

Chapter 19

Technology and the Role of Proof: The Case of Dynamic Geometry

Nathalie Sinclair and Ornella Robutti

Abstract This chapter brings together two intersecting areas of research in mathematics education: teaching and learning with dynamic geometry environments (DGEs) and the teaching and learning of proof. We focus on developments in the literature since 2001 and, in particular, on (a) the evolution of the notion of “proof” in school mathematics and its impact on the kinds of research questions and studies undertaken over the past decade—including increasing use of DGEs at the primary school level; and (b) the epistemological and cognitive nature of dragging and measuring as they relate to proof.

Section A: Introduction

This chapter brings together two intersecting areas of research in mathematics education: teaching and learning with dynamic geometry environments (DGEs) and the teaching and learning of proof. Given that both of these areas of research have been predominantly related to the study of geometry, our chapter focuses mainly on this strand of the curriculum.

We begin by making a comment on terminology, and some brief remarks on recent history. In this chapter we use the term “dynamic geometry” broadly. The phrase was originally invented (and trademarked by Key Curriculum Press) to describe *The Geometer’s Sketchpad* (Jackiw, 1991). *Sketchpad* and *Cabri-Géomètre* (Baulac, Bellemain, & Laborde, 1988) were independently invented in the late

N. Sinclair (✉)
Simon Fraser University, Burnaby, BC, Canada
e-mail: nathsinc@sfu.ca

O. Robutti
Università di Torino, Turin, Italy

M. A. (Ken) Clements et al. (Eds.), *Third International Handbook of Mathematics Education*, 571
Springer International Handbooks of Education 27, DOI 10.1007/978-1-4614-4684-2_19,
© Springer Science+Business Media New York 2013

1980s, and were the first software systems to offer the dragging capacity described by the term. There now exist many software packages that use these seminal ideas. Our paper loosely targets all of them, but given the importance of intellectual attribution in academic writing, we feel it important to acknowledge similarly the sources of software innovation within our field. We have opted for the term *dynamic geometry environment* (which has been used at least since 1996) over *dynamic geometry software* to underscore the fact that we are dealing with micro-worlds (including pre-existing sketches and designed tasks) and not just a software program.

The *Second International Handbook of Mathematics Education* (Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003) included several chapters focussing on the use of digital technologies. Within these, Chapter 9 (Hoyles & Noss, 2003) considered DGEs in particular and documented the way in which research was focussing less on geometric constructions and more on the way DGEs mediate explanation, verification and proof. The authors commented on the “commonplace” understanding that students often spontaneously articulate when using DGEs, but opined that it was “not obvious that the facility to drag and conjecture will necessarily encourage an engagement with proof” (p. 335). They cited research showing how novel activities aimed at generating surprise and uncertainty can strengthen students’ need for deductive proof (Hadas, Hershkowitz, & Schwartz, 2000); how 12-year-old students can move from everyday to mathematical explanations through DGE interactions (Jones, 2000); and, how DGEs can help 14–15-year-old students connect their informal geometric explanations with logical, deductive arguments as long as the tasks used are undertaken with teacher support, including a teacher-introduction to writing proofs (Healy & Hoyles, 2001).

The present chapter thus focusses on developments in the literature since 2001 and, in particular, on (a) the evolution of the notion of “proof” in school mathematics and its impact on the kinds of research questions and studies undertaken over the past decade—including increasing use of DGEs at the primary school level; and (b) the epistemological and cognitive nature of dragging and measuring as they relate to proof—thus taking up the Hoyles and Noss issue.

In preparing this chapter, we found several other overview studies in each of our primary areas of focus. Mariotti (2006) provided an overview of the past 30 years of research reports presented at the annual conferences of the International Group for the Psychology of Mathematics (PME) on “Proof and Proving in Mathematics Education,” and this included a significant section on the role of DGEs. Similarly, Hollebrands, Laborde, and Sträßer (2008) offered a chapter on “Technology and the Learning of Geometry at the Secondary Level,” which focussed primarily on DGEs and included a significant section on proof and proving. This chapter also includes a detailed account of the main theoretical approaches currently being used in research on the role of DGEs in proving. The two main approaches are instrumental genesis, which draws on the work of Rabardel (1995), and Verillion and Rabardel (1995), and has been elaborated by Mariotti (2002), and semiotic mediation, which draws on the work of Vygotsky (1978). These two approaches are discussed in detail in Drijvers, Kieran et al.’s (2010) overview chapter in the ICMI Study on Technology.

Briefly, a DGE can be seen, in an instrumental framework, as an “artefact” in the hands of the users. The DGE artefact can be transformed into an “instrument” by the user, according to the “schemes of use” that are activated. And the instrument obtained thus differs among users and contexts according to the particular schemes of use activated. Several studies have documented how this works in the contexts of DGE (see, e.g., Olivero, 2006; Baccaglini-Frank & Mariotti, 2010). If the students “internalize” (Vygotsky, 1978) the use of a DGE, the artefact DGE becomes a means of semiotic mediation (Mariotti, 2010) that gives rise to problem resolution.

In Section B below, we outline the perspective on proof that has emerged in the mathematics education literature during the past decade. We connect this literature to the previous theorizing on the epistemological and cognitive status of DGE diagrams and tools. While some of this theorizing dates back to the late 1990s, it was not taken up in the *Second International Handbook* and its connection to emerging perspectives on proof is new.

In section C, we identify and connect previous studies on the use of DGEs in proving in terms of the particular and distinctive tools shared by all the more widely used DGEs. We have chosen this somewhat unusual approach in order to emphasize, as well as better understand, the epistemological, cognitive and didactic implications of these tools—implications that have been studied in much more depth over the past decade than in previous research. This section will enable us to investigate the effects of DGEs on the proving process.

Having analyzed the tools in this way, we turn in section D to two case studies in which we illustrate instances of the proving process in which these tools play a significant role. Our goals are threefold: at one level, we aim to exemplify the most common theoretical perspectives, methodologies and protocols currently used—instead of merely reporting results from existing studies, we would like to show the context in which these results have emerged, as they are central to the findings proffered. At a second level, these case studies can permit us to enter into a level of detail about the proving process using DGEs that would otherwise not be possible. At a third level, given our focus on the particular tools of section C, we hope that the case studies more authentically illustrate the way in which these tools are often used in concert by learners.

In the final section, we highlight some of the major themes of sections C and D, and point to emerging questions and theoretical approaches that are likely to take centre stage in the next decade, with a particular focus on the role of teachers in classrooms using DGEs.

Section B: Proof as a Process

Policy makers, curriculum designers and researchers concurrently agree with the necessity of proof not only at the high levels of schooling, where it has traditionally occupied an important position, but also, for better continuity, at the primary level (see Bartolini Bussi, 2009; Stylianides, 2007; Stylianou, Knuth, & Blanton, 2009).

In many countries, teachers are being encouraged to offer activities favouring exploration, conjecture, argumentation, discussion and also proof (see, e.g., Centre de Recherche sur l'Enseignement des Mathématiques, 1995; Japanese Society of Mathematics Education, 2000; Ministry of Education and People's Republic of China, 2001; National Council of Teachers of Mathematics, 2000; Unione Matematica Italiana, 2004).

The notion of proof in school mathematics has changed enormously during the past century—see Herbst (2002) and Sinclair (2003, 2008) for overviews. The shift is due in part to a greater awareness of the way in which proving is done in the mathematics discipline (see Borwein & Bailey, 2008; Hanna, Jahnke, & Pulte, 2010) and, in part, we believe, to the affordances of new digital technologies, which greatly facilitate experimentation (de Villiers, 2010).

In this chapter, we have opted to frame our discussion about proof and proving in terms of the notion of the “proving process,” which has been defined as a process of constructing and entering the relation “B follows from A” until the agents are satisfied with the explanation for the truth of the statement (see Rav, 1999). This explanation is a proof if it satisfies the rules of logical consequence.

The proving process consists of two phases: the formulation of a conjecture and the construction of a proof (Arzarello, Olivero, Paola, & Robutti, 1999; Leung & Or, 2007). In the first phase, a person explores the situation and formulates a conjecture, and searches for elements (e.g., properties) that will be organized later on. In the second phase, these elements are to be put in order according to the rules of logical consequence. As we emphasize below, the shift to the focus on the “proving process” is especially relevant to DGEs in the sense that the first phase of the process differs radically from pencil-and-paper environments, which, in turn, affects the way in which the second phase evolves.

