Σπηλιώτης : Στοχαστικές διαφ. εξισώσεις

Le Gall : Stoch. calculus . . .

Baldi : . . . .

M. Steele : Stoc. Calculus and financial applications

1ο μέρος

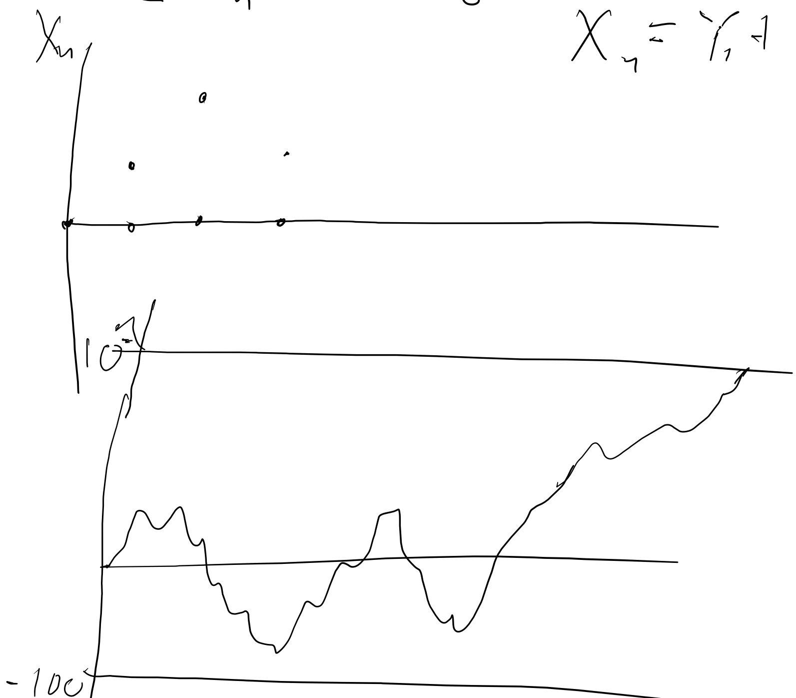
Δεσμευμένη μέση τιμή

Martingales

$(X_n)_{n \geq 0}$

$EX_n = EX_0$

$X_n = Y_1 + \dots + Y_n$



2ο μέρος

Κίνηση Brown

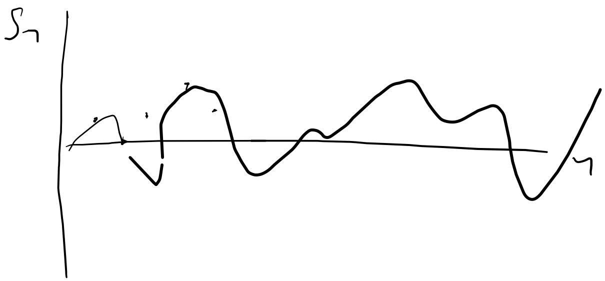
Ολοκλήρωμα Itô

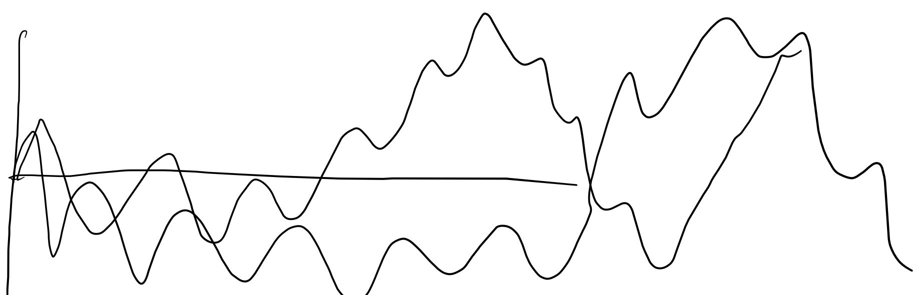
$$Y_i = \begin{cases} -1 & \text{με πιθ } 1/2 \\ 1 & \text{'' } 1/2 \end{cases}$$

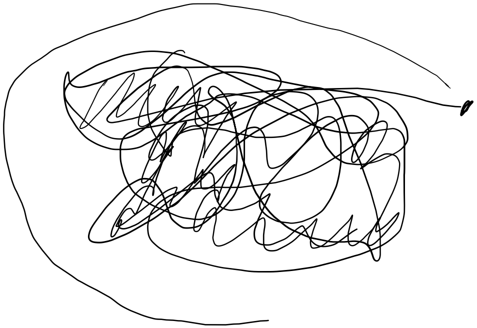
$$S_n = Y_1 + \cdots + Y_n, \quad n \geq 1$$

$Y_i(\omega)$

$S_n(\omega)$

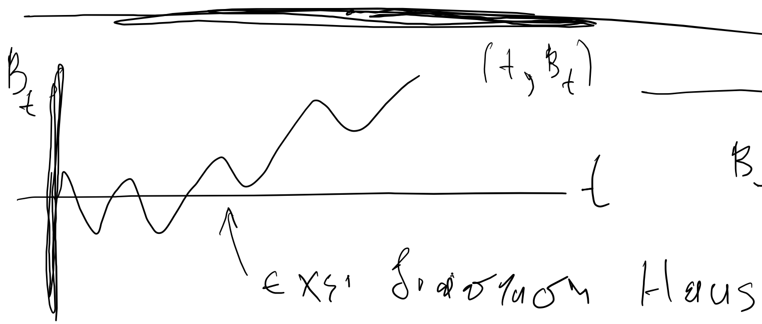






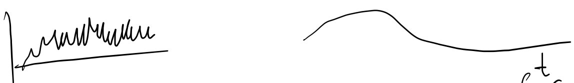
2 διάστατη Κ.Β.

