

A SECONDARY SCHOOL STUDENT'S UNDERSTANDING OF THE CONCEPT OF FUNCTION – A CASE STUDY

ABSTRACT. This article is concerned with an average student's understanding of the notion of function. It presents an analysis of a dialogue held with Kasia, a sixteen-years-old pupil of a general academic secondary school. The conversation concentrates on a non-standard problem – a functional equation. The theoretical framework used here is the PROCEPT THEORY formulated by Gray and Tall (1993, 1994). The article diagnoses Kasia's procept of function revealed during the dialogue. Her difficulties with understanding this notion can be described as based on a very limited procept of function and a misinterpretation of the symbols used in the functional equation. Three kinds of sources of the difficulties are identified: firstly, the intrinsic ambiguities in the mathematical notation, secondly, the restricted context in which some symbols occur in teaching and a limited choice of mathematical tasks at school, and, thirdly, Kasia's idiosyncratic interpretation of mathematical tasks. Moreover, the paper describes several important changes in understanding of symbols and of the concept of function that took place in the course of the dialogue. This article shows that we can use functional equations as both a research and a didactic tool in mathematics education.

KEY WORDS: case study, difficulties, function, functional equation, procept, sources, symbolism, understanding

1. INTRODUCTION

Function is one of the basic concepts of mathematics, amazing in the diversity of its interpretations and representations. Much time and attention have been spent on it in the didactic process, yet it remains a difficult concept.

Students face many obstacles trying to understand functions. Among the most objective ones are epistemological obstacles identified and described by Sierpinska (1992). On the one hand, they are related to philosophy of mathematics and mathematical methods and various unconscious schemes of thinking; on the other hand, they are related to the concept of function and related terms (definition, number, variable, coordinates, graph of the function).

Another difficulty in students' understanding of the concept of function stems from its dual nature. Indeed, the function can be understood in two essentially different ways: structurally – as an object, and operationally –



as a process (Sfard, 1991). From the structural point of view, the function is a set of ordered pairs (Kuratowski and Mostowski, 1966, p. 73), and from the operational one it is a computational process or well defined “method for getting from one system to another” (Skemp, 1971, p. 246). Those two ways of understanding functions, although apparently ruling out one another, should, however, complete each other and constitute a coherent unity – like two sides of the same coin (Sfard, 1991).

Gray and Tall (1994) point out that the notation of function, for example, $f(x)=2x+3$ tells us two things at the same time – how to calculate the value of the function for particular arguments (and therefore evokes the process) and encapsulates the whole concept of function for any given argument (thus presenting the object). The notation of function is ambiguous in yet another way. Sierpiska (1992) emphasizes that flexibility in understanding is necessary because, for example, $f(x)$ represents both the name of a function and the value of the function f . Interpretation depends on the context, which can confuse a non-advanced student. Conditions for success or failure in teaching such a difficult concept have been investigated. One of them is undoubtedly the teacher’s thorough knowledge and understanding of their subject. The problem of in-service and pre-service teachers’ understanding of the concept of function has been researched, for example by Even (1990), who enumerates aspects of teachers’ subject knowledge about functions. Sierpiska (1992) formulated both an analysis of the epistemological obstacles accompanying an understanding of the concept of function, and conditions for understanding this concept. This cognitive analysis of the concept of function was followed by conclusions and suggestions concerning teaching.

Sfard (1991) analyzed the process of forming the concept of function from historical and psychological points of view. She claims that function is a concept which is first acquired operationally, and the transition to its structural form – the mathematical object – takes place in three stages: *interiorization, condensation and reification*.

On the other hand, Vinner (1983, 1991) presented a model of cognitive processes based on the relation between the definition of the concept and concept image. On the basis of this model, different categories of students’ definitions and concept images were singled out and compared both for grades 10 and 11 (Vinner, 1983) and for college students and junior high school teachers (Vinner and Dreyfus, 1989). Following the research some didactic conclusions were drawn. Moreover, Eisenberg presented certain phenomena in the process of learning and analyzed them from psychological and historical perspectives (1991).

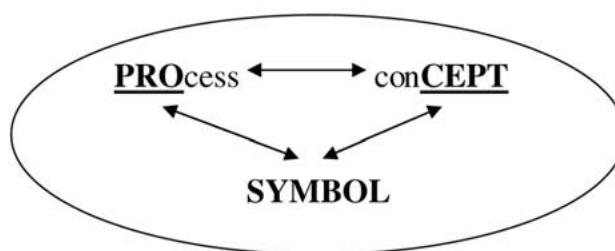


Figure 1. Elementary PROCEPT.

There have been numerous attempts to define levels of understanding of the concept by students. Such levels were singled out for example by Dyrszlag (1978), Bergeron and Herscovics (1982) and Vollrath (1984). Furthermore, certain categories of understanding and students' activities related to the concept of function were defined (Sierpiska, 1992).

In this paper, I would like to attempt to address the question: how is the concept of function understood by an average student and to what extent is a function for him or her a complete fully developed *mathematical object*? To be more precise, on the basis of the theory formulated by Gray and Tall (1993, 1994), I will ask the question: what is the likely procept of function held by such a student and to what extent does it differ from what the teachers would like the student to have?

2. THEORETICAL FRAMEWORK: THE PROCEPT THEORY

I will start with a brief outline of Gray and Tall's theory (1994). This theory posits a duality between process and concept in mathematics. One way in which this duality becomes apparent is that a single symbol is often used to represent both a process (such as the division of two numbers 1:2) and the product of that process (the fraction 1/2). The authors coined a new term: *procept*. In order to define *procept*, the authors introduced the term *elementary procept*. It consists of three components: **a process** that produces a mathematical **object (or concept)** and **a symbol** that represents either the process or the object (Gray and Tall, 1994) (see Figure 1).

On the other hand, PROCEPT consists of a set of *elementary procepts* standing for the same object (see Figure 2).

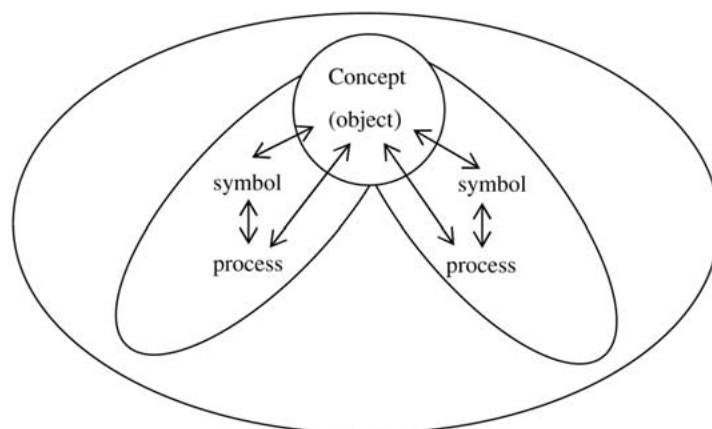


Figure 2. PROCEPT (consisting of two elementary procepts).

