation, it is impossible to grasp the sense of a *mathematical proof* without linking it to the two other elements involved: a *statement* and overall, a *theory*. Thus, proof is considered in itself, without any reference to the other key elements without which the idea of proof evaporates

In other words, we speak of a proof when there is a statement to which it provides validation, but also when there is a theoretical framework within which such support makes sense. Leaving statement and theory implicit is certainly comprehensible if we take the perspective of mathematicians, but it may become a serious flaw if we take an educational perspective and consider the students' point of view.

What may have become an automatic and unconscious implicit reference for experts, cannot be expected to be the same for novices. Grasping the specificity of the mathematical proof in respect of generic argumentation may be difficult and not spontaneous. Certainly, when students are concerned, the peculiarity of a *theoretical validation* perspective cannot be taken for granted, on the contrary what can be expected is the confusion between an *absolute* and a *theoretically-situated* truth,

corresponding to the two main functions of proof-explication and validation (for a full discussion see Mariotti 2006).

Hence, taking an educational perspective, and in order to express the contribution of each component involved, the following characterisation of Mathematical Theorem was introduced, where a proof is conceived as part of a system of elements:

The existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules are so intimately interrelated so that what characterises a Mathematical Theorem is the system of statement, proof and theory. (Mariotti et al. 1997, 182)

Traditionally in school practice, the reference to the Theory within which a proof make sense, is neglected and except for the case of Geometry, the theoretical context in which theorems are proved normally remains implicit. Consider, for instance, Calculus courses and textbooks, this is one of the few cases when proofs are provided, however the axiomatic reference system is largely left implicit, and a careful analysis shows that most of the proofs would rather be classified as argumentations.

Coming back to the definition of Mathematical Theorem, it is important to state that what, for brevity, is referred to as Theory, has a twofold component. On the one hand, Axioms, Definitions and already proved Theorems constitute the means of supporting the single steps of a proof; on the other hand, meta-theoretical rules assure the reliability of the specific way to accomplish this support, in other words, how the Axioms and Theorems, belonging to one Theory, can be used to validate a new statement.

Actually, as clearly pointed out by Sierpinska, acting at a meta-theoretical level constitutes the very essence of a theoretical perspective:

[T]heoretical thinking is not about techniques or procedure for well-defined actions, [...] theoretical thinking is reflective in that it does not take such techniques for granted but considers them always open to questioning and change. [...] Theoretical thinking asks not only, Is this statement true? but also What is the validity of our methods of verifying that it is true? Thus theoretical thinking always takes a distance towards its own results. [...] theoretical thinking is thinking where thought and its object belong to distinct planes of action. (Sierpinska 2005, 121–3)

In the school context, the complexity of this meta theoretical level seems to be ignored. It is commonly taken for granted that students' ways of reasoning are spontaneously adaptable to the sophisticated functioning of a theoretical system. Thus, not much is said about it, and deduction rules and their function in the development of a Theory are rarely made explicit². As a matter of fact, there are at least two aspects of acting at a meta level that need to be made explicit. One is the acceptability of some specific deductive means, the other in the fact that no other means, except those explicitly shared, is acceptable.

If these two aspects are left implicit, it may prevent students having any access to control over their arguments. The control remains entirely in the hands of the teacher, resulting for students in a general feeling of confusion, uncertainty and lack of understanding.

Students' introduction to a theoretical perspective: examples from a teaching experiment

As pointed out by different authors (Jones, Gutiérrez and Mariotti 2000; Laborde 2000) a DGS may offer powerful resources for designing situations for enhancing students' learning of proof:

[...] the findings concerning the failure to teach proofs, the recognition of the multiple aspects of proving, and the existence of DG tools, lead naturally to the design of investigative situations in which DG tools may foster these multiple aspects. (Hadas et al. 2000, 130)

The example that I am going to present concerns a long term teaching experiment at the upper secondary level centred on the use of a particular DGS, Cabri Géomètre (Laborde and Bellemain 1995) aiming to initiate and develop a culture of Theorems within what we can call the field of experience of Cabri Construction (Mariotti 2000).

The main objective was that of exploiting the didactical potential offered by a DGS, and in particular the didactical potential offered by the use of the dragging function with regard to meanings related to the notion of Theorem presented above.

In order to better explain the principles that inspired the design of the teaching experiment I need to clarify some elements of the theoretical framework within which the didactical potential of Cabri can be described and its utilisation in classroom activities explained. Firstly I shall give an account of the Theory of Semiotic Mediation (TSM) (Bartolini Bussi and Mariotti 2008), then present our instructional experience.