The proving process also involves back-and-forth moves between the spatio-graphical field and the theoretical field (Laborde, 2004). Given the back-and-forth nature of the proving process, it is not viable to abandon visual representations in order to encourage the move toward the theoretical field—instead, learners need to be able to coordinate different semiotic resources (Duval, 2006). Once again, as we elaborate below, the status of the dynamic diagrams supported in DGEs, as well as their insistence on the spatio-graphical field, presents both opportunities and challenges when working with learners on the proving process.

The Proving Process in DGE

In terms of the back-and-forth nature of the proving process, we focus on the status of geometric objects in DGEs. Laborde (2000) considered dynamic diagrams (such as a dynamically draggable parallelogram) as scaffolding the drawing/figure divide in the sense that it remains a material object (albeit virtual on the screen), but the invariances it carries in dragging can represent its basic properties.

But Jones (2000) argued that learners in DGEs “can get ‘stuck’ somewhere between a drawing and a figure” (p. 58). This led Battista (2008) to conclude both

that the drawing/figure distinction might not be sufficient in working with geometry objects and that more attention needs to be paid to the ways in which learners *perceive* and think about DGE objects. We focus here on the latter issue: Battista drew attention to two subtly different perspectives. One was captured by Marrades and Gutierrez (2000) when they wrote: “The main advantage of DGS learning environments” is that “students have access to a variety of examples that can hardly be matched by non-computational or static computational environments” (p. 95). This theoretical perspective tends to treat examples as *representations* of the figure. The second theoretical perspective, articulated by Laborde (1992) and Battista (2008), focusses on the continuous transformation of the draggable object rather than on the set of characteristics that can be abstracted from a given set of examples. This second perspective treats the object as an entity whose behaviour can be investigated and described—not as an entity that can be represented by particular diagrams.

It is still unclear whether learners somehow naturally see the draggable diagrams as a series of examples or as one continuously changing object, and whether this depends on their previous exposure to the static geometric discourse of the typical classroom (see also Lehrer, Jenkins, & Osana, 1998). Whatever the case may be, there is strong evidence to show that dynamic diagrams support students’ transition from an exclusively spatio-graphical field to a more theoretical one by helping them attend to the visual invariance of the dynamic diagram, which can be verbally mediated through classroom discussion or teaching intervention (Battista, 2008; Laborde, Kynigos, Hollebrands, & Sträßer, 2006; Sinclair, Moss, & Jones, 2010).

Dynamic draggable diagrams can strongly affect the proving process in the sense of mediating what kinds of conjectures will be made (first phase), and what kinds of properties will be identified and then organized (second phase). In this sense, the diagram constructed in a DGE and the schemes of use activated on it by the students can be considered an actual mediator between the first and second phases, thus providing continuity—despite the epistemological discontinuity—between a conjecture (hypothetic statement) and the corresponding proof (logical deductive sequence of statements). The internalization of the schemes of use in the artefact and its transformation into an instrument help them in building the proof as final product (Arzarello, Micheletti, Olivero, Paola, & Robutti, 1998; Boero, Garuti, Lemut, & Mariotti, 1996; Olivero, 2002).

The role of DGEs in the proving process also depends on the underlying goal of proof, as perceived by teachers and students. Some have suggested that the use of DGEs could inhibit proof since students may think that a fact is empirically evident, and they do not feel the necessity of proving it (Frant & de Costra, 2000). However, as many studies have shown, exploratory experiences with DGEs do not necessarily jeopardize the development of deductive proof (Bruckheimer & Arcavi, 2001; Christou, Mousoulides, Pittalis, & Pitta-Pantazi, 2005; Guven, Cekmez, & Maratas, 2010; Hadas et al., 2000; Healy & Hoyles, 2001; Mariotti, 2006; Oner, 2008, 2009). The work of de Villiers (1990, 1997, 1998), in articulating the different functions of proof, has brought some nuance to the question of whether or not DGEs inhibit proof. He listed the following functions: verification, explanation, systematization, discovery, communication and intellectual challenge. The explanatory function of proof has been particularly prevalent in the DGE literature. In particular, proving

through surprise involves appealing to the explanatory function of proof, as described by de Villiers (1990, 1997, 1998). For example, instead of questioning the conviction empirical methods can give, de Villiers invited students to accept the evidence, but then to ask *why* the relationships they were seeing must hold. In answer to that question they were expected to find a logical chain for going from hypothesis to thesis.

DGEs are used not only in Euclidean geometry but also in non-Euclidean geometry as a means of offering representations and interactive models (such as the Poincaré disc model of hyperbolic geometry) that are more powerful than paper (Jones, Mackrell, & Stevenson, 2010). Hollebrands, Conner and Smith (2010) examined the structure of arguments that students created using a DGE in light of Toulmin's theory and presented evidence of the supporting value of DGEs in such a context.

This section has focussed on the general role of DGEs in the proving process and, in particular, the important role of open problems and the teacher's insistence on explanation in supporting this proving process. In the next section we examine in more detail the nature of the dynamic geometry tools that relate to the process of proving.

Section C: Roles of Different Instruments Typical in DGEs

The role of the tools students have at their disposal in a DGE (as for example dragging, measures, locus, trace, and so on) can be that of a bridge between the spatio-graphical and the theoretical way of looking at a diagram. This point of view is supported by data from teaching experiments at different school levels (Arzarello et al., 1998; Jahn, 2002; Laborde, 1998, 2004; Olivero & Robutti, 2007; Sinclair et al., 2010; Vadcard, 1999). In the next three sections, we focus on two central tools of DGEs, that of dragging and measuring, isolating each of them (even if they can be used together in exploration), in order to provide a more fine-grained analysis of the epistemological, cognitive and didactical implications of each one.

Focus on the Dragging Tool

Dragging is the most central and distinctive tool of DGEs and allows users to select one or more objects and to move them continuously on the screen. We are concerned with the way the dragging may mediate the process of proving, in particular focussing on the epistemological and cognitive implications of it. (There are also didactical implications, but these will be considered in section D.)

Epistemological implications of dragging. Dragging changes the figural aspect of a construction such as an equilateral triangle (the representation changes) but not the conceptual one (since all the properties of the equilateral triangle are being maintained). This duality does not arise in a static pencil-and-paper environment,

since the figural aspects are handled in a visual register and the conceptual one in the discursive register.

Since geometric proofs are meant to concern theoretical objects—and not just specific drawings—the role that dragging can play in managing the figural/conceptual duality is of particular interest. Making a conjecture about any configuration depends on the expectation of some sort of invariance under figural change. And any conjecture about an equilateral triangle must assume that the conjecture will hold true for any configuration of an equilateral triangle.

Taken as a tool of semiotic mediation, Mariotti (2006) wrote that dragging fosters students' access to the world of Geometric theory, citing Jones' (2000) study of students working with quadrilaterals, with the task of constructing a quadrilateral and modifying it into a special case with dragging. By dragging a rectangle, say, the students could observe that "A rectangle ... becomes a square" and "You can make a rectangle into a square by dragging the side shorter ...". At the end of the unit, when asked to classify quadrilaterals into a family tree, the students were also able to answer questions of the form "why is one quadrilateral a special case of another?" However, they persisted in saying that the set of more general quadrilaterals to which they were referring excluded the special case. This reveals an epistemological gap between the pragmatic, inductive argument enabled by the dragging (a square is a special case of a rectangle because you can turn the rectangle into a square) and the property-based, deductive inference required for a proof (a square is a special case of a rectangle because a square has four right angles).

Erez and Yerulshalmy's (2006) study showed that with some guidance from the teacher, drawing attention to the relevant properties, 5th grade students (aged 10 and 11) can, with dynamic geometry, view a square as a rhombus. In his study of Grade 5 children using the *Shape Maker* microworld, Battista (2007) theorized the effectiveness of dragging in terms of a two-folding assumption: first, subconscious visual transformations "are one of the mental mechanisms by which we spatially structure shapes" (p. 150). For example, the opposite sides of a parallelogram are subconsciously seen as parallel through a translation of one side onto another.

Battista's second assumption, which he called the *transformational-saliency hypothesis*, related more centrally to dragging. This hypothesis essentially stated that people notice invariance. As students drag the rhombus maker, they notice what stays invariant, namely the fact that all four sides are equal. For Battista, it was not just that one *might see* invariance in dragging but that one cannot help but notice it. He thus conjectured that "investigating shapes through *Shape Maker* transformations make the essence of the properties more psychologically salient to students than simple comparing examples of shapes as in traditional instruction" (p. 152). Dragging thus changes the way shapes are perceived, moving from a static visual apprehension to a temporal attention to what remains invariant. This hypothesis might explain the tendency students have to compare shapes using transformations, as reported both in DGE environments (Jones, 2000; Sinclair et al., 2010) and in non-DGE ones (Lehrer et al., 1998).