3. METHODOLOGY

I have recently begun introductory research into the understanding of functions using random research. This is a method for qualitative pedagogical research, whose goal is *not verification, but discovery, a description of that which is not necessarily typical, but unique and individual* (Turlejska, 1998, p. 86). Qualitative research allows for an exemplification of a certain law or the building up of general knowledge, in which individual experience and individual diagnosis form the basis for generalization and objectivization of knowledge.

As an instrument of research, I chose a set of non-standard tasks, in which the pupil should both show familiarity with different representations of the idea of function and show his or her understanding. Among the many tasks dealing with this idea there are exercises on functional equations and inequalities. Such exercises have recently appeared in mathematical competitions for secondary school students all over the world. In such tasks, function is used as a preset, completely formed idea, serving as an unknown. These tasks require the pupils to look at function from a different perspective – they are thus difficult, and addressed to pupils with a talent for mathematics. It seems possible, however, to use this type of exercise to test the image of the concept of function held by an average secondary school pupil. This was the goal of my research. I was not interested in the pupil's ability to solve the problem on his or her own, but rather in observing the process of finding a solution. The problems encountered in the process of solving such a non-standard task could provide information on the student's understanding of function. Solving standard tasks in this area would not necessarily show such an understanding of the idea of function.

In the present article I will present an example of this use of tasks dealing with functional equations. It is a dialogue held with Kasia (pronounced “Kashia”) – a sixteen-year-old pupil of a general academic secondary school in Cracow. She is a pupil of average ability in mathematics. Kasia attends a general academic class with additional lessons in Greek and Latin. Her teachers classify her as a “typical humanist”.¹ I have known her well for three years. I am familiar with her progress in learning mathematics and have good rapport with her. Kasia is open, talkative, and willing to speak about her own thoughts and thought processes.

At the time of the interview Kasia has been learning about functions for three years. She knows the formal definition of function, and is familiar with different representations and examples. Kasia uses this idea over and over in mathematics lessons, so its image is constantly changing. This is why I will limit myself to formulating conclusions on Kasia's understanding of the idea of function during a dialogue, which lasted 42 minutes and dealt with the following task:

Give an example of a function f such that for any real numbers x, y in the domain of f the following equation holds: $f(x+y) = f(x) + f(y)$.

This dialogue was recorded and an exact tape script was written, including pauses and nonverbal responses. The statements made by me as the experimenter and those made by Kasia were numbered. Taking into account the non-standard task type, I assumed that I would have to intervene during the solving of the problem.

4. DESCRIPTION OF THE STUDENT'S BEHAVIOR

4.1. Overview

When Kasia began to solve the problem, she had difficulty understanding what she was being asked to do. She pondered over what exactly it was: “to give any number or any formula”. She started with looking for “a certain x, y ”. After several attempts substituting numbers as the arguments of the function she concluded that “it's about the formula”, but giving the examples she enumerated two different functions: $f(x) = x^2 - 2x + 3$ and $f(x) = x^2 + 5$. Then she tried to treat the functional equation as a formula of a function. The student discovered the relation between the functional notation and the argument of the function; however, she managed to do it only after a lengthy conversation about the symbolism and as a result of the experimenter's intervention. Only then, in her 69th statement did she show that she understood the task content:

EXP₆₉: You are supposed to give an example of what?

K₆₉: **Of a function, and not of an argument² of a function!**

Next, Kasia began her solution by examining examples of linear functions, until she provided and verified the solution in the form of the function $f(x) = 2x$. Our conversation continued as follows:

EXP₉₈: Do you think this is the only function that satisfies the equation?

K₉₈: **No. I think if there were only x , that would fit, and if there were something next to x , that would work, too.³**

EXP₉₉: And if we had 0 next to x , what function would we have then?

K₉₉: **None.**

EXP₁₀₀: None? . . .

K₁₀₀: **$f(x)$ would equal 0. It would overlap with one of the axes – the x -axis.**

EXP₁₀₁: The graph of the function would overlap that axis?

K₁₀₁: **Yes, and it would be correct too. $f(x)=0, f(y)=0, f(x+y)=0, 0+0=0$.**

As we can see, the pupil found all the solutions with the exception of $f(x) = 0$ on her own. However, we assume, possibly being over-optimistic, that when Kasia said “something next to x ” she meant real coefficients, but unfortunately we cannot be certain of that. The experimenter’s surprise at “none [no function]” made Kasia re-think and verify her answer. She accepted and examined the function with the formula $f(x)=0$.

To analyze the dialogue between Kasia and myself as an experimenter, we can divide the dialogue into short episodes and analyze each as it pertains to the understanding of the idea of function as a procept (see Figure 2). Let us have a look at an excerpt:

EXP₆₀: Does the symbol f in the equation $f(x+y) = f(x) + f(y)$ refer to the same function?

K₆₁: (after a long pause) **But function as what? As just a formula?**

EXP₆₁: How do you understand the word “function”?

K₆₂: **It’s a formula that leads us to drawing a parabola or something like that.**

EXP₆₂: In that case, will it be the same formula for $f(x)$ and $f(y)$?

K₆₃: **No.**

Interpreting the above statements, we could place them in a table such as in Figure 3.

However, this kind of tables cannot be the only method of analysis since very often the symbol evoked in the student other procedures and directed her towards other concepts, which had no direct connection with that symbol. It is therefore worthwhile examining the chronological sequence of

SYMBOL	PROCESS	CONCEPT
Exp₆₀: Does the symbol f in the equation $f(x+y) = f(x) + f(y)$ refer to the same function? K₆₃: No. (Most likely a failure to understand the symbol f as the same function defined with different arguments.)	K₆₂: [A function] is a formula that leads us to drawing a parabola or something like that. (An obvious procedural aspect of the understanding of function. The pupil associates function with the drawing of its graph.)	K₆₁: (after a long pause) But a function as what? As just a formula? (It is only in Kasia's 69th statement that an autonomous reflection on the concept appears. The pupil thinks about what function is. The expression <i>just a formula</i> suggests that by "function" the pupil understands something more than a formula.) (She possibly understands the concept as: Function = formula + graph)

Figure 3. Kasia's procept of function – analysis of the excerpt from the dialogue.

associations in the student, as they can provide us with information about the inception of the procept of function in the student. Let us analyze from that angle several initial excerpts from the conversation. They illustrate Kasia's understanding of concepts when she was starting to deal with the problem.