The theoretical framework: the Theory of Semiotic Mediation

Taking quite a broad perspective, TSM aims to describe how the use of specific tools can be exploited by the teacher to foster the teaching-learning process; it combines a semiotic and an educational perspective, and elaborates on Vygotskij's notion of semiotic mediation (Vygotskij 1978), considering the crucial role of *human mediation* (Kozulin 2003, p. 19) in the teaching-learning process. Interpreting the teaching-learning process from a semiotic perspective means recognising the central role of semiotic processes and describing teaching-learning as an evolution (transformation in a given direction) of signs. According to the semiotic approach developed by other researchers (Radford 2003, Arzarello 2006) and inspired by C. S. Pierce, we use the term sign consistently with the idea of the indissoluble relationship between signified and signifier, and the idea that meaning originates in the intricate interplay of signs (Bartolini Bussi and Mariotti 2008); for a thoughtful discussion see also (Sfard 2000, 42 and following).

Following the distinction introduced by Rabardel (1995) we use the term artefact in order to distinguish between the tool itself and the specific way of using that tool in order to accomplish a particular task. TSM focuses on the production of signs, as they originate in relation to personal meanings emerging from the use of a specific artefact, and on the process of transformation of such signs through social interaction in the class community. When using an artefact for accomplishing a task, students' personal meanings emerge. Such meanings can be related to mathematical

meanings, but establishing such a relationship is not a spontaneous process for students. On the contrary, it can be assumed as an explicit educational aim by the teacher, who can *intentionally* orient her/his own action towards promoting the evolution from personal meanings towards mathematical meanings. Evolution may occur through social interaction, so that in a mathematics class context, signs produced by students and expressing the relationship between the artefact and the task in which it is used, can evolve into signs expressing the relationship between the artefact and mathematical knowledge (Figure 1).

How to organise and foster this semiotic process has been the crucial issue of our studies, based on long term teaching experiments (see for instance, Mariotti et al. 1997; Bartolini Bussi 1996; Bartolini Bussi et al. 1999; Mariotti 2000; Mariotti and Cerulli 2001), from which the theoretical frame originated and developed around two key elements: the notion of semiotic potential of an artefact and the notion of didactic cycle (for a full discussion see Bartolini Bussi and Mariotti 2008; Mariotti and Maracci 2010). Here, I will focus on the notion of semiotic potential that will frame the discussion of our example.

The notion of semiotic potential

As said above, the use of an artefact to accomplish a particular task may evoke a specific mathematical knowledge (Hoyle 1993). In fact, going beyond the immediate sense of its use, an expert – a mathematician, and in particular a teacher – may recognise mathematical ideas when solving a specific problem with an artefact. For example, positional notation and the polynomial notation of numbers may be evoked by an abacus and by its use in counting or making additions; similarly, as we will see in the following, constructing a stable figure in a DGS may evoke classic “rule and compass” Geometry. In the frame of TSM, the following definition is given:

A double relationship may occur between an artefact and on the one hand the personal meanings emerging from its use to accomplish a task (instrumented activity), and on the other hand the mathematical meanings evoked by its use and recognizable as Mathematics by an expert. We defined this double semiotic link as the *semiotic potential* of an artefact. (Bartolini Bussi and Mariotti 2008, 754)

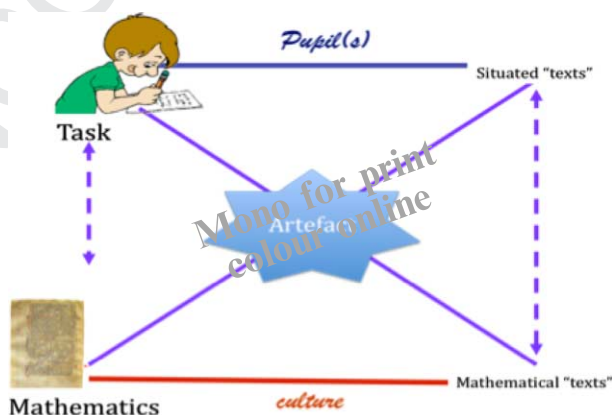


Figure 1. The Theory of Semiotic Mediation (TSM) model for the teaching-learning process.

Such a definition aims to make explicit the twofold relationship that may link an artefact to both the plane of individuals and that of culture, to personal meanings and to mathematical meanings. At the same time it aims to draw attention to the need for a clear distinction between what is related to the use of an artefact, and for this very reason related to the individual accomplishing a task, and what is related to mathematical content as cultural attainment.

The notion of semiotic potential captures the idea that an artefact may be used not only by the student to accomplish a task, but also by the teacher as a vehicle for learning: once the use of a certain artefact is introduced in the classroom, the teacher may exploit its semiotic potential to foster students' mathematical learning, in other words, the artefact can be used by the teacher as *tool of semiotic mediation*.