Leung's (2008) approach to theorizing dynamic geometry experiences—and particularly the temporal and spatial changes involved in dragging—also accorded

a central role to invariance. Leung (2008) used Marton & Booth's (1997) theory of variation to analyze the different ways in which dragging might help a learner come to know mathematics. According to Leung (2003), "in DGE, it is possible to define a way of seeing (discernment) in terms of actually seeing invariant critical features (a visual demarcation or focussing) under a continuous variation of certain components of configuration" (p. 198). He elaborated Marton and Booth's notion of *simultaneity*—experiencing different temporal or spatial "instances" of a phenomenon at the same time—which he saw as a promising agent to help bridge the gap between experimental and theoretical mathematics. Leung described two different types of simultaneity: (a) the spatial type involves experiencing different features of the same thing simultaneously; and (b) the temporal type involves experiencing instances that have previously been encountered at different points in time, at the same time (for example, seeing three animations that had been viewed one after the other, all at the same time). Interestingly, these loosely correspond to Battista's two perspectives on dynamic diagrams: the spatial type corresponds to the multiple example perspective and the temporal type to the continuous transformation perspective. In Leung's example of the temporal simultaneity, two things were varying simultaneously and the observer was trying to identify some kind of invariance. The spatial simultaneity will involve comparing and contrasting the similarities and differences among the four types dragging experiences. Dragging is central to both types of simultaneity, as is the experience of invariance. Indeed, simultaneity can only enable discernment and awareness if invariances can be seen, an assumption that relates strongly to Battista's *transformational-saliency hypothesis*.

Leung's (2008) case study showed how these two types of simultaneity not only lead to the generation of a conjecture but also to the construction of what he calls a "DGE proof" (p. 146). Although Leung did not dwell on the transition from conjecture to "DGE proof," it is worth attending to the epistemological nature of the transition, particularly in terms of the role of dragging. The conjecture identifies an invariance. In Leung's case study, very briefly, the problem is as follows: For a quadrilateral $ABCD$, under what conditions is $\angle ABC = 2\angle ADC$ (see Figure 19.1a). Through explorations in dragging, Leung saw that if A, B, C are left fixed and D is dragged, then $\angle ABC = 2\angle ADC$ when D seems to be on a circle passing through A and C . Moving to a DGE proof then, involves first constructing the circle c_4 through A, B and C , then constructing the perpendicular bisector of AC , then finding the intersection E of this perpendicular bisector with c_4 (see Figure 19.1b). If D is merged to the circle passing through A and C , centred at E , then $\angle ABC = 2\angle ADC$.

The role of dragging has shifted from interrogation to declaration. The construction process is *declarative*, in that it asserts the circle, as well as *demonstrative* in that it shows how the geometric configuration can be put into action (much like Euclid's first proposition, which shows how an equilateral triangle can be constructed). As Jackiw (2006) wrote, the DGE proof follows a long tradition of mechanical demonstration in the history of science and mathematics, one that is replete with personal agency and aesthetic satisfaction. While not providing the propositional sequence of deductions required in a formal, written proof, the "DGE proof" is epistemologically much closer to the explanation than to verification and,

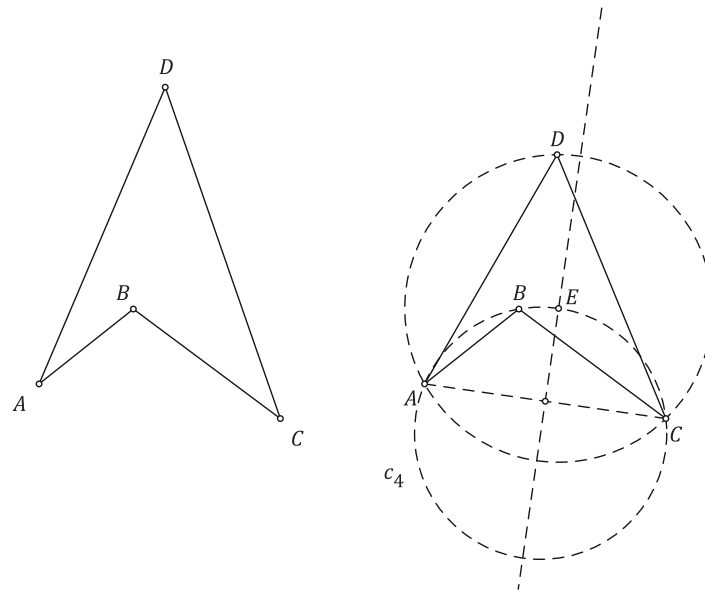


Figure 19.1. (a) The general configuration; (b) Under what conditions is $\angle ABC = 2\angle ADC$.

Cognitive implications of dragging. Dragging can be seen as semiotic tool used to express meanings and solve problems (Mariotti, 2002). It can also reveal students' thinking processes as they work on tasks. By observing students solving open problems, Arzarello, Olivero, Paola, and Robutti (2002) identified several distinct dragging modalities that were used, and that revealed specific features of their thinking. They have been widely cited and used by researchers trying to understand students' problem solving and proving activity. We highlight some of them here in order to stress the links to proving. Consider the blackbox situation given in Figure 19.2. If students are asked to identify the transformation leading from A to A' , they may begin by wandering dragging in which they drag point A more or less randomly on the screen. They have no plan yet, and no conjectures about how A and A' are related. In their wandering dragging, they may notice that there are places where the two points meet. They may therefore engage in "dummy locus" dragging in which they attempt to drag A along a path that keeps A and A' together; here they are following a hidden path (the dummy locus, in this case a line) even without being aware of it. In order to keep track of these places where A and A' overlap, they may engage in line dragging, in which they draw new points on the spots of overlap, or in guided dragging, in order to obtain and observe a particular situation (for example moving A towards A'). They will thus see that the spots form a line, at

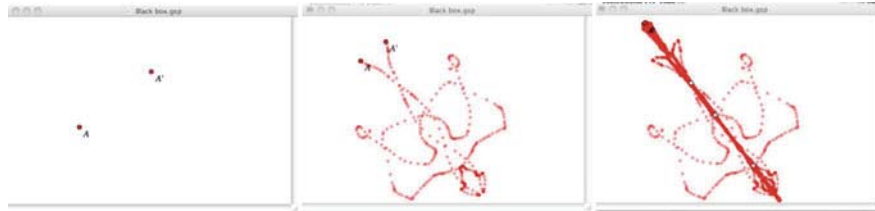


Figure 19.2. Moving from wandering to dummy dragging.

which point they may construct a line and use linked dragging to merge the point to the line and move it along that line.

These drag modalities structure a situated hierarchical scheme that, as suggested in the above example, are indicative of different modes of thinking such as exploring, forming conjectures, testing conjectures, and verifying conjectures.

The final type of dragging identified by these researchers is called the *dragging test*, and involves dragging points in order to verify whether a figure keeps its initial properties: if a student constructs an equilateral triangle, he/she then uses the drag test to see whether moving one vertex maintains the triangle as equilateral.

These modalities can be grouped in two broad categories, characterizing the moves between the spatio-graphical field and theoretical field: dragging for exploring/conjecturing and dragging for validating a conjecture or proof. In the first case, students drag objects observing the figure in search for regularities and invariants (e.g., wandering dragging): once they have found one and they express it through a conjecture, they shift from the spatio-graphical field to the theoretical field. And vice versa, they shift backward if they drag objects to check a conjecture already discovered (e.g., guided dragging).

To this list of drag modalities, Leung (2008) added one that relates to the parametric colour tool, which enables users to make the colour of an object depend on a measured quantity. For example, one could have the colour of a circle depend on its distance from a fixed point: dragging the circle, its colour would thus change. While this *spectral dragging* often involves moving the mouse (as in the example given above), it is not fundamentally about motion; rather, its power is more in the chromatic feedback offered. Leung's study focusses less on the cognitive dimension of dragging than, as he says, on the mathematical relevance of dragging strategies. He identified different types of dragging that are based on different ways of attending to variation: dragging for contrast, for separation, for generalization and for fusion. As with the dragging modalities of Arzarello et al. (2002), these are all related to conjecturing and proving.

Focus on the Measurement Tool

Measuring is a powerful tool of dynamic geometry environments, and also a complex one that requires appropriate management and interpretation, and may be

used by students with varying degrees of confidence and awareness. Actually, students using a DGE in geometry tasks can utilize measures in different ways, according to the description given for dragging: in the spatio-graphical field, if they are more concerned with the perception of the drawing, or in the theoretical field, if they are more concentrated with the properties of the figure. According to their use of measurements, students can construct mathematical meanings, formulate conjectures, check them or use them to construct a proof, in a continuous process from exploration to the final product of a formal proof.