Changes in understanding observed during the conversation will be presented in a more detailed way in the next section. In the section below I will present only those associations, which were evoked by the terms Kasia encountered in the content of the task. It seems interesting to examine such sequences of associations, as we can find the beginnings of procepts in them. Let us have a closer look at some passages from the dialogue:

4.2. Details

4.2.1. *The sequence of associations evoked by the word 'EXAMPLE' in the formulation of the problem*

In trying to understand the problem, Kasia was asking:

- K₃: **Can I take x and y to be any number? Or any formula? Is it about a number or a formula?**
- EXP₃: Read the problem.
- K₄: (reading). . . **I think it's about a formula. . .**
- EXP₄: A formula of what?
- K₅: **A formula of a function. . . So, so it's a formula. . . And not a particular number?**
- EXP₅: And what do you think?
- K₆: **I don't know, because it could be both ways. Because at first I thought it's about a particular number x or y . I would substitute them with two numbers, for example 3 and 5. We would have 8 here and, here, $3+$. . . 5?**

Kasia wants to deal with something concrete, this is why she directs her attention towards numbers. Most likely the words “give an example” prompt her to do so. The formula of the function is not ‘concrete’ enough for her. This is why the procedure of substituting the numbers as arguments of the function and then adding them is initiated. For a while Kasia is not sure what she is supposed to do. Several readings of the problem are not enough to convince her that it is the function that is looked for. The experimenter, seeing Kasia's confusion, asks:

- EXP₇: And how would you read what you have just written? (*Pointing to $f(3+5)=f(3)+f(5)$*)
- K₈: **Function f of three plus five is function f of three plus function f of five.**

The student reads out loud the symbolic notation without any hesitation but purely mechanically. She probably does not understand its meaning. The student concentrates on numbers and does not think about the role of the letter f in the notation, although she reads it as ‘function’. However, she feels no need to refer to the concept of function itself. It is only the experimenter's question:

- EXP₈: And how do you understand the expression: “Function f of 3”?

that evokes from Kasia the reflection:

- K₉: **Umm. . . It makes me think about a . . . a “zero of the function”, or something like that. . .**

Among the many terms related to the concept of function the only association that comes to the student's mind, is "the zero of the function". She is not able to interpret the symbol $f(3)$ correctly. She is not happy with the symbol to such an extent that she concludes that the task is to provide the 'formula' and not a 'specific number'. The reasoning is based on very risky premises:

K₉: (continued) ... **Because usually when we write, we write a formula. I mean we don't write a specific number in parentheses. So I think it is about a formula, because it would be difficult to write something like that in such a form.**

This passage shows two parallel associations Kasia has when she is wondering whether to give 'the formula or a specific number'. The verbal symbol 'function' together with the symbol $f(x)$ in the functional equation evokes in her the concept of 'the formula of a function.' However, Kasia does not want to follow this track. The other sequence of associations is stronger, namely, the very word 'example' evokes the concept of a number and leads to the procedure of substituting 3 for x and 5 for y , which produces the symbol $f(3)$, which in turn evokes the concept of the zero of a function. An outline of the sequence could look as follows:

Example $\rightarrow$ a specific example $\rightarrow$ substitute numbers for x and $y \rightarrow f(3)$ is a zero of the function

Unfortunately, in the case of the given task, this sequence of associations leads Kasia into a dead-end.

4.2.2. *Sequence of associations triggered by the word 'EQUATION' in the formulation of the problem*

The student goes on handling the task:

K₉: (...) **Now I would say something about the formula – instead of those 3 and 5... It's just that I don't know how! Just a moment. . . (Reads the problem once again). But this is ... AN EQUATION?! (Indignantly)**

EXP₉: How do you think, is this an equation?

K₁₀: **Not really, because when there's an f here, it can't be an equation.**

EXP₁₀: Why not?

K₁₁: **Because if on the left-hand side we leave this (pointing to $f(x+y)$) and here we substitute two formulas of functions – for $f(x)$ a formula of a function and for $f(y)$ a formula, too – only then we would have it in the form of an equation.**

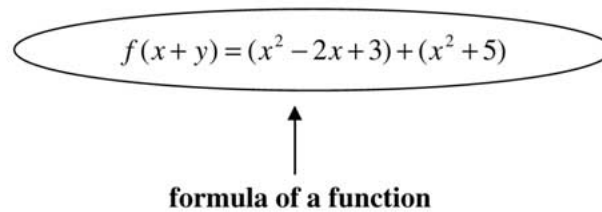


Figure 4. 'Holistic' understanding of the formula of a function.

EXP₁₁: Would you substitute the same thing for $f(x)$ and $f(y)$?

K₁₂: **No, something different.**

EXP₁₂: Something completely different?

K₁₃: **Yes, for example, $f(x+y) = (x^2 - 2x + 3) + (x^2 + 5)$**

For Kasia, an equation is probably any two algebraic expressions containing letters x or y connected by the equals sign. However, she does not allow for a situation where there would be f on both sides of the equation. The 'equation' she came up with: $f(x+y) = (x^2 - 2x + 3) + (x^2 + 5)$ clearly has the shape of a formula of a function, and not by coincidence. Indeed, the student identifies the concept of a formula of a function with the concept of equation, which she later expresses by saying that the letter f represents "the beginning of an equation, that is, of a formula of a function" (K₂₄). It is possible that if the experimenter asked her to "solve the equation", it would have triggered a procedure – for example, calculating the values of the function for specific arguments.

An outline of the above sequence of associations could be presented as follows:

Equation $\rightarrow$ contains unknowns $x, y \rightarrow$ cannot contain f on both sides $\rightarrow$ the formula of a function is an equation

4.2.3. Sequence of associations triggered by the symbols f and $f(x)$ and $f(y)$

Let us look at Kasia's associations evoked by the functional symbolism in the initial stages of the dialogue. The experimenter repeatedly inquires about Kasia's interpretation of the symbol f . At the beginning her explanations are rather vague:

K₁₇: **This is about some kind of beginning or the main point or something like that. 'Cause when you give an example you write it like this: $y = \dots$ or $f(x) = \dots$. So it is a beginning of a new thought or a new task.**

Later Kasia verifies her answer and is more specific by saying that the symbol f stands for "the beginning of (...) a formula of a function" (K₂₄).