The analysis of the semiotic potential is to be considered an a priori phase that constitutes the core of any teaching sequence design. Exploiting the semiotic potential of an artefact requires that the expert (for instance, the teacher) is aware of its potentialities, both in terms of evoked mathematical meanings and in terms of emergent personal meanings during the activity in the classroom. On the one hand, it involves the orchestration of didactic situations where students face designed tasks that are expected to mobilise specific schemes of utilisation, and consequently, situations in which students are expected to generate personal meanings. On the other hand, it involves the orchestration of social interactions during collective activities, where the teacher has a key role in fostering the semiotic process required to help personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives (for a discussion on the teacher's role see Bartolini Bussi 1998; Mariotti 2009; Mariotti and Maracci 2010).

Within the frame of the TSM, I am going to discuss how a DGS offers a good context for approaching proof. I will focus on the description of the semiotic potential offered by a DGS, that is on how some of its tools and modes of use may originate specific meanings that evoke significant mathematical meanings related to the notion of Theorem as a system of statement, proof and theory.

Geometrical construction and the semiotic potential of a DGS

Let us start from the core of the analysis that lies in the relationship, immediately evoked in the mind of any mathematician, between Cabri figures and geometrical constructions. Such relationships can be elaborated, from the point of view of semiotic mediation, through both an epistemological and a cognitive perspective. That leads to outlining the semiotic potential of the artefact Cabri in respect of mathematical meanings related to Theorem.

Construction problems constitute the core Euclidean Geometry: for instance, the first proposition, proved in the Elements (Book I, Prop. 1), concerns the construction of an equilateral triangle given one of its sides. The theoretical nature of a geometrical construction is clearly stated, in spite of the apparent practical objective, i.e. the execution of a drawing following the construction procedure (Mariotti 2007). As Vinner clearly points out:

The ancient Greek undertook a challenge which in a way represents some of the most typical features of pure mathematics as an abstract discipline. It is not related to any practical need. (Vinner 1999, 77)

Actually, the use of specific artefacts, the ruler and the compasses, can be considered at the origin of the set of axioms defining the theoretical system of Euclid's Elements, and any *geometrical construction* corresponds to a theorem that validates the construction procedure that solves the corresponding construction problem.

After the advent of Dynamic Geometry Systems, geometrical constructions have been enjoying a new revival: one can simulate the use of ruler and compasses of classic Geometry through the use of virtual tools that produce lines and circles on the screen. Consider a Dynamic Geometry System (DGS) like Cabri Géomètre (Laborde and Bellemain 1995): a sub-set of the tools available in a Cabri menu can be related to a corresponding set of construction tools in Euclidean Geometry (Laborde and Laborde 1991).

The set of tools available in the Cabri menu in this respect corresponds with classic Geometry theory. Let us elaborate further on this correspondence.

Compare the classic world of paper and pencil and the new world of Cabri figures: the novelty of a DGS consists in the fact that screen drawings can be acted upon, using the dragging modality. Dragging the basic point from which the construction is initiated, the whole drawing is transformed, however all the properties defined by the construction procedure are maintained, that is remain invariant. Moreover, any property that is a consequence of the constructed properties is also maintained, e.g. invariance. This functioning is the cause of the perceptual permanence of those properties that make the user recognise the movement of the same figure, and the reason why the stability of the drawn figure, with respect to dragging, constitutes the standard test of correctness for any construction task in a DGS.

Although they appear simultaneously, the invariant properties of any DG figure are related according to a hierarchy: some of them are defined by the commands used in the construction procedure, others are consequences of them. Such a hierarchy of properties corresponds to a relationship of logical dependence between them, thus any construction has a counterpart in the statement of a Theorem. The properties stated by the procedure's commands constitute the premise of the statement and those which are their consequences constitute its conclusion, while the simultaneity of the two kinds of invariants corresponds to the logical link between the premise and the conclusion. The stability of the figure obtained has a counterpart in the fact that there must be a proof validating the procedure with respect to the theory that, as said above, may be put in correspondence with the specific Cabri tools available in the menu. This correspondence allows perceptual control by dragging to be put into a relationship with theorems and definitions within the theory of Euclidean Geometry (Hölzl et al. 1994; Mariotti 2000; Jones 2000)³.

In summary, as far as the tools available in a DGS, like Cabri, are concerned, a double relationship is recognisable that may constitute its semiotic potential. On the one hand, DGS tools are related to the construction task that can be solved through them, resulting in the appearance of a drawing on the screen, and to the stability of such a drawing by dragging; on the other hand, specific DGS tools can be related to geometrical axioms and theorems that constitute the Geometry theory

within which the solution of the corresponding construction problem must be proved. Thus, the semiotic potential of a DGS resides in the twofold relation that its tools and their functioning have with the mathematical meanings related to the notion of Theorem: such a meaning emerges from the use of the virtual drawing tools for solving construction problems, controlled by the dragging test, and can be related to the theoretical meaning of geometrical construction as it is defined within Euclidean Geometry, i.e. it can be related to the meaning of Theorem.

Exploiting the semiotic potential in the classroom

According to the analysis developed above, and taking the pedagogical assumption of exploiting such a semiotic potential of a DGS (Cabri), a long term teaching experiment was designed and implemented for several years and in several classes.