Epistemological implications of measurements. One of the specific features available in a DGE, the measuring tools, within the context of open geometry problems, can offer deep insight on the figures and ideas for the formulation of conjectures, especially if used with the dynamicity offered by dragging. However, from an epistemological point of view, the double meaning of measuring (in mathematics and in physics) can also cause misunderstanding or conflict in the proving process, in the interplay between theoretical and graphical aspects of figures. Actually, if we consider the meaning of measures in mathematics, we have a function that gives a unique real number associated to a quantity (that can be rational or irrational): for example, in the case of diagonal of a square measured with its side, it is irrational (the square root of 2). Measuring in physics and experimental sciences has a different epistemological status, because it depends on the kind of tool and its sensibility, on the number of measurements, and on calculation of uncertainty. In this case, measuring never provides an irrational number, but always a rational number, with a precision (number of significant digits) that depends on the tool and on the process, and gives as result a number in an uncertainty interval.

When using a DGE, things are more complex, and go further than this double meaning, because measurements are given as in physics (using a virtual ruler), but the software simulates a geometric environment (the Euclidean one) where the measure is mathematics (Olivero & Robutti, 2001). In a DGE, users can construct geometric figures according to Euclidean properties, but it is not the Euclidean plane, because there is no continuity, but a finite number of pixels. In this plane, computations do not have infinite precision, but they are approximate. For these reasons, coordinates of points, amplitudes of angles, and measurements in general are written with a finite number of decimal digits. The user can increase that number of digits, however the precision of the tool does not increase, because there is a technological limit to the precision of the measuring tools: the dimension of the pixel.

The outcome of these features is that many different things may happen when students are measuring in a DGE (length of segment, amplitude of an angle, and so on). For example, one might see on the screen two equal lengths corresponding to segments that are known to be not equal from a theoretical point of view, or the additive property of measures is not always satisfied (Figure 19.3). These examples show that some geometric properties may appear not to be satisfied when looking at measurements in a DGE.

When students approach problems in a DGE and find a situation like the one in Figure 19.3, they may remain stuck and attribute the mistake to the software, or may re-consider again their work from the beginning, thinking they have made a mistake.

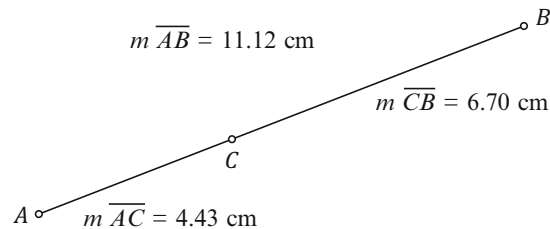


Figure 19.3. The length of AB is not equal to the sum of the lengths of AC and CB.

Otherwise, if they are aware of the epistemological implications of the use of measures in a DGE (approximation, pixels, calculation, and so on), they can look at their figure in a theoretical way, in order to interpret it with the constraints of software use. From a didactic point of view, de Villiers (1999) showed that one can also use the finite limitations of a DGE to design specific situations where students are not absolutely certain in order to motivate a need for proof as a means of further verification.

Cognitive implications of measures. The epistemological implications described above imply a complexity where measure interpretation and management surely creates opportunities for construction and exploration, but it can also be a source of conflict. Actually, the double meaning of measure in a DGE is not always transparent to students, and can sometimes act as a black box, where students accept as “true” a value of a measurement they read on the screen, without treating it contextually with some uncertainty or approximation. Students’ competence in managing measures depends on their awareness of their epistemological status, for which the teacher’s role is fundamental.

As for dragging, also for measure in a DGE, the different uses of the tool can correspond to different cognitive activity, from a spatio-graphical level to a theoretical one, or backwards. In fact, students may use the measure tool at times for discovering and conjecturing and at times for validating a conjecture. This was shown by Vadcard (1999), in his study on the use of measuring in *Cabri*, which distinguished between *mesure exploratoire*, used mainly as a heuristic tool and *mesure probatoire*, used as a checking tool. Furthermore, it is possible to split each category in different modalities (Olivero & Robutti, 2007). The first category (*mesure exploratoire*), related to the shift from the spatio-graphical field to the theoretical field, can be divided into different modalities. One of them is *wandering measuring*, when students do not have any precise ideas about the configuration—so they explore the situation randomly: they take measurements of some elements of the configuration, in the same way as they might use wandering dragging, in order to identify quantitative relations, invariants, congruencies, etc. Of course, it seems to be important *which* quantitative relationships will be identified: as Sinclair (2002) argued, only certain ones will be deemed interesting enough—for example, in wandering dragging that involves measuring angles, certain values (such as 90° , 120° , etc.) will seem much more relevant than others, because of an aesthetic component in conjecturing,

in the sense that only some configurations, invariances, or measures are worth attending to. If students trust measurements and they believe measurements are absolutely exact, they stay at a spatio-graphical field. Instead, if they use the information provided by the reading of the measurements to formulate a conjecture in the standard form “if ... then ...” and in a general case (“generic example,” in the sense of Balacheff, 1987), they connect the spatio-graphical field with the theoretical one.

Another modality is *guided measuring* (echoing guided dragging), when students do a guided exploration of the configuration, examining particular cases one after the other: measurements are used in order to obtain a particular figure from a generic configuration; for example a generic quadrilateral can be transformed into a parallelogram by looking at the length of the opposite sides. This modality can be useful to put in order (Guala & Boero, 1999) a set of different cases, with the aim of exploring them—from, for example, the most particular to the most general.

A third is *perceptual measuring*, when students use measurements as a means of checking the validity of a perception: having the intuition on a property, but not being sure of it, they use measurements to validate the perception, by transforming a qualitative relationship into a quantitative one, and remain in the spatio-graphical field. They may jump to the theoretical field if they then transform that perception into a conjecture.

On the opposite side, the second category is related to the shift from the theoretical field to the spatio-graphical field, when students need to check the validity of an intuition, a conjecture, even a proof, and use *mesure probatoire*. In the *validation measuring* modality, students use measures to check whether a conjecture should be accepted or refuted. This use is very similar to the dragging test, for checking a construction. Another modality is *proof measuring*, not frequently used, where students go back to the DGE after having constructed a proof, in order to check it or gain conviction. In this case, new experiments are made in the DGE and measurements are used to “validate” the proof from an experimental point of view, or to have data against the proof itself.

Students’ competencies in using measurements discriminate their approach to proof. If they read values in the DGE, giving them a proper approximation and range of uncertainty, it could be useful both in the passage from spatio-graphic to theory and backwards, otherwise some problems may occur (one is described in the data below). In terms of the use of the measuring, researchers have studied the ways students use measures in the proving process (perceptually or theoretically?), with their awareness in doing so (knowing its epistemological status and its role in a proof or not?), and the relation of their use of measures to the construction of a proof (support or against?). We exemplify some findings of this research in section D.

Other DGE Tools

The dragging and measurements tools are central in DGEs, and have been studied extensively, but they are not the only ones: there are many other tools at a learner’s disposal including trace, locus, transformation, animation, parametric colours,

macro-constructions or custom tools, iteration and graphing. Mariotti (2002) described the way in which tracing and locus can work together with the former providing exploratory evidence of a relationship and the latter a confirmatory construction. While trace guides a student to the solution of the problem, locus supports investigation on specific characteristic of the figure or relationships among its components. Locus and tracing have been shown to be effective in problems where students construct and use transformation, and are engaged in finding invariants (Jahn, 2002), or in problems where they are involved in guessing what kind of transformation is at the basis of a given construction—as in Laborde's (2001) black boxes. While dragging and measurement will remain the focus of attention, we anticipate more work in the next decade on the use of these other tools in the proving process.

Section D: Protocols in DGE Research Around Proving

In this section we offer two case studies of student interactions with a DGE. In section C we focussed on the two tools of dragging and measuring, charting out their cognitive and epistemological implications across a range of topic areas and grade levels—particularly as they relate to the proving process. Here we consider particular classroom situations dealing with specific content, tasks and grade levels, as well as specific theoretical lenses that can be used to study geometric thinking. The first example, reported by Sinclair et al. (2010), involved primary grade children engaging in geometric argumentation in the context of a whole classroom exploration around identifying triangles. Dragging is a central feature of this example but the analysis focusses less on the modality of dragging used than on the discursive changes in students' talk about triangles. The second example, reported by Olivero and Robutti (2007), involved high school students working at a more sophisticated level of the proving process, in the context of a teaching experiment in which pairs of students solved an open problem that centrally involved the use of the measure tool.