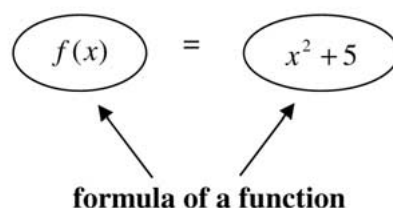


Figure 5. 'Fragmentary' understanding of the formula of a function.

Questioned about that symbol again she says impatiently that it stands for: "a function!" (K₂₆). She is able to put into words the inscription $f(3+5) = f(3) + f(5)$ using the word *function*, but, as it was pointed out earlier, she does not understand it. We can therefore put forward a hypothesis that for Kasia the symbol is just an abbreviation for the word itself and has no content, it is not the designate of a concept. Therefore on one occasion the student understands the symbol f as an abbreviation for the word *function* and on another as the beginning of the formula of a function. The question arises: what would be a complete formula of a function for the student? Kasia explains that, for her, "the equation, that is, the formula of a function" is "for example, $f(x+y) = (x^2 - 2x + 3) + (x^2 + 5)$ " (K₁₃). In this case, for the student, the formula of a function is an inscription containing the symbol '=' and the letter f on its left side and an algebraic expression with the variable x on the right side. She treats the formula as 'a whole' (Figure 4), iconically, as it were, without looking at it analytically, as an expression written using an operational symbolism.

On the other hand she says that she wants to substitute two formulas – "for $f(x)$ the formula of a function and for $f(y)$ a formula, too" (K₁₁); she appears, therefore, to talk about the algebraic expression itself as a formula of a function. In another part of the dialogue Kasia says:

EXP₁₇: Is the symbol $f(x)$ just a sign of the beginning of a new problem?

K₁₈: **This is the formula of a function. I suppose.**

Here, for Kasia, the symbol $f(x)$ stands for *a formula of a function* and she treats it as a name, which is interchangeable with the algebraic expression specifying the function formula (Figure 5).

The next part of the dialogue shows further associations that Kasia makes. The student gives an example of a function $f(y) = x^2 + 5$. She treats the symbol $f(y)$ as an integral whole that begins the formula of a function, which is different from $f(x)$. To verify, if the student can, indeed, see no connection between the notation and the argument of the function the experimenter asks a direct question:

EXP₄₁: Do you think that the symbol y standing on the left-hand side of the formula has nothing to do with the variable of the function – with what we'd have written on the right-hand side of the formula?

(...)

K₄₄: **No, not really.**

EXP₄₄: Can you see no connection whatsoever?

K₄₅: **No, I can't.**

EXP₄₅: So it would make no difference if I wrote $f(b) = x^2 + 5$, right?

K₄₆: **Right.**

(...)

EXP₄₇: So what could the letter b mean in this notation?

K₄₈: **If we have no drawing, then [it means] nothing.**

EXP₄₈: Absolutely nothing?

K₄₉: **Almost nothing, b defines the function. It's the same as $y = x^2 + 5$.**

The student appears to treat the symbols $f(x)$, $f(y)$, $f(b)$ as three different names of the same function ('formulas of the function'). On the other hand, we can see that Kasia is referring to the geometric interpretation of the function. Kasia says that "if we have no drawing, [it means] almost nothing." Earlier she revealed what kind of drawings she had in mind:

K₁₈: **$[f(x)]$ is not a formula of a function, I think. Because $f(x)$ is the same as y and y is the abscissa. No! The ordinate.**

EXP: And what if instead of this equation (pointing at $f(x+y)=f(x)+f(y)$) I wrote $f(a+b)=f(a)+f(b)$?

K₁₉: **It would be the same as $f(x+y)=f(x)+f(y)$.**

EXP₁₉: Does b represent here the ordinate of a specific point?

K₂₀: **It depends on the drawing. Everything is a matter of convention in mathematics. We could draw an axis and mark that we have a here and b there. (She draws a co-ordinate system and marks its axes.) And I think it would be the same.**

In Kasia's case the symbol y is very strongly associated with the ordinate of a hypothetical point on the coordinate axis – probably a point of a graph of the function. This association is so strong that in the notation $f(b)$ she interprets b as the ordinate of the point, too. Undoubtedly, the initial form of the functional equation influenced her. Possibly that association would not have occurred if the functional equation had had the form $f(a+b)=f(a)+f(b)$ from the very beginning. However, there can be no doubt that the graph of a function is central to Kasia's procept of function. Kasia's associations