The teaching sequence was designed according to the TSM (for details see Mariotti 2000; 2001), and the use of the artefact was centred on a specific type of task, the *construction task*, requiring students:

- to produce a specific Cabri figure that is stable under dragging, and write the description of the procedure used to obtain it;
- to produce a validation of the ‘correctness’ of the construction realised.

Such a construction task was thus composed of two requests, the former corresponding to acting with the artefact, the latter to producing a written text referring to such actions. Note that producing a written text consists in both describing and commenting on the procedure carried out. The request to *validate* the solution made sense with respect to the Cabri environment corresponding to the need explaining and getting insight into the reason why the figure on the screen passes the dragging test.

The teaching sequence had the structure of an iteration of *didactic cycles*. Each cycle started with the students’ solution of the task, and was followed by semiotic activities – individual and social – aimed at generating signs whose meaning evolved from a personal and contextual one to a mathematical one. According to the model provided by the TMS, evolution occurred through social interactions orchestrated by the teacher in true Mathematics Discussions (Bartolini Bussi 1998; Mariotti 2001). If, at the very beginning, the term *construction* made sense only in relation to using particular Cabri tools and to passing the dragging test; later on, the meaning slowly evolved, acquiring also a theoretical component. As the meaning of construction evolved, the practical/empirical meaning of producing a stable image on the screen gave way to the theoretical meaning of geometrical construction, expressing the need for a theoretical control of the construction procedure within the Geometry theory.

Such evolution was made possible by exploiting, with the guidance of a teacher, the semiotic potential of Cabri in respect to the different components of the notion of Theorem. First of all, let us consider the correspondence between Euclidean Axioms and specific Cabri tools and their modes of use. Starting from an empty menu, the choice of the first appropriate set of tools was discussed, as well the correspondence of such a set with a set of *construction axioms*. This set of axioms constituted the first core of the Geometry theory which any validation should

refer to. Then, as long as new constructions were produced, the corresponding statements were validated, and the corresponding theorems added to the theory. Students could experience and participate in two parallel processes: on the one hand, the enlargement of the Cabri menu with the addition of new tools, and on the other hand the corresponding development of a Geometry Theory with the addition of new theorems. We can say that students were introduced to the mathematical meaning of Theorem, because students not only produced new statements and their proofs, but also had the opportunity to become aware of the theory within which the proofs made sense, and of how such a theory was developing.

The results of longstanding teaching experiments attest the emergence of intermediate meanings, rooted in the semantic field of the artefact, and their evolution into mathematical meanings, consistent with Euclidean Geometry. In particular, the experience in the classroom highlighted how meanings emerging from the use of tools in the constrained world of Cabri were effective in developing and interweaving the senses of proof and theory. The use of any single tool mediates the meaning of the application of an axiom or an already accepted theorem, while the idea of a Cabri menu, that is the set of available tools, mediates the meaning of theory. In particular, key aspects of a theory, such as its conventionality and its evolutionary nature, clearly emerged during collective discussions where students experienced both establishing and developing a Geometry theory, through exploiting the possibility of personalising the menu by selecting the tools to be used.

I will say no more about this example because I have already discussed the results of these experiments on many occasions (Mariotti 2000, 2001, 2007, 2009). Rather, in the following section, I will widen discussion about the semiotic potential of the dragging tool in relation to the notion of statement. I will elaborate on the use of dragging to solve open problems, that is the use of dragging in the exploration of a specific configuration, with the aim of producing a conjecture, that is a conditional statement.

More on the semiotic potential of a DGS: dragging as a semiotic mediator of conditionality

The mathematics education community shares the opinion that solving open-ended problems plays a fundamental role not only in fostering conceptualisation, but particularly in developing students' sense of proof. Open-ended problems are characterised by raising uncertainty and conflicts, sometimes also surprise. In order to solve an open-ended problem, the conjectures produced are often accompanied and supported by explanations and validations articulated in spontaneous arguments. Thus, a rich and meaningful activity can be designed on which the teacher may base a didactic intervention aimed at fostering a productive relationship between spontaneous argumentation processes and theoretical validation (Arsac and Mante 1983; Arsac 1992; Hadas et al. 2000; Boero, Garuti and Lemut 2007; Pedemonte 2008). Let us consider open-ended problems in a DGS and analyse the potential offered not only in producing a solution, but in developing the meaning of conditional statements.

Solving open problems in a DGS

The term open-ended problem has often been used in mathematics education literature (see for instance Arsac and Mante 1983; Silver 1995). Usually, it refers to a task that puts a question without revealing or suggesting the expected answer. When the solver is explicitly asked to formulate a conjecture, we speak of conjecture open problems. This is a very common case in Geometry, when within a specific situation, corresponding to a well-defined geometrical configuration, the solver is asked to make a conjecture that usually assumes the form of a conditional statement between possible properties of the given configuration. When the solution is searched within a DGS, like Cabri, the solver is expected to explore the configuration dynamically. This means that the solver has to interpret perceptual data that comes from observing the screen while the figure is being dragged, and transform them into a geometrical statement. In other words, a process of exploration is productive if the solver is able to *transform perceptual data into a conditional relationship between geometrical properties*.