Dragging at the Elementary School Level

This case study involves children 4–5 years old who had never seen the software before, and had not engaged in any formal work on geometry at that point in the year (January). The task was designed in such a way to promote exploration, conjecturing and argumentation. Although the episode is not directly about proof, we include it because we see it as belonging to the initial stages of the proving process, including the move to the theoretical and the use of definitions. The episode involved students engaged in identifying triangles. We focus attention on the different ways that students talked about the shapes they see and their developing discourse as they encountered a broader range of dragged triangles.

The teacher began by asking “What is a triangle?” The children use their hands and bodies to form triangles. There was no reference to specific properties of the shape. The teacher then constructed a triangle with its vertex facing down using the segment tool. Most of the children did not think this was a triangle. Some turned around, and said it was a triangle only if they looked at it upside down. The teacher then dragged one of the vertices of the triangle.

-
- | | | |
|-----|----------|---|
| 34. | T: | I’m going to ask you to tell me what you see when I’m doing this. |
| 35. | Ss: | Ooh. Wow. [<i>some laughing</i>] |
| 36. | T: | What do you see? |
| 37. | Abigail: | That’s actually a triangle. |
| 38. | Leah: | A triangle. |
| 39. | John: | It is. |
| 40. | T: | Why is it a triangle? [<i>continuing to drag</i>] |
| 41. | Robert: | You can stretch it out. |
| 42. | T: | And that’s what makes it a triangle? |
| 43. | Robert: | Yeah. |
| 44. | Dasia: | Every triangle |
| 45. | John: | You’re going to make |
| 46. | T: | Just wait. Dasia, what are you saying? |
| 47. | Dasia: | Every triangle could be, um, a different shape but it just has three corners. |
-

Although brief, this exchange was long enough to show a process of discourse change in real time. The change was signalled with a collective expression of surprise (see “Ooh. Wow” in [35]). Abigail’s statement “That’s actually a triangle” [37] signalled that what so far was considered to be a triangle only *potentially*, pending a person’s position with respect to the shape, now became a triangle as-is, unconditionally. Since it was now obvious, even if not actually shown, that this animated thing could easily reincarnate into the canonical shape she had always considered as a triangle, that canonical shape became but one of the possible images of the *moving* object called triangle. The follow-up comments by Leah [38] and John [39] suggested the same interpretation. Most notably, Robert explicitly signalled having changed his mind when he explained that whatever he could see on the screen, including the original “upside-down” shape, was a triangle simply because it could be obtained from one basic triangular shape by “stretching it out” [41]. A significant aspect of the children’s talk was that it shifted toward the hypothetical and the abstract: so far, a triangle used to be something that was present visually, but the children now spoke of it as an object that changes over time.

Dasia’s talk stood out in three respects. First, she spoke in plural, mentioning many triangles rather than just one (see her expression “every triangle” in [44]). Second, she referred explicitly to properties of a triangle (see her mention of “three corners” in [47]; this is similar to, but still different from Abigail’s earlier mention of three sides). Finally, she turned Robert’s statement around into a more general description of a triangle. On the basis of [47] we can conclude that having three corners was, for her, the only condition a shape had to fulfil to be called “triangle.” With the word “just” in her requirement of “just has three corners” Dasia communicated the idea that a triangle can have many other characteristics (be upside down, stretched, etc.), but having three corners was the only property that was really indispensable.

Wishing to elicit the difficulty that children might have in identifying stick-shaped triangles as triangles, the teacher asked the students to close their eyes so that she could drag the triangle into a more “monstrous” (very long and skinny) shape. When asked whether it is a triangle, some students say “no” and others “yes.” After some discussion, Morgan remained the only dissenter.

-
- | | | |
|-----|-----------|---|
| 82. | Dasia: | It's a triangle for me though. |
| 83. | Robert: | It's a triangle for me too. |
| 84. | Other Ss: | Me too. |
| 85. | T: | But we have to try and explain why it's a triangle for Morgan. You have to convince her. Why do you think it's a triangle because she's not sure? |
| 86. | Dasia: | Because it has three corners. |
| 87. | T: | What else does it have? |
| 88. | Nadia: | Three lines. |
| 89. | T: | We need to come up with a definition we all agree on and so far your definition is three vertices, or you can call them corners, and three sides. |
| 90. | Michael: | Well it's a triangle as long as it has a point and as long as it had three dots and as long as it has three lines. |
| 91. | Leah: | Yeah whatever you do with it, it's still a triangle. |
-

Dasia, whose utterance [86] showed once again that for her deciding whether a drawing was a triangle required just checking whether it was showing three corners. In contrast, Morgan's use of triangle included a broader class of transformations of the canonical one, the routine of identification was still exclusively visual, with no appeal to properties. The rest of the class, however, could now speak about a *family* of triangles rather than a single one, and the family was unified by common properties. Dasia's and Michael's utterances [82] and [90], respectively, showed an interesting discursive gap: whereas Dasia appealed only to properties that were both necessary and sufficient, Michael added to the “three dots” the requirement that there must also be “a point” and “three lines,” thus making his statement more of a description than a definition.

The children's talk bore clear evidence of its being developed in a DGE. With the phrases “as long as,” (Michael, [90]) and “still” (Leah, [91]) the children communicated their reliance on transformability in trying to identify a triangle—thus supporting Battista's hypothesis. Leah, in particular, used a dynamic, material language in which a triangle was something you could do to things. Her “whatever you do” referred to what could be done with one of the triangle's corners. Unlike Michael, she did not draw on the properties; her statement spoke to the variability she thought the triangle had.

We see this episode as relevant to the proving process in that the development of the notion of triangle is not achieved through memorization of vocabulary but, instead, through a back-and-forth between the visual and discursive registers. In prompting the students to use words, to compare their visual identifications with their verbal statements, and to consider counter-examples, the teacher supported the use of explanation. The students moved toward deductive statements such as “that shape is a triangle *because* it has three sides.” Further, this excerpt revealed the way in which the work on identifying shapes as simple as triangles, in the primary grades,

can be vectored toward the development from description to definition—a crucial one in the proving process.

Measurement in a High School Setting

Students in the 10th grade were involved in a teaching experiment aimed at introducing the proving process as a continuing activity from construction of a figure in a DGE and its exploration until the formulation of a proof, via some conjectures. The teacher emphasized the need for proof as the explanation of *why* a statement is valid within some hypothesis. The students, adopting this didactic contract, knew that once they have found one or more conjectures in the geometric situation, they had to justify and explain them. The research study started from the hypothesis (Arzarello et al., 1998, 2002) that the DGE can support a cognitive continuity from conjecturing to proving, whereas conjecture and proof have two different epistemological statuses

In the classroom sessions (each of them lasted about two hours) the students were engaged in solving an open problem working in pairs with *Cabri* and sharing their solution at the end of the group session. Discussion was coordinated by the teacher. Data collected were field notes and videotapes of the work, made in *Cabri* and in paper and pencil. With these data, researchers can pay attention to students' approaches to measurements and to their cognitive role during the activity. The data showed students' frequent passages from the spatio-graphical to the theoretical field.

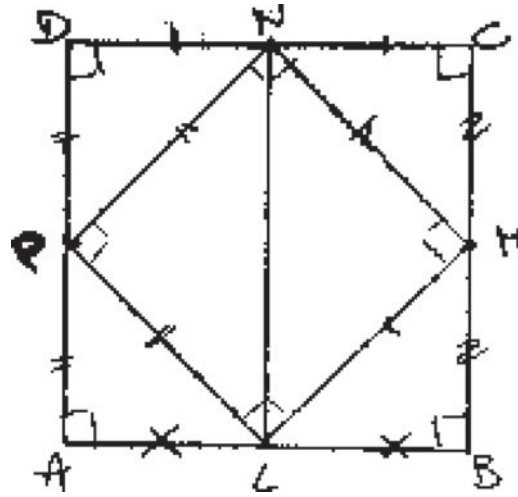
In the example we present here, two students (Alessandra and Tiziana) in their working group explored a geometric situation, made conjectures and proved one of them. After having done this, they used measurements to check their proof. So, they did a passage from their theoretical approach (doing the proof) to a spatio-graphical approach again (checking the proof with measures), which showed the use of the measurement tool in modality *proof measuring* of the list above. At this point, students were surprised to find that the two results, coming from theory and experiment, were not coherent (Olivero & Robutti, 2002, 2007). These students were medium achievers; they had not used *Cabri* very often before, and had no substantial experience with open problems before this one.