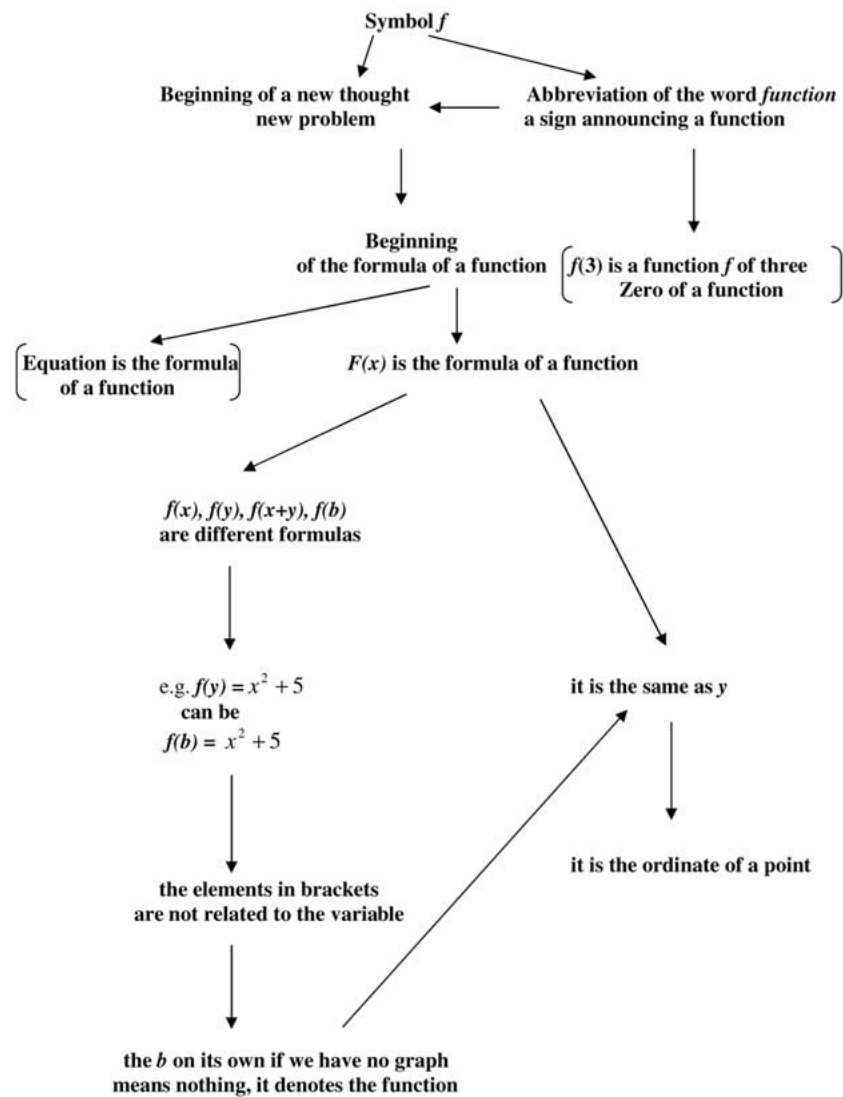


Figure 6. Associations triggered by the symbols f , $f(x)$, $f(y)$ (arrows stand for an after-effect of time).

enumerated above may be presented in the Figure 6. The associations examined earlier are in brackets.

4.2.4. *Sequence of associations triggered by THE FUNCTIONAL EQUATION given in the problem*

Let us examine yet another set of associations triggered by the functional equation in the problem: $f(x+y) = f(x) + f(y)$. Kasia explains that the left-hand side of the equation is different from the right-hand side:

K₆₄: (...) **Because here it's like. . . separated. . . Oh! (Appears satisfied) It is as if it's multiplied. It is not the same.**

The student made an association with the distributive law of multiplication. The student sees an analogy between the notation of the equation and the notation of the law of numbers: $f(x+y) = fx + fy$. She does not think about the meaning of the symbols and her attention is drawn only to the iconic similarity of the symbolism. This is how she justifies that we do not have the same thing on both sides of the equation. Therefore, according to Kasia, the equality has to be proved. This conviction is very strong. Even after finding a solution to the problem, the student described her understanding of the problem in the following way:

K₁₁₅: **But we could just as well write it as an ordinary equation.**

EXP₁₁₆: How?

K₁₁₆: **For example, $3 \cdot (2 + 1) = 3 \cdot 2 + 3 \cdot 1$. Actually, this is not about the function but the equation. Right? The main thing here is the equation and not the function.**

EXP₁₁₇: Is that so?

K₁₁₇: **I think so. Because we have to express the equation using the function. It was about checking if the left-hand side equals the right one.**

Kasia's reasoning is still at the level of numbers and unknowns, at the level of an ordinary equation (identity). The student is not willing to move to the level of thinking about functions as variables satisfying a given property. According to Kasia it is not a function that satisfies the equation. In the problem the function is, for her, just a way of 'encoding' the identity. The sequence of associations examined above may be presented in the following way:

The functional equation from the problem $\rightarrow$ the distributive law of multiplication $\rightarrow$ one has to check if the left-hand side equals the right-hand side $\rightarrow$ the main thing here is the equation and not the function

4.3. *Changes in the student's understanding of symbols and of the concept of function that took place in the course of the dialogue*

As a result of the dialogue several important changes in Kasia's understanding of symbols related to functions took place. Of course, we are not able to identify and diagnose all the changes because we do not have sufficient tools at our disposal. Let us have a look at two main changes revealed by Kasia during the dialogue.

4.3.1. *Discovering that the symbol $f(y)$ is related to y*

Let us remember that, at the beginning, the student gives ' $f(y) = x^2 + 5$ ' as an example of a function. She does not see a connection between the notation $f(y)$ with the argument of the function but treats it as an integral whole, which begins the formula of a function, different from $f(x)$. Kasia does not see the clash in the notation related to the symbol y , which she treats, at the same time, as the value of the function. The experimenter, seeing that she has led herself into a dead-end, intervenes in her reasoning. She asks Kasia to calculate the value of that function for the argument 1, and then for the argument z . The student performs those activities with no difficulty although the second request seems odd to her. Nevertheless, she writes $y = z^2 + 5$. In the course of the conversation the student discovers the connection between the symbol $f(x)$ and what is written in the brackets:

EXP₅₄: And how would you write, in a different way, that $z^2 + 5$ is the value of that function (*pointing to the symbol f*) for the argument z ?

K₅₅: (*repeats slowly*) **The value of that function for the argument z . . . It'll be $z^2 + 5 = f(z)$.**

EXP₅₅: Correct. Coming back to our notation $f(y) = x^2 + 5$. . .

K₅₆: (*interrupting*) **It can equally be $y^2 + 5$.**

EXP₅₆: Yes. The notation $f(z)$ suggests. . .

K₅₇: (*interrupting*) **Suggests what should be there in the equation!**

EXP₅₇: Almost. It suggests that z is the argument of the function f .

K₅₈: **Yes. So here we will have y .** (*changes into $f(y) = y^2 + 5$*). **So here everything is mixed up.**

The expression the student uses is interesting: "It can equally be. . ." Possibly she does not want to admit that the initial notation she used was not correct. However, it is also possible that she thinks it was correct, and changing the letters is yet another useless convention in mathematics. Kasia noticed that what is inside the brackets should appear on the right side of the formula. Unfortunately, she focuses her attention only on that. Indeed, she worked out on her own that the symbol $f(z)$ denotes

the value of the function f for the argument z but she discovered it only with considerable help from the experimenter. Later she does not listen to the explanation (EXP₅₇) about the argument of the function – her “Yes” is devoid of reflection. She might be thinking of the variable as a placeholder: $f(\square) = \square^2 + 5$. There is evidence for that in the following passage from the dialogue:

- K₇₃: **So $f(x)=2x+1$, and, to make $f(y)$ similar, I'll take $x+5$.**
 EXP₇₄: So $f(x)$ is $2x+1$, and $f(y)$ is x plus 5?
 K₇₄: **Aha! It has to be $y+5$.** (*She corrects the inscription to $f(y)=y+5$*)

As it has to be y , the student changes it into y . She is adjusting to the convention although she does not feel the convention is necessary. Later she reveals in the conversation that she still cannot see the connection between the variable and the argument of the function. The experimenter wants her to discover that $f(x)$ and $f(y)$ represent values of the same function for different arguments:

- EXP₇₅: (. . .) Calculate $f(a)$.
 K₇₅: **$f(a)=2a+1$**
 EXP₇₆: Exactly, that's correct.
 K₇₆: **But there I wrote that $f(y)=y+5$!** (*With resentment, emphasizing the letter y*)
 EXP₇₇: You wrote $y+5$, and not $2y+5$.
 K₇₇: **But it is not the same!**
 EXP₇₈: And is it supposed to be the same or not?
 K₇₈: **No, it cannot be the same!**
 EXP₇₉: Why?
 K₇₉: **Because here we have x and here y (*pointing to $f(x)$ and $f(y)$*) – it cannot be the same.**
 EXP₈₀: But here we have x and here a (*pointing to $f(x)$ and $f(a)$*) and in your opinion it can be “the same”.
 K₈₀: **Depending on what we have in brackets, we can substitute in here (*in the formula*).**
 EXP₈₁: So what should be the value of $f(y)$?
 K₈₁: **I don't know.**

Kasia focuses her attention only on having the same variables; she has just learnt that. On the other hand, the conviction that $f(x)$ is a function different from $f(y)$ is fossilized in her mind and leads to a conflict. She is still treating those symbols ‘holistically’.