Other studies concerning open problems within other contexts (see for instance, the experiments of Boero and colleagues (Boero and Garuti 1994), in the field of experience of sun shadows, highlighted the crucial role played by the dynamicity of the investigation process that seemed to foster transformational mental processes (Simon 1996; Harel and Sawder 1998) and provide key elements to produce conditional statements. Actually, as reported Boero and his colleagues, the conditional statement seemed to emerge as “crystallisation” of a dynamic exploration. Such a crystallisation was described as composed by a complex process that culminate in isolating a specific moment in the dynamic flux of the phenomenon when the occurrence of a fact has the occurrence of another fact as consequence. The formulation of a conjecture expresses such a crystallisation in a conditional statement (Boero and Garuti 1994; Boero, Garuti and Lemut 2007). Given such a key role of dynamicity, it seems reasonable to address the issue of conjecture production through dynamic explorations in a DGS.

Thus, in the broader perspective of the solution of open problems, let us consider the specific request of formulating a conjecture concerning a specific configuration in a DGS.

The seminal work of Arzarello, Olivero and colleagues (Arzarello et al. 1998, 2002; Olivero 2001, 2002) as well that of other researchers (Hölzl 1996; Leung and Lopez-Real 2002; Lopez-Real and Leung 2006; Leung 2008; Healy 2000), has shown the potentialities of different dragging modalities in supporting the conjecturing process. Starting from their results, I want to discuss the potential offered by a DGS not only in supporting and fostering conjecturing processes, but also in mediating the mathematical meaning of *conjecture* and particularly in mediating the meaning of *conditional statement* in the Geometry context. In the frame of the TSM, I will outline the semiotic potential of particular modalities of dragging with respect to the notion of conjecture as a conditional statement. Dragging modalities can be considered as specific artefacts used to solve an open problem. Meanings emerging from their use may be related to the mathematical meaning of a conditional statement that expresses the logical dependency between a *premise* and a *conclusion*.

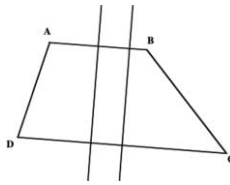


Figure 2. After the construction of the bisectors their parallelism appears.

Invariants by dragging and conditional statements

As already said, when a figure is acted upon by dragging its basic points, the properties stated by the primitives used in the construction are maintained, but the same will happen also to all those properties that are a consequence of the construction properties within the theory of Euclidean Geometry (Laborde and Sträßer 1990). Consider the configuration obtained through the following steps: Construct:

- a quadrilateral ABCD, with AB parallel to DC;
- the perpendicular bisectors of AB and CD.

While dragging the Cabri figure, the constructed properties are preserved-side AB remains parallel to side CD and the each side remains perpendicular to its bisector-but a new invariant clearly emerges: the parallelism between the two bisectors. This can lead us to formulate the following conjecture: “if two sides of the quadrilateral are parallel, then the corresponding perpendicular bisectors are parallel.”

In other words, the simultaneity of invariants that can be observed during the dragging corresponds to the general validity of a logical implication between specific properties of a geometrical figure. However, because of their simultaneity, the hierarchy between the invariants of a certain configuration, and consequently, it may be difficult to identify the conditional link between the corresponding properties. The hierarchy between the invariants of the Cabri-figure is imposed on them by specific construction steps, but it may be difficult to keep control of the origin of the figure and identify the logical status of the different invariants: those referring to the single steps of the construction and those appearing as their consequence; the former will be the premise, the latter the conclusion of the conditional statement that expresses the conjecture.

However, during a dynamic exploration, only basic points – those from which the construction originates – can be selected and dragged, while the other elements move as a consequence of their motion. As a matter of fact, two different types of movement occur:

- a *direct motion*, that is the variation of an element in the plane under the direct control of the mouse; and
- an *indirect motion*, that is the variation of any other element as a consequence of direct motion.

Thus, there is the possibility of discriminating between the two movements and enactively perceiving motion dependency. In other words, the a-symmetry of such movements induces an a-symmetry between the two kinds of invariants, and allows the solver to distinguish between *direct invariants* and *indirect invariants*.

Furthermore, dynamic dependency between invariants may be interpreted by the solver in terms of logical consequence between geometrical properties, eventually expressing it as a conditional statement between a premise and a conclusion.