The problem given to the students is “Varignon’s problem,” formulated as follows: Draw any quadrilateral $ABCD$. Draw the midpoints L, M, N, P of the four sides.

1. Which properties does the quadrilateral $LMNP$ have?
2. Which particular configurations does $LMNP$ assume?
3. Which hypotheses on the quadrilateral $ABCD$ are needed, in order for $LMNP$ to assume those particular configurations? Make conjectures and prove them.

At the beginning the two students explored the situation in *Cabri* and formulated the following conjecture: if the external quadrilateral is a square, the internal quadrilateral is a rhombus. After that, they proved the conjecture, based on Figure 19.4 drawn on paper. Note that it is not surprising that the students first call the “inside”

Figure 19.4. Proving that
“If $ABCD$ is a square then
 $HKLM$ is a rhombus.”



shape a rhombus, because their shape identification strategies were visually oriented, namely based on a visual association with a prototypical shape—they do not call it a “square” because its sides are not vertical and horizontal.

-
- 181 Tiziana All this stuff...these...they are congruent (the halves of the sides of $ABCD$).
[Then Tiziana writes down the thesis: LM equals MN , equals NP , equals PL . Meanwhile Alessandra uses a ruler to measure the sides of $LMNP$.]
- 189 Tiziana So NC is congruent to MC PAL (angle) is congruent to NCM (angle) because $ABCD$ is square [she writes down all this]. And so it has got right angles, right? [She marks the right angles in $ABCD$]. We didn't write this in the hypothesis, but shall I explain this?
- 193 Tiziana So PL equals MN . The same for PDN triangle and LBM triangle then PN equals LM . Should I do a cross comparison? PDN triangle and PAL triangle then PN equals PL . What's missing? These two are done, these two are done
- 195 Tiziana NCM triangle and MBL triangle so NM congruent to ML
- 197 Tiziana They all have equal sides. So it is a rhombus! OK!
-

The students constructed a proof with all the logical passages required at their school level. They used the congruence of triangles in which the figure was decomposed (Figure 19.4) and the congruence theorem SAS. At the end of their proving process, they were sure of the validity of their conjecture, because they had proved it, i.e., they had deduced the thesis (the quadrilateral inside the square is a rhombus) from the hypothesis (the starting quadrilateral is a square)—thus appealing to the verification function of proof.

To be convinced of their theoretical approach, at this point, after the construction of their final proof, the two students went back to *Cabri* to look for validation—that

is, empirical evidence. So, a new episode started up, marking a passage from the theoretical to the spatio-graphical field:

207	Tiziana	Try to make $ABCD$ a square
208	Alessandra	[Drags A and B trying to obtain a square with a side of 6.17 cm]
209	Tiziana	Is it a rhombus?
210	Alessandra	Well ... [Alessandra points at the sides of $LMNP$]. A rhombus has got equal sides oh no! It's not a rhombus! (see Figure 19.5)

The two students dragged $ABCD$ into a square (using guided dragging, along with a guided use of measurements), to check if $LMNP$ is a rhombus. And to do this, they dragged $ABCD$ not randomly, but checking measures of its sides, in order to obtain equal values. They were transforming a generic quadrilateral $ABCD$ into a square just by dragging (and looking at the side measures), not by constructing it. Moreover, they used the *proof measuring* modality in that they wanted to find the empirical evidence of their theoretical finding. This passage from theoretical to spatio-graphical was not so unusual (even for mathematicians!) and could give different results. The two students were surprised when they found that such numbers were not equal (PN and PL , for example, in Figure 19.5). These students had more confidence in the measurements taken by *Cabri* than in the correctness of their proof. In fact, they immediately checked each passage of the proof but concluded that the software was right—and so they rejected their proof:

257	Tiziana	Because if this is the midpoint [<i>she points at P</i>] then it divides this side in two equal parts.[<i>She points at AD and AP and PD</i>].
263	Tiziana	So it should be: if it is a square, the quadrilateral inside is a square too.
264	Teacher	Right!
265	Tiziana	Why doesn't the figure show that?
274	Teacher	I'll try to pose the problem from a different perspective: this figure. [<i>Points at the Figure 19.4 on the screen</i>] Is it a square? Really?

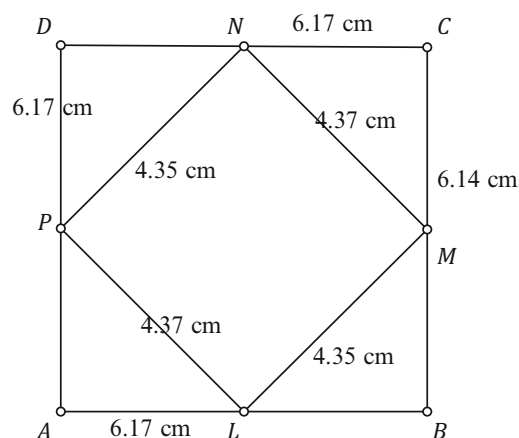


Figure 19.5. Measuring the sides of the inner quadrilateral.

The teacher's intervention focusses students' attention on the link between the spatio-graphical and the theoretical, helping them to realize that their figure (Figure 19.5) was *not* properly a square. Furthermore she suggested that they *construct* a square, and check if it was in accordance with the proof. In the sense of Laborde, these students were looking at a square as a sketch, not at a *Cabri*-square (a dynamic construction). After the teacher's suggestion, the students constructed a square and thus overcome the conflict. The use of measurements described above seemed to detach the students completely from the theoretical field and made them dependent on the spatio-graphical one.

For these students, the intervention of the teacher was crucial for overcoming the conflict and using the figure in a theoretical way, accepting the proof and going further in solving the problem. This case study thus illustrates the epistemological and cognitive implications of the measuring tool in relation to the proving process and offers a specific example of the kind of work that a teacher must do to support this process.

Section E: Conclusion and Implications

Didactic Implications of DGEs

As observed in Hoyles and Noss's (2003) *Second Handbook* chapter, research has focussed not just on the affordances of the tool, but also on the role of the teacher in the classroom. In both of the case studies shown above, the teacher played a crucial role in emphasizing relationships between objects, while trying to introduce students to the theoretical world of geometry, which is similar to Mariotti's (2000) thesis. However, as shown in Sinclair and Yurita (2008), it can be challenging for teachers to play this role, particularly when their way of thinking and talking about geometry is static in nature. Although it is clear that teachers must help students move towards explanation and deduction, there is still a great deal of research to be done on how teachers should deal with the epistemological and cognitive implications of DGE tools such as dragging and measuring. Should, for example, the various dragging modalities be modelled by the teacher? How might a teacher choose tasks that can scaffold dragging modalities in such a way to better support the proving process? Do teachers need to be more careful about how and when they introduce measure tools? How can teachers carry out in class an instrumentation of the different tools present in a DGE, in the sense of giving students schemes of use of these tools and awareness of their use according to the kind of problem (construction, exploration, modelling, and so on)?

The orchestration of different materials and technologies (Trouche, 2004), and the use of new teaching practices (working groups, discussions, distance education), are difficult and can create confusion for teachers. However, many national and international projects of teacher education are directing their efforts at supporting teachers in changing (see Sinclair, Arzarello, Gaisman, & Lozano, 2009). More research, along

the lines of Laborde's (2001), needs to focus on developing better teacher integration, including supporting the development of new tasks that are suitable to DGE use and new modes of assessment that do not simply fall back to pencil-and-paper testing.

Research Implications of DGEs

Hoyles and Noss (2003) stated that "attention is turning away from the investigation of the process of construction and conjecturing with DGS, and towards consideration of how the new tools mediate the nature of explanation, verification and even proof" (p. 335). Since the early 2000s, we have seen a strong emphasis on how those two processes are related, which has led to an increasing range of contexts of study, across the grade levels (from primary school to university level, including pre-service teacher education—see Lassak, 2009). This may lead to more longitudinal research on the way in which the proving process might develop over the school curriculum—such longitudinal studies have been quite rare in the field of technology, but they would be particularly helpful in elucidating the nature of the proving process. In particular, as anticipated in the first case study, we need new models, or learning trajectories, that can account for the impact of DGE use in the development of student thinking.

Attention is also turning to the ways by which DGEs can mediate the objects of geometric investigation and the ways students can think about these objects, as discussed in section C. With the fine-grained analyses of dragging and measuring already undertaken, we tried to show that the dynamicity and the measure give students the opportunity to pass from the exploration phase, characterized by conjectures, to the proving phase, characterized by the construction of proof, with continuity.