SYMBOL	PROCESS	CONCEPT
K₇₃: $f(x)=2x+1$ $f(y)=x+5$ K₈₂: $f(y)=2y+1$...But it was supposed to be a separate formula... (with dissatisfaction)	EXP₈₂: What does the function f (pointing to $f(x)=2x+1$) do to its argument? K₈₂:...It multiplies it by 2 and adds 1...	K₈₂: $f(y)=2y+1$...But it was supposed to be a separate formula... (with dissatisfaction)
Probably the student treated the y in the notation $f(y)$ as an argument of the function, at the same time seeing here two different symbols, which she still regards as symbols of two different functions $f(x)$ and $f(y)$. There is a clash with her earlier associations. It seems that the student discovered that we are dealing here with one function, but she doesn't know yet why it is so. The student still does not understand the symbol $f(y)$ fully, although we can observe a change in its understanding.	The experimenter's intervention was significant, but the student accepted the suggestion that the function should be treated as a computational process. Undoubtedly, we would not have got that statement if the student had not understood the function in this way. The student revealed that she knew how to calculate the value of the function for the argument x.	We can see understanding of the formula itself, however it was not provided spontaneously by the student. She probably understands the connection between an argument and the value of the function determined by the formula. Does Kasia understand this formula as encapsulating the whole function? The student's statement is very general, which may suggest an affirmative answer. Moreover the student noticed that it is the same function as before and it was supposed to be a different one ("a separate formula"). However, one has to be careful in forming on that basis any far-fetched hypothesis.

Figure 7. Kasia's procept of function – analysis of the excerpt from the dialogue (Table II).

4.3.2. Understanding of the symbol $f(x)$ as the value of the function f for the argument x

Another excerpt from the dialogue will be analysed in a chart (Figure 7).

Then we see a breakthrough in Kasia's understanding of the formula of a function, triggered by the conversation quoted above and by yet another intervention by the experimenter:

EXP₈₄: If we had a 'separate' formula, it would be a different function – not the function f - but a different one, for example h , g or f^* .

K₈₄: **Aha! (Joyously) Now I understand.**

EXP₈₅: How do you understand it?

The rest of the conversation is analyzed in Figure 8.

SYMBOL	PROCESS	CONCEPT
K₈₅: Oh... It was about this f all along, and I didn't get it! K₈₆: f determines what should stand on the right side. The form will not change but x or y will change, depending on what is in the brackets. It is now that the student understands the meaning of the symbol f in the function formula and the meaning of the symbol $f(x)$. However, the understanding seems to be instrumental.	K₈₆: (...) depending on what is in the brackets, x or y will change. The student sees the procedure of substituting arguments of the function, understands the formula of the function in a procedural way and probably understands the function itself as a computational process.	K₈₅: The thing is that this f is a kind of determinant. K₈₆: The determinant of all the rest. Kasia understands the concept of the function formula. According to Kasia, function f determines only the form of an algebraic expression describing the function formula. Using Kasia's language we could put forward the hypothesis that for her: "Function" = the determinant of the form of function formula.

Figure 8. Kasia's procept of function – analysis of the excerpt from the dialogue.

5. CONCLUSIONS

The analysis of the whole dialogue allows us to describe Kasia's procept of function and formulate conclusions about her difficulties in understanding this notion. We propose that these difficulties stem from (a) Kasia's misinterpretation of the symbols used in the functional notation and (b) her very limited *procept* of function.

5.1. Misinterpretation of the symbols used in functional notation

This section contains a summary of Kasia's understandings of the symbols used in functional notation, and an identification of their possible sources. We posit three kinds of sources:

1. The intrinsic ambiguities of the mathematical notation;
2. The restricted contexts in which some symbols occur in teaching, and a limited choice of mathematical tasks at school;
3. Kasia's idiosyncratic interpretation of school mathematical tasks.

These factors are not mutually exclusive; on the contrary, they influence one another and very often contribute to Kasia's understanding of the symbolism, as shown in the descriptions below.

Understanding S1⁴. '*f*' as a label

At the beginning of the conversation, while the function symbol f is identified correctly, it does not carry any content, it is perceived as a label, or an abbreviation of the word 'function'. Sometimes it represents only "the beginning of a thought or new task" (K₁₇). Usually the tasks concerning functions that Kasia was solving began with the notation $f(x)=\dots$ or $y=\dots$. They were probably not varied as far as content is concerned and rather schematic, and used to practice a chosen skill. So at the beginning of the conversation the student associates that notation only with a new task, probably related to a function. (Sources 2, 3)

Understanding S2. Association of a symbol like ' $f(3)$ ' with the zero of the function

Kasia is not able to interpret the symbol $f(3)$ correctly. It was very rarely that she had encountered such notation: "Because usually when we write, we write a formula. I mean we don't write a specific number in parentheses. So I think it is about a formula" (K₉). What "we usually write" and do in mathematics lessons is very important for the student. It is more important than thinking about the meaning of the symbol. Kasia does not want to go into the heart of the matter; she chooses the easiest way of reasoning, which can be summarized as "If something occurred rarely or never in mathematics classes, it is not correct". Probably such an approach to the problem is a result of her 'coping strategy'. Kasia associates the symbol $f(3)$ only with the zero of the function. This is probably because during mathematics lessons she has only come across this notation when testing whether a given argument is the zero of a function. Kasia did not remember its meaning, only the context in which it was used. Could she then understand it correctly in a situation related to the zero of the function? It is highly unlikely. To interpret it correctly she would have to see it as the value of the function for the argument 3, which she did not show in dealing with the task. (Sources 2, 3)

Understanding S3. ' $f(x) = [\text{algebraic expression}]$ ' as the formula of the function