Results coming from previous studies on conjecturing in a DGS can be reinterpreted through the distinction between direct and indirect movements, and leads us to reconsider one specific modality of dragging, previously described as *Dummy locus dragging* (or *Lieu muet dragging*) (Arzarello et al. 1998; Arzarello et al. 2002; Olivero 2002). Such a modality consists of realising a constrained movement of a configuration by acting on one of the basic points with the intention of maintaining a specific property, that is in order to make a specific property become ‘invariant’. The constrained movement corresponds to the logical action of adding a new hypothesis to the properties assumed by construction, while the new invariant, an *Indirectly Induced Invariant*, corresponds to a consequence of all of them.

We call this modality of exploration Maintaining Dragging (MD) (Baccaglini-Frank 2010). Mathematicians often say “exploring under which condition ... a certain property occurs”.

According to the previous analysis using MD to solve a conjecture problem, it is possible to distinguish clearly between the premise and the conclusion of the conditional statement that is the outcome of the exploration as the student can carry it out. The student can directly and intentionally control such a distinction during the exploration: the conclusion will be the property that the solver decides to maintain, the premise will be the property corresponding to the constrained movement.

In summary, as far as dragging for conjecturing is concerned, there are two main modalities characterised by the intention with which the solver selects and searches for different types of invariants:

- free dragging: looking for indirect invariants as consequences of the direct invariants;
- constrained dragging: looking for possible construction invariants that may cause a specific indirect intentional invariant to happen.

The simultaneity of invariants combined with the control of the different status of each kind of invariant is the counterpart of the logical dependency between a premise, corresponding with the direct invariants, and a conclusion, corresponding with the indirect invariants (intentionally induced or not).

Taking the perspective of semiotic mediation, we claim that the different dragging modalities, together with the different types of invariants that originate from their use in solving a conjecture-production task, offer a rich semiotic potential in respect of the notion of conditional statement as the logical link between a premise and a conclusion. The semiotic potential is recognisable in the following relationships between:

- *direct and indirect invariants* and respectively *premise and conclusion* of a *conditional statement*;
- the *dynamic sensation of dependence between the two types of invariants* and the geometrical meaning of *logical dependence between premise and conclusion*.

A study, carried out at the upper secondary level (Baccaglini-Frank 2010), focused on the analysis of the process of exploration that can be expected when using the

MD. The results of this study, and in particular a model of the process of conjecture-generation based on the use of dragging and MD, have been presented and discussed in several papers to which I refer for a detailed discussion (Baccaglini-Frank 2010; Baccaglini-Frank and Mariotti 2009; Baccaglini-Frank et al. 2009; Baccaglini-Frank and Mariotti 2010). In the following I will present some examples aimed at illustrating the unfolding of the semiotic potential described above.

Some results from the teaching experiment

The teaching experiment involved students from three Italian high schools (Ita. licei scientifici) between the ages of 15 and 18, who had been using Cabri in the classroom for at least one year prior to this study. The experiment consisted of two phases. The first was the teaching phase, in which students were introduced to the different modalities of dragging, and specifically to MD; the second phase was an observation phase, in which students were interviewed in pairs, while solving conjecture open problems (Baccaglini-Frank 2010).

Results from the study have shown the unfolding of the semiotic potential of dragging and in particular of MD. Actually, it was possible to observe the emergence of different meanings related to use of the dragging tool that could be related to the notion of conjecture as a conditional statement relating a premise and a conclusion. We observed how invariant properties appearing in the exploration assumed a specific logical status according to their status in the dragging activity, and how they could be related logically to formulate the conditional statement. In the examples we will consider the following problem.

Problem 2

Draw a point P and a line r through P .

Construct the perpendicular to r through P and construct a point C on this line.

Construct the symmetrical image of C with respect to P and call it A .

r separates the plane in two semi-planes. Choose a point D on the semi-plane that contains A .

Construct the line through D and P .

Construct the circle with centre in C and radius CP .

Let B be the second intersection of the circle with the line through D and P .

Consider the quadrilateral $ABCD$.

Make conjectures about the types of quadrilaterals it can become, trying to describe *all the ways* in which it can become a particular type of quadrilateral.

From dragging to conjecture: the emergence of the meanings of conditional statements

In this section, I will present three examples, drawn from the study mentioned above. The aim is to illustrate what can be observed during explorations by dragging and its interpretation according to previous semiotic analysis as the emergence of meanings that can be related to the notion of conditional statement.

The first example presents the very beginning of an exploration, and shows both how such exploration is oriented by students' awareness of direct and indirect motion and how it depends on the properties that are stated by construction.

The excerpt (drawn from the data reported and discussed by (Baccaglini-Frank 2010, 267)) clearly shows how during an exploration, while trying to move the different points of a construction, the solvers may identify direct and indirect movements, together with idea of dependence and independence (of points). Both referring to the construction procedure and to the dependency/independency by dragging, the status of invariants is slowly revealed. The two students, Giu & Ste, are solving Problem 2.

Excerpt 1

[1] Giu: So D is independent and it stays on its own ...