Over the past decade, researchers have identified some of the more subtle aspects of learning to use and teach with DGEs. For example, the literature suggests that teachers need to help students develop "schemes of use" (Rabardel, 1995)—not just how to drag a point, but why a point or an object is not draggable, how and when to use a measure to infer a property, how to read a measure in a proper way, and how to use dragging to obtain a particular configuration. This awareness of the schemes of use enables students to master the problem, from the hypothesis to the thesis, during the various solution phases (Sträßer, 2009). More research is needed to study how (and how much) teachers should "instrument" their students in the use of technologies. A delicate balance is needed since, on the one hand, it is important to introduce schemes of use in a cognitive and metacognitive way, but on the other, this activity should not become a sequence of instructions and rules, but an investigation methodology. In fact, tools are active elements of the culture in which they are used as they involve cognitive structures (schemes) and also introduce cultural systems that are products of cultural evolution. For this reason, we have to examine the closeness of the notion of "scheme" to that of "situated abstraction" (Noss & Hoyles, 1996) as a complex product of activity, context, history and culture. And we have to examine the complexity of the relation between humans and technology (Borba &

Villarreal, 2005)—not only as one subject interacting with one tool, but as *more subjects* interacting together and with *more tools*.

For what concerns *more subjects*, Trouche (2004) notes that: “an instrument is the result of a construction by a subject, in a community of practice, on a basis of a given artifact, through a process, the instrumental genesis” (p. 289). The social dimension of the tool can be analyzed according to different theoretical perspectives, such as semiotic mediation (Mariotti, 2010) or the human, embodied, cultural and multimodal perspectives put forward by Radford (2010) and Arzarello and Robutti (2010). These frames consider all the signs introduced by students (verbal utterances, gestures, bodily movements, actions on the computer, etc.), during a classroom activity to be relevant for understanding cognitive processes.

In terms of *more artefacts*, we again point to Trouche’s emphasis on the importance of “instrumental orchestration,” which involves the design of tasks, the guidance of students’ instrumental genesis, and the environmental organization (Trouche, 2004). Instrumental orchestration is achieved by “didactic configurations” (layout of available artefacts in the environment), “exploitation modes” of these configurations (decisions for carrying out the activities with artefacts) and “didactical performance” (ad hoc decisions taken while teaching on how actually to perform in the chosen didactic configuration and exploitation mode). Recent trends in research study the types of instrumental orchestration that emerge in technology-rich classroom teaching and to what extent are teachers’ repertoires of orchestrations related to their expressed views on mathematics education and the role of technology (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010). They should inform research on the use of DGEs in the proving process.

References

- Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analysing the transition to formal proofs in geometry. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 24–31). Stellenbosh, South Africa: International Group for the Psychology of Mathematics Education.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (1999). I problemi di costruzione geometrica con l’aiuto di Cabri. *L’insegnamento della matematica e delle scienze integrate*, 22B(4), 309–338.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM—Zentralblatt für Didaktik der Mathematik*, 34(3), 66–72.
- Arzarello, F., & Robutti, O. (2010). Multimodality in multi-representational environments. *ZDM—The International Journal on Mathematics Education*, 42(7), 715–731.
- Baccaglioni-Frank, A., & Mariotti, M. A. (2010). Generating conjectures in dynamic geometry: The Maintaining Dragging Model. *International Journal of Computers for Mathematical Learning*, 15(3), 225–253.
- Balacheff, N. (1987). Processus de prevue et situations de validation. *Educational Studies in Mathematics*, 8, 147–176.
- Bartolini Bussi, M. (2009). Proof and proving in primary school: An experimental approach. In F.-L. Lin, F.-J. Hsieh, G. Hanna, & M. De Villiers (Eds.), *Proceedings of the ICMI Study 19 Conference: Proof and proving in mathematics education* (Vol. 1, pp. 53–58). Taipei, Taiwan: National Taiwan Normal University.

- Battista, M. T. (2007). The development of geometric and spatial thinking. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Reston, VA: National Council of Teachers of Mathematics.
- Battista, M. T. (2008). Representations and cognitive objects in modern school geometry. In G. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Cases and perspectives* (Vol. 2, pp. 341–362). Charlotte, NC: Information Age Publishing.
- Baulac, Y., Bellemain, F., & Laborde, J. M. (1988). *Cabri-Géomètre, un logiciel d'aide à l'apprentissage de la géométrie: Logiciel et manuel d'utilisation*. Paris, France: Cedic-Nathan.
- Bishop, A. J., Clements, M. A., Keitel, C., Kilpatrick, J., & Leung, F. K. S. (Eds.). (2003). *Second international handbook of mathematics education*. Dordrecht, The Netherlands: Kluwer.
- Boero, P., Garuti, R., Lemut, E., & Mariotti, A. M. (1996). Challenging the traditional school approach to theorems: A hypothesis about the cognitive unity of theorems. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 113–120). Valencia, Spain: International Group for the Psychology of Mathematics Education.
- Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking*. New York, NY: Springer.
- Borwein, J., & Bailey, D. (2008). *Mathematics by experiment: Plausible reasoning in the 21st century*. Wellesley, MA: A. K. Peters/CRC Press.
- Bruckheimer, M., & Arcavi, A. (2001). A Herrick among mathematicians or dynamic geometry as an aid to proof. *International Journal of Computers in Mathematics Learning*, 6, 113–126.
- Centre de Recherche sur l'Enseignement des Mathématiques (CREM). (1995). *Les mathématiques de la maternelle jusqu'à 18 ans. Essai d'élaboration d'un cadre global pour l'enseignement des mathématiques*. In L. Grugnetti & V. Villani [Italian edition, *La matematica dalla scuola materna all'Università*]. Bologna, Italy: Pitagora Editrice.
- Christou, C., Mousoulides, N., Pittalis, M., & Pitta-Pantazi, D. (2005). Proofs through exploration in dynamic geometry environments. *International Journal of Science and Mathematics Education*, 2, 339–352.
- de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17–24.
- de Villiers, M. (1997). The role of proof in investigative, computer-based geometry: Some personal reflections. In J. King & D. Schattschneider (Eds.), *Geometry turned on* (pp. 15–24). Washington, DC: Mathematical Association of America.
- de Villiers, M. (1998). An alternative approach to proof in dynamic geometry? In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 369–393). Mahwah, NJ: Lawrence Erlbaum.
- de Villiers, M. (1999). *Rethinking proof with the Geometer's Sketchpad*. Emeryville, CA: Key Curriculum Press.
- de Villiers, M. (2010). The role of experimentation in mathematics and mathematics education. In G. Hanna & H. N. Jahnke (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives* (pp. 205–222). Basel, Switzerland: Springer Books.
- Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, 75(2), 213–234.
- Drijvers, P., Kieran, C., Mariotti, M., Ainley, J., Andresen, M., Chan, Y., Dana-Picard, T., Gueudet, G., Kidron, I., Leung, A., & Meagher, M. (2010). Integrating technology into mathematics education: Theoretical perspectives. In C. Hoyles & J. B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain* (pp. 89–132). New York, NY: Springer.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in the learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.
- Erez, M. M., & Yerulshalmi, M. (2006). “If you can turn a rectangle into a square, you can turn a square into a rectangle . . .”: Young students experience the dragging tool. *International Journal of Computers for Mathematical Learning*, 11, 271–299.