Kasia interprets the symbol $f(x)$ and the algebraic expression determining the function as *the formula of a function*. At the same time the actual formula of the function has the same name. She uses those interpretations interchangeably. This way the student copes with the ambiguity of the symbol $f(x)$, as a function and as the value of the function for the argument x , calling it all *the formula of a function*. She does not see in that symbol the value of the function for the argument x . Possibly the fact that for Kasia the algebraic formula defining the function represents the formula of the function results from mental shortcuts sometimes used by mathematicians and teachers themselves who say for example: "Given the function $x+2$

...” or “the function f is given by the formula $x+2$ ”. Mathematicians are not confused but novices are, and the apparent terminological chaos may result in difficulties with understanding the concepts. (Sources 1, 2, 3)

Understanding S4. *‘ $f(x)$, $f(y)$, $f(x+y)$ ’ are seen as three different functions*

Initially Kasia sees the symbol $f(y)$ as an integral whole. In the symbolic notation she sees no connection with defining the variable, thus $f(x)$, $f(y)$, $f(x+y)$ are treated as three different functions. Unfortunately this conviction is so strong that even after having dealt with the task she thinks that the problem was about three unknown functions.

EXP₁₁₉: So what have you found in the task?

K₁₁₉: **The formulas of the functions $f(x)$, $f(y)$, and $f(x+y)$.**

EXP₁₂₀: And if we know $f(x)$, don’t we know $f(y)$ and $f(x+y)$ as well?

K₁₂₀: **Yes, we do. So it’s enough to find $f(x)$. But I don’t quite understand what’s the use of it – why it is written in such a way that there are three unknowns, when in fact there’s only one. It’s really complicated! (Laughs)**

Kasia’s interpretation is probably influenced by the fact that, in her mathematics classes, the student was rarely faced with the situation where the argument of the function was marked by a letter different than x . Moreover she must have not known or not remembered functions’ names different from f . The conversation partially confirms this thesis because when the experimenter explained how we mark different functions the student experienced an illumination (“Aha!” experience). So she got used to how the function formula ‘begins’ and she would not think about the meaning of that symbol. And as the symbol $f(x)$ ‘looks’ different than $f(y)$, for the student those were two different functions. Possibly, Kasia does not think about the logic of the operational symbolism of functions at all, but accepts them unconditionally – she adjust herself to the convention. However, we do not have enough evidence to support this hypothesis at this point. (Sources 2, 3)

Understanding S5. *‘ $f(y)$ ’ is the ordinate of some hypothetical point*

The symbol y in the notation $f(y)$ is treated as *the same as $f(x)$* or as the ordinate of *some hypothetical point*. The ambiguity of the mathematical notation undoubtedly plays a significant role here: writing y in the context of functions we sometimes refer to a value of a function, sometimes to the ordinate of a point in the co-ordinate system, and in our task – to an argument. The problem with accepting that y is an argument is a matter of custom since this symbol is usually used to denote the value of a function. This is a typical and natural difficulty. (Sources 1, 2)

Understanding S6. *A functional equation is not an equation*

The notation $f(x+y) = f(x) + f(y)$ was not initially seen as an equation by Kasia, but the notation $f(x+y) = x^2 - 2x + 3 + x^2 + 5$ was seen as an equation (K_{13}) and so was any formula of a function (K_{24}). Another problem with interpreting the concept of an equation appears here. We should not be surprised that the student does not accept a functional equation as an equation in which the function is the unknown, as she had not come across such form before. However, we may be surprised that a formula of the function is an equation for her. We can see here her own interpretation of the concept of equation. It is likely that for her an equation consists of any two algebraic expressions or symbols she encountered in the course of studying, which contain the letter x . Since she had never come across a notation in which the letter f appeared on both sides of the equation mark, she does not treat that notation as an equation. (Sources 2, 3)

Understanding S7. *Analogy of form of expressions is a basis of meaning making*

The student sees an analogy between the notation of the functional notation and the notation of the distributive law in numbers $f(x+y) = fx+fy$. Kasia again does not think about the meaning of the symbols analytically but notices only an iconic similarity between algebraic expressions. (Source 3)

Understanding S8. *“The main thing in ‘ $f(x+y)=f(x)+f(y)$ ’ is the equation, not the function”*

In the conversation summarizing our dialogue, Kasia stated: “The main thing here is the equation, not the function”. She expressed in this way her understanding of functional equations as a way of writing the ordinary equations (identities) with which she was familiar. Undoubtedly it is an interesting idea and we cannot negate this way of reasoning. In school she has been dealing with equations for so many years that she feels comfortable with them. So she changes the unknown situation into a familiar one. (Sources 2, 3)

The above summary of Kasia's understandings shows how serious can be the difficulties of a student of average mathematical abilities. It is worth noting that, as far as Kasia's difficulties with symbols are concerned, in as many as seven cases they are caused by restricted contexts in which some symbols occur in teaching, limited choice of mathematical tasks at school and Kasia's own interpretation of mathematical tasks – thus sources 2 and 3. Let us quote another Kasia's statement confirming this hypothesis. After having finally understood the task, she says with satisfaction:

K_{71} : **Oh, so it's not going to be a quadratic function.**

EXP_{72} : Why?

K_{72} : **I think that it's because it would be too complicated.**

The student is used to the situation where school examples can be reduced to simple calculations and that is what she expects from every mathematical task. It seems that the ambiguity of functional notation is not the main obstacle here because we diagnosed source 1 only in two cases. However, that does not mean that the ambiguity of mathematical symbolism is not significant in understanding concepts. Twice during the dialogue (in the context of interpreting symbols) Kasia said, “everything is a matter of convention in mathematics” (K₂₀, K₄₇). This suggests that notational ambiguity and its relativity with respect to the context of use may blur, for some students, the fact that mathematical symbolism is still an operational symbolism and not a random set of ad hoc abbreviations. Source 1 may thus cause serious difficulties with understanding concepts.

5.2. *Very limited procept of function*

Let us try to describe the procept of function revealed by Kasia.