[2] Ste: ... however ... yes

[3] Giu: Yes ... exactly.

[4] Ste: A depends ... [they speak together]

...

[8] Giu: A is dependent on C because it is at the same distance so it remains like that.

[9] Ste: It's an axial symmetry, so I can't do anything about it.

[10] Giu: Good for you.

[11] Giu: uh, B is dependent both from C and from D, right?

[12] Ste: But can I move it? ... no

[13] Giu: Of course not!! [they laugh]

[14] Ste: right, actually ...

[15] Giu: Because if you move C ...

...

[23] Giu: it moves ... so B is dependent from D and from C ...

[24] Ste: from C and from D.

[25] Giu: from C and D exactly. [Ste takes the mouse back]

[26] Giu: Therefore we need to find the way to ... so the possible conditions are [on] C and D, because only moving C and D we can have something that changes.

[27] Ste: right.

[28] Giu: Otherwise ...

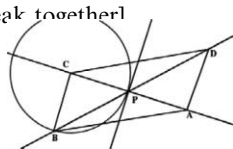
[29] Ste: right, actually here I can't do ... oh no, I can ...

[30] Giu: eh, I can chaaange [he drags P] yes

[31] Ste: Yes, because that point too can move this line here, so ...

[32] Giu: But it is like changing D.

The solvers note the dependency and independency of different points.



They refer to the steps of the construction and affirm that A is dependent and for this reason it cannot be dragged.

Similarly, they seem unsure as to whether B is dependent from C as well or not, so they test it by dragging C.

They refer to the construction of B and also conclude that B is "dependent" and therefore not draggable.

They observe that C does influence B, and decide that the points that influence the behaviour (in order to "have something that changes" [26]) of the Cabri-figure are C and D.

They finally notice that P is also an independent (base) point, but decide not to use it for the exploration, since they see dragging it as "like changing D".

The solvers note the dependency and independency of different points ([1]–[4]). The solvers carry out a careful analysis of the different points. Dependency and independency are related to the possibility or impossibility of direct dragging, but also to the fact of being a base point or a constructed point.

Such a preliminary analysis, concerning direct and indirect movements, independency and dependency between the movements, establishes a relationship between changes (and invariants) in the dragging. Such a relationship of dependence between movements can be related to the logical order of a conditional statement, the order between what will be the premise and what will be the conclusion. The students show they are aware of the status of the different points, and of the consequences of their movements, but that specific exploration modalities can foster the appearance of specific properties.

The second example aims to show how the students control the use of different modalities of dragging, and how they relate such modalities to the status of invariant properties, while the emergence of a conditional relationship is realised when dragging is maintained. F and G are solving Problem 2. Student F is performing the dragging

Excerpt 2

[1] F: wait, it is a ... let's try to for example make it become a parallelogram.

[2] G: No ... yes, go.

[3] F: Like this.

[4] G: So, for it to be a parallelogram ... I think it always is a parallelogram.

[5] F: Let's try?

[6] G: No, no, there, it's a parallelogram ...

[7] F: because like this it's ...

[8] G: I understand! so, C ...

...

[12] G: therefore it's enough that PB is equal to PD.

After a first exploration where students, dragging freely on the screen, searched and found invariants related to the construction, F proposes using the Maintaining Dragging (MD) modality, starting by trying to make ABCD a parallelogram ([1]). His initial dragging suggests to G, only for an instant, that the property of being a parallelogram might be construction-invariant: he says: "I think it always is ABCD parallelogram" ([4]), using the adverb "always" to express the generality of the property. F proposes a check and starts to drag freely; further movement of the base point leads the students to discard such a hypothesis ([5]). However, the possibility of obtaining a parallelogram in a specific configuration is recognised as possible ("there, it is a parallelogram ...") and G seems to identify a specific condition to obtain a parallelogram related to the movement of C (he exclaims: "I understand!" In Italian: "ho capito!" [8]), and eventually a condition is identified for obtaining a parallelogram.

This short exchange shows how students may relate the different status of invariant properties to different modalities of dragging: free dragging is related to properties that are always true ("it is always ..."), constrained dragging ("make it become ...") is related to the realisation of a specific conditions ("therefore it's enough that ...").

As discussed above, during the exploration of the figure by using the dragging mode, we can observe the emergence of a relationship between a cause and a consequence: that is, a relationship between a specific movement and the maintained invariant. Such a relationship can be referred to the meaning of conditionality as the relationship between a premise (the cause) and a conclusion (the consequence).

The two students are familiar with using the mouse, and seem to have no difficulties with carrying out MD, easily controlling the status of the different invariants.

This may not be the case for other students who get into trouble with the realisation of MD. However, this may turn out to be an advantage for our purpose, rather than an obstacle. Actually the difficulty in maintaining a certain property invariant can require specific attention, and consequently s the emergence of the meaning of dependence. As is shown in the following excerpt, after the difficulty of realising the MD, and when finally simultaneity is achieved, the distinction between what is the cause and what is the consequence becomes even more evident.