- Frant, J. B., & de Costra, R. M. (2000, July). *Proofs in geometry: Different concepts build upon very different cognitive mechanisms*. Paper presented at the ICME 9, TSG12: Proof and Proving in Mathematics Education, Tokyo, Japan.
- Guala, E., & Boero, P. (1999). Time, complexity and learning. *Annals of the New York Academy of Sciences*, 879, 164–167.
- Guven, B., Cekmez, E., & Maratas, I. (2010). Using empirical evidence in the process of proving: The case of dynamic geometry. *Teaching Mathematics and Its Applications*, 29(4), 193–207.
- Hadas, N., Hershkowitz, R., & Schwartz, B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, 44, 127–150.
- Hanna, G., Jahnke, H. N., & Pulte, H. (Eds.). (2010). *Explanation and proof in mathematics: Philosophical and educational perspectives*. New York, NY: Springer.
- Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potentials and pitfalls. *International Journal of Computers for Mathematical Learning*, 1(3), 235–256.
- Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education*, 33(3), 176–203.
- Hollebrands, K. F., Conner, A. M., & Smith, R. C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. *Journal for Research in Mathematics Education*, 41(4), 324–350.
- Hollebrands, K., Laborde, C., & Sträßer, R. (2008). Technology and the learning of geometry at the secondary level. In K. Heid & G. Blume (Eds.), *Research in technology and the teaching and learning of mathematics: Research syntheses* (Vol. 1, pp. 155–203). Charlotte, NC: Information Age Publishing.
- Hoyles, C., & Noss, R. (2003). What can digital technologies take from and bring to research in mathematics education? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Second international handbook of mathematics education* (pp. 323–349). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Jackiw, N. (1991, 2001). *The Geometer's Sketchpad* [Computer Program]. Emeryville, CA: Key Curriculum Press.
- Jackiw, N. (2006). Mechanism and magic in the psychology of dynamic geometry. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the aesthetics: New approaches to an ancient affinity* (pp. 145–159). New York, NY: Springer.
- Jahn, A. P. (2002). “Locus” and “Trace” in Cabri-géomètre: Relationships between geometric and functional aspects in a study of transformations. *ZDM—Zentralblatt für Didaktik der Mathematik*, 34(3), 78–84.
- Japanese Society of Mathematics Education. (2000). *Mathematics programme in Japan*. Tokyo, Japan: Author.
- Jones, K. (2000). Providing a foundation for deductive reasoning: Students’ interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, 44, 55–85.
- Jones, K., Mackrell, K., & Stevenson, I. (2010). Designing digital technologies and learning activities for different geometries. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain New ICMI Study Series* (Vol. 13, pp. 47–60). Dordrecht, The Netherlands: Springer.
- Laborde, C. (1992). Solving problems in computer based geometry environments: The influence of the features of the software. *ZDM—Zentralblatt für Didaktik der Mathematik*, 92(4), 128–135.
- Laborde, C. (1998). Relationship between the spatial and theoretical in geometry: The role of computer dynamic representations in problem solving. In J. D. Tinsley & D. C. Johnson (Eds.), *Information and communications technologies in school mathematics* (pp. 183–195). London, UK: Chapman & Hall.
- Laborde, C. (2000). Dynamical geometry environments as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics*, 44(1–2), 151–161.
- Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Géomètre. *International Journal of Computers for Mathematical Learning*, 6, 283–317.

- Laborde, C. (2004). The hidden role of diagrams in pupils' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 1–21). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Laborde, C., Kynigos, C., Hollebrands, K., & Sträßer, R. (2006). Teaching and learning geometry with technology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 275–304). Rotterdam, The Netherlands: Sense Publishers.
- Lassak, M. (2009). Using dynamic graphs to reveal student reasoning. *International Journal of Mathematical Education in Science and Technology*, 40(5), 690–696.
- Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Mahwah, NJ: Lawrence Erlbaum.
- Leung, A. (2003). Dynamic geometry and the theory of variation. In N. A. Pateman, B. J. Dougherty, & J. T. Zillox (Eds.), *Proceedings of PME 27: Psychology of Mathematics Education 27th International Conference Volume 3* (pp. 197–204). Honolulu: University of Hawaii.
- Leung, A. (2008). Dragging in a dynamic geometry environment through the lens of variation. *International Journal of Computers in Mathematics Learning*, 13, 135–157.
- Leung, A., & Or, C. M. (2007). From construction to proof: Explanations in dynamic geometry environments. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 177–184). Seoul, Korea: International Group for the Psychology of Mathematics Education.
- Mariotti, M. A. (2000). Introduction to proof: The mediation of dynamical software environment. *Educational Studies in Mathematics*, 44(1–2), 25–53.
- Mariotti, M. A. (2002). Influence of technologies advances on students' math learning. In L. English (Ed.), *Handbook of international research in mathematics education* (pp. 695–723). Mahwah, NJ: Lawrence Erlbaum.
- Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 173–204). Rotterdam, The Netherlands: Sense Publishing.
- Mariotti, M. A. (2010). Proofs, semiotics and artefacts of information technologies. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives* (pp. 169–190). Dordrecht, The Netherlands: Springer.
- Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87–125.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Lawrence Erlbaum.
- Ministry of Education, People's Republic of China. (2001). *Full-time obligatory education mathematics curriculum standards (experimental version)* [in Chinese]. Beijing, China: Beijing University Press.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- Noss, R., & Hoyles, C. (Eds.). (1996). *Windows on mathematical meanings—Learning cultures and computers*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Olivero, F. (2002). *The proving process within a dynamic geometry environment* (Doctoral thesis). University of Bristol, UK.
- Olivero, F. (2006). Students' constructions of dynamic geometry. In C. Hoyles, J.-B. Lagrange, L.-H. Son, & N. Sinclair (Eds.), *Proceedings of the 17th International Congress on Mathematical Instruction Study Conference "Technology Revisited"* (pp. 433–442). Hanoi, Vietnam: Hanoi Institute of Technology and Didirem University.
- Olivero, F., & Robutti, O. (2001). Measuring in Cabri as a bridge between perception and theory. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 9–16). Utrecht, The Netherlands: International Group for the Psychology of Mathematics Education.
- Olivero, F., & Robutti, O. (2002). An exploratory study of students' measurement activity in a dynamic geometry environment. In J. Novotná (Ed.), *Proceedings of CERME 2* (Vol. 1, pp. 215–226). Prague, Czech Republic: Charles University.

- Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*, 12(2), 135–156.
- Oner, D. (2008). A comparative analysis of high school geometry curricula: What do technology-intensive, standards-based, and traditional curricula have to offer in terms mathematical proof and reasoning? *Journal of Computers in Mathematics and Science Teaching*, 27(4), 467–497.
- Oner, D. (2009). The role of dynamic geometry software in high school geometry curricula: An analysis of proof tasks. *International Journal for Technology in Mathematics Education*, 16(3), 109–121.
- Rabardel, P. (1995). *Les hommes et les technologies—Approche cognitive des instruments contemporains*. Paris, France: A. Colin.
- Radford, L. (2010). The anthropological turn in mathematics education and its implication on the meaning of mathematical activity and classroom practice. *Acta Didactica Universitatis Comenianae*, 10, 103–120.
- Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica III*, 7, 5–41.
- Sinclair, N. (2002). The kissing triangles: The aesthetics of mathematical discovery. *International Journal of Computers for Mathematical Learning*, 7(1), 45–63.
- Sinclair, M. P. (2003). Some implications of the results of a case study for the design of pre-constructed dynamic geometry sketches and accompanying materials. *Educational Studies in Mathematics*, 52(3), 289–317.
- Sinclair, N. (2008). *The history of the geometry curriculum in the United States*. Charlotte, NC: Information Age.
- Sinclair, N., Arzarello, F., Gaisman, M. T., & Lozano, M. D. (2009). Implementing digital technologies at a national scale. In C. Hoyles & J.-B. Lagrange (Eds.), *Digital technologies and mathematics teaching and learning: Rethinking the terrain* (Vol. 13, pp. 61–78). New York, NY: Springer.
- Sinclair, N., Moss, J., & Jones, K. (2010). Developing geometric discourse using DGS in K-3. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 185–192). Belo Horizonte, Brazil: International Group for the Psychology of Mathematics Education.
- Sinclair, N., & Yurita, V. (2008). To be or to become: How dynamic geometry changes discourse. *Research in Mathematics Education*, 10(2), 135–150.
- Sträßer, R. (2009). Instruments for learning and teaching mathematics. An attempt to theorise about the role of textbooks, computers and other artefacts to teach and learn mathematics. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 67–81). Thessaloniki, Greece: International Group for the Psychology of Mathematics Education.
- Stylianides, A. J. (2007). The notion of proof in the context of elementary school mathematics. *Educational Studies in Mathematics*, 65, 1–20.
- Stylianou, D., Knuth, E., & Blanton, M. (Eds.). (2009). *Teaching and learning proof across the grades: A K-16 perspective*. New York, NY: Routledge.
- Trouche, L. (2004). Managing complexity of human machine interactions in computerized learning environments: Guiding student's command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9(3), 281–307.
- Unione Matematica Italiana. (2004). G. Anichini, F. Arzarello, L. Ciarrapico & O. Robutti (Eds.), *Matematica 2003. La matematica per il cittadino. Attività didattiche e prove di verifica per un nuovo curriculum di Matematica (ciclo secondario)*. Lucca, Italy: Matteoni Stampatore.
- Vadcard, L. (1999). La validation en géométrie au collège avec Cabri-Géomètre: Mesures exploratoire et mesures probatoires. *Petit X*, 50, 5–21.
- Verillion, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, 10(1), 77–101.
- Vygotsky, L. S. (1978). *Mind in society. The development of higher psychological processes*. Cambridge, MA: Harvard University Press.