Understanding F1⁵. *Absence of a notion of function: Thinking in terms of numerical equations and unknowns*

The student avoids the concept of function and even talking about it or saying the word ‘function’. She focuses her attention on other concepts. Such behavior was observed repeatedly in the course of the dialogue. It can be suggested that function is for her not an *encapsulated* concept, but it is still in the process of construction. Kasia’s reasoning is on the level of numbers and unknowns, on the level of numerical equations (identities) and she does not want to move to the level of thinking about a function. The difficulty experienced by Kasia is one of the epistemological obstacles in understanding the concept of function described by, e.g., Sierpiska (1992). This obstacle – an unconscious scheme of thought – was described as *thinking in terms of equations and unknowns to be extracted from them*. Overcoming this obstacle requires *discrimination between two modes of mathematical thought: one in terms of known and unknown quantities, the other – in terms of variable and constant quantities* (pp. 37–38). In this task the difficulty is even greater and indeed triggered by the wording of the task. The word ‘equation’ used in the content of the task makes Kasia think about unknowns in an equation, usually x and y . It interferes with perceiving them as arguments of the function.

Understanding F2. *Function as the beginning of a new thought or new task*

Function is initially seen as “the beginning of a new thought or new task” (K₁₇). It is only the first signal for beginning a certain procedure, known to the pupil, of solving the problem. Later the symbol f means to her “the beginning of the equation, that means the formula of the function”

(K₂₄). So the student associates the function with the process of 'solving an equation'. Maybe this 'solving an equation' is looking for a value of the function for different arguments, the zero of the function etc. in order to draw a graph. However the association may be related to any task, concerning functions solved by the student before.

Understanding F3. *Function as a formula*

For Kasia, the concept of function is often indistinguishable from the concept of the formula of a function. Sierpiska (1992) singled out *discrimination between a function and the analytic tools sometimes used to describe its law* as one of the conditions indispensable for understanding functions (p. 45). In the case of our dialogue the lack of this discrimination may be linked to the type of the task, in which one is supposed to give an example of a function by providing its formula. However, because we had access only to what Kasia was saying, and not to what she was thinking, the use of 'function' for 'formula of function' could have been a mere metonymical figure of speech (*pars pro toto*).

Understanding F4. *Function as that which "determines all the rest" in the formula*

Function, in Kasia's understanding, also appears as "a determinant of all the rest" in the formula – and thus a form of an algebraic expression describing the formula of a function. Kasia says that "x or y changes, depending on what is in the parentheses" (K₈₆).

Understanding F5. *Function as a computational process*

About a function defined by the formula $f(x) = 2x + 1$, the student states: the function f multiplies it [the argument] by 2 and adds 1. . . (K₈₂). She thus treats the function as a computational process. She revealed that she knows how to calculate the value of function for a given argument, that she understands the formula of the function. It seems that this formula encapsulates for Kasia the whole function; nevertheless other excerpts from the dialogue suggest that one should be cautious in formulating this hypothesis.

Understanding F6. *Function as a kind of formula which leads to drawing a graph*

We witnessed several times that Kasia was clearly aiming at a geometrical interpretation of the concept. She twice mentioned the ordinate of a point in the co-ordinate system and associated function with its graph:

K₆₂: [Function] is a kind of formula that leads to drawing a parabola or something like that.

Here, Kasia clearly associates the concept of function with the process of drawing its graph. The formula is necessary to draw the graph, so she understands it as a process. It is only after we have drawn the graph that

SYMBOL	PROCESS	CONCEPT
e.g. $f(x) = x^2 + 5$	Computing the value of the function	"Function" = formula only
e.g. $f(x) = x^2 + 5$ or $f(y) = y^2 + 5$	Determining the value of the function	"Function" = something that determines the form of the formula of the function
e.g. $y = x^2 + 5$ or $f(x) = x^2 + 5$	Drawing the graph of the function	"Function" = formula + graph
e.g. $y = x^2 + 5$ or $f(x) = x^2 + 5$	Drawing the graph of the function	"Function" = graph only

Figure 9. Kasia's procept of function revealed during the dialogue (consisting of four elementary procepts).

the task has been completed. Sfard (1991) emphasizes that the graph of the function is static; it facilitates holistic perception and is conducive to a structural understanding of the concept. And, indeed, when the task is almost completed Kasia says:

K₁₀₀: **$f(x)$ would be equal to zero. It would match one of the axes – the x-axis.**

In this statement Kasia equates the zero function with its graph.

The student had difficulty accepting the function $f(x)=0$ and initially she says that it is not a function. It is probable that her difficulties were triggered by the association *zero = nothing*, so they have their sources in her understanding of the concept of number. The student accepted the function only after referring to a geometrical interpretation of the concept.

In sum, it appears that Kasia's understanding of function was based on four elementary procepts of function, described in Figure 9.

Obviously other elementary procepts were also included in the student's procept of function (for example: the procept of the formula of a function or the procept of equation). Moreover other symbols, processes and concepts have been revealed during the dialogue. Those are only the beginnings of other elementary procepts related to a greater or lesser extent to the procept of function.

6. FINAL REMARKS

The dialogue provided a wealth of information on the student's understanding of the concept of function and the symbolism related to it. Kasia's concept of function is not yet a fully-fledged mathematical object.

Interpretation of the functional symbolism presents the greatest difficulty for Kasia. However, the fact that Kasia does not understand the idea on a level of its symbolic and formal complexity does not mean that she does not understand it at all. Her procept of function, although limited, is not empty. It contains many valuable elementary procepts and is continuously developing. Nevertheless, the understanding reached by the pupil in her school work was insufficient for solving the task.

The dialogue shows the influence of the typical nature of school tasks leading to standard procedures. A situation may well occur in which a pupil achieves good grades in mathematics without understanding the concept of function. This example demonstrates beyond doubt the possible unwelcome consequences of biased and careless selection of tasks in the process of teaching.

The dialogue also provided a wealth of information about the understanding of the concept of equations and numbers and pointed to other difficulties (in solving this type of task), whose analysis could be the topic of further consideration.

It appears interesting to carry out research aimed at finding out what image of the concept of function is held by those learning mathematics at different levels of teaching and presenting different scale of mathematical abilities, using the same instruments as were used with Kasia and performing a comparative analysis. It could enrich our knowledge about the concept examined in the present study.

Therefore, it seems that the tasks concerning functional equations can be used to examine understanding of chosen aspects of the concept of function.

NOTES

1. The term commonly used in Polish schools to refer to a person who is intelligent and good at humanistic subjects but not very successful in mathematics. In the context of teaching mathematics the term has a slightly condescending overtone and evokes pejorative connotations. A *typical humanist* often declares a dislike of mathematics.
2. In Polish schools elements of the domain of a function are called "arguments of the function" – the term is already commonly used at secondary school level (12–15 years old).
3. Kasia means directional coefficients a of linear functions $f(x)=ax$.
4. 'S' stands for 'symbolism'; in this section we enumerate Kasia's understandings of the functional symbolism.
5. 'F' stands for 'function'. In this section we enumerate Kasia's understandings of the notion of function.

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