Excerpt 3

[5] Isa: I am trying to make a parallelogram, uh, to put two sides parallel.

[6] Isa: and so AD and BC.

[7] I: uhm [Isa is dragging but the property is not maintained, the interviewer expresses perplexity]

[8] Isa: So now I need to go back a second ... no, no, no, no ... [Isa is aware of her failure]

[9] I: eh, it's hard when you go close...

[10] Isa: alright, anyway, here it should ... how nice! ... be here. [Isa seems happy about the success]

[11] Isa: There ...

[12] I: Let's continue over here...

[13] Isa: So, like this ... uhm ... here it becomes easier ... There, more or less

[14] I: uhm.

[15] Isa: So, let's see to try it. So, if I construct, uh, if I move D on a circle with centre in A, and, theoretically, radius AP ... [Isa construct the circle and moves D on it]

...

[17] Isa: ... I find that the quadrilateral is a parallelogram, ...

After deciding to use MD to maintain the parallelogram, Isa experiences difficulties ([7]–[8]) in realising the conditions in which the quadrilateral will be a parallelogram, thus to be sure of her conjecture she constructs a circle on which point D should move.

Actually, the need for certainty can lead the solver to change the status of the soft invariant, transforming the *cause* of the temporary maintenance into a *robust constructed property*. As a consequence of the new constructed properties, this change is expected to cause the Indirectly Induced Invariant to become a robust invariant too. Thus, in this transformation of the status of the invariants, the meaning of premise referring to constructed invariants may be related to the meaning of premise referring to a condition under which a certain property is maintained.

In summary, specific dragging modalities offer a rich semiotic potential. Meanings emerge from haptic (touch-based) experience of direct and indirect movements, and these can be related to the idea of dependency between invariant

properties that appear simultaneously on the screen. Thus, simultaneity combined with the control of direct and indirect movement leads to the different status of invariants; specifically, the two kinds of invariant can be characterised by referring to their specific status in the exploratory activity, and these characteristics make them clearly recognisable by the students. Such a web of meanings provides the base for developing the mathematical meanings of logical dependency between a *premise*, corresponding to construction invariants and a *conclusion*, corresponding to derived invariants, and generally speaking the mathematical meaning of *conditional statement*.

Conclusions

The important educational value of explorations using Dynamic Geometry environments has been widely claimed (Edwards 1999; Talmon and Yerushalmy 2004; Restrepo 2008), specifically for the development of a sense of proof (Schwartz and Yerushalmy 1987; Chazan 1993; de Villiers 1998; Hadas et al., 2000; Christou et al. 2004) However, the analysis developed in this paper and the examples discussed lead us a step further: not only may the use of Cabri tools and different dragging modalities lead students to solve construction tasks or conjecture-generation tasks successfully, but it may also offer teachers significant support with the difficult task of introducing students to a theoretical perspective, and specifically to the notion of Theorem.

As shown above, a rich web of meanings relating the different components of the notion of Theorem can emerge in using Cabri to solve both construction and conjecturing tasks. The semiotic potential of a DGS – of its single tools and its dragging modalities – can be exploited by the teacher as a tool of semiotic mediation, helping students develop mathematical meanings related to the notion of Theorem (statement, proof and theory), and in particular mathematical meaning related to the notion of conditional statement, such as expressing the logical dependency between premise and conclusion.

This evolution of personal meanings emerging from the activities with the artefact Cabri towards the mathematical meanings is neither spontaneous nor taken for granted, the role of the teacher becomes fundamental for orchestrating the semiotic process that take place during collective activities. Evidence has been found not only of the possibility of such evolution, but also of the modalities of teachers' interventions (Bartolini Bussi and Mariotti 2008; Mariotti 2009; Mariotti and Maracci 2010). Research is still in progress: teaching experiments have been designed according to the didactical cycle described in the Theory of Semiotic Mediation, with the aim of exploring the whole process. Focussing in particular on the actions of the teacher, we aim to better describe how the complex web of meanings emerging from activities with the artefact may be transformed into mathematical meanings.

Notes

1. An earlier version of this paper was presented at Seventh Congress of the European Society for Research in Mathematics Education (CERME7) in Rzeszów, Poland, in February 2011 (Mariotti 2011).

2. An exception is that of mathematical induction, which is explicitly treated, and to which a specific training is devoted. But, mathematical induction is very rarely presented in comparison with other modalities of proving, which are commonly considered natural and spontaneous ways of reasoning.
3. Actually a DGS provides a larger set of tools, including for instance ‘measure of an angle’, ‘rotation of an angle’ and the like. This implies that the whole set of possible constructions do not coincide with that attainable only with ruler and compass (see Stylianides and Stylianides 2005 for a full discussion).

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