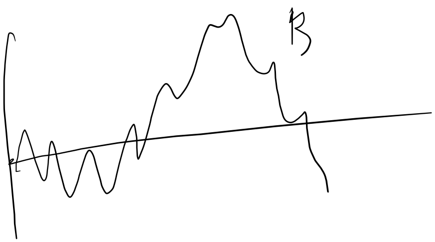
(B_t) $B_t \in \mathbb{R}^2$



$B_t \in \mathbb{R}$

έχει διάσταση Hausdorff $\frac{3}{2}$





$$\int_0^t f(s)\, dg(s)$$

$B.$

Revuz-Yor: Continuous martingales and Brownian motion

$(\Omega, \mathcal{F}, P)$

ω πληροφορία

$\mathcal{F} \subset \mathcal{P}(\Omega)$

$\Omega \in \mathcal{F}$

$A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$

$A_i \in \mathcal{F} \; \forall i \in \mathbb{N} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$

$\Omega = \{1, \quad , 6\}$

$\mathcal{F}_1 = \{\emptyset, \Omega, \{3\}, \Omega \setminus \{3\}\}$

$\mathcal{F}_2 = \mathcal{P}(\Omega)$ $\quad \{1\}, \{2\}, \ldots \{6\}$

ω

X σύνολο

$\mathcal{E} \subset \mathcal{P}(X)$

$$\sigma(\mathcal{E}) = \bigcap_{\substack{\mathcal{A} \subset \mathcal{P}(X) \\ \sigma\text{-άλγ.}, \ \mathcal{E} \subset \mathcal{A}}} \mathcal{A}$$

π.χ. $X = \{1, \ , 6\}$

$\mathcal{E} = \{ \{1, 2\}, \{2, 3\} \}$

$\{1,2\}, \{2, 3\}, \{2\}, \{1\}, \{3\}$

$\{1, 2, 3\}, \emptyset, X$

$$\sigma(\mathcal{E}) = \{ \emptyset, X, A, A \cup \underline{\{4, 5, 6\}} : A \subset \{1, 2, 3\} \}$$

$\{1, 2, 3\}$

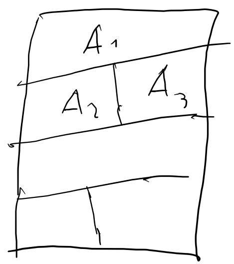
Σ-άλγεβρα από διαμέριση

X, $\mathcal{C} \subset \mathcal{P}(X)$ αριθμήσιμη διαμέριση του X.

$\mathcal{C} = \{A_i : i \in I\}$ I αριθμ.

$A_i \neq \emptyset \ \forall i$

$\bigcup_{i \in I} A_i = X$



$\sigma(\mathcal{C}) = \{\bigcup_{i \in J} A_i : J \subset I\}$ $\mathcal{R}$

$\sigma(\mathcal{C}) = \bigcap_{\mathcal{A} \supset \mathcal{C}} \mathcal{A}$

– $\mathcal{R} \subset \sigma(\mathcal{C})$

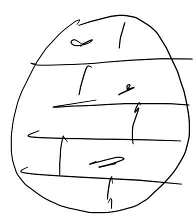
$\mathcal{A} \supset \mathcal{C}$ $A_i \in \mathcal{A} \Rightarrow \bigcup_{i \in J} A_i \in \mathcal{A} \Rightarrow \mathcal{R} \subset \mathcal{A}$

— $\sigma(\mathcal{C}) \subset \mathcal{R}$

$\mathcal{R}$ σ-αλγ. και $\mathcal{R} \supset \mathcal{C}$

Αν $A \in \mathcal{R}$ $A = \bigcup_{i \in J} A_i$

$X \setminus A = \bigcup_{i \in I \setminus J} A_i$



Μετρήσιμες συναρτήσεις

$(X, \mathcal{A})$ $\quad (Y, \mathcal{B})$

$f: X \to Y$ $\quad \mathcal{A}/\mathcal{B}$ μετρήσιμη αν

$f^{-1}(B) \in \mathcal{A} \quad \forall B \in \mathcal{B}$

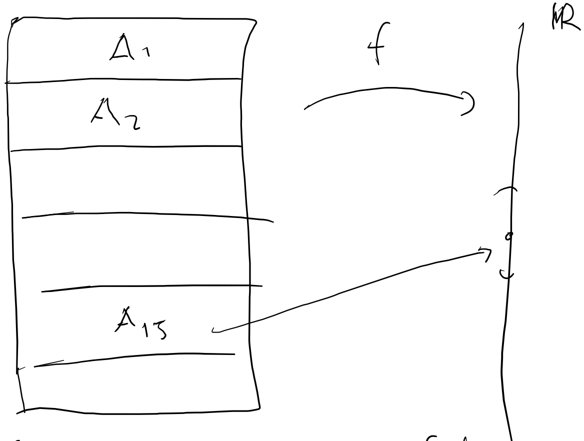
$f: X \to \mathbb{R} \quad \underline{\mathcal{B} = \mathcal{B}(\mathbb{R})}$

<u>Παράδειγμα</u> $\quad X$ σύνολο, $\mathcal{C} = \{A_i : i \in I\}$

αριθμ. διαμέριση του X.

Ποιες $f: X \to \mathbb{R}$ είναι $\sigma(\mathcal{C})/\mathcal{B}(\mathbb{R})$

μετρήσιμες;



f μετρήσιμη $\iff$ $f|A_i$ σταθερή $\underset{= a_i}{}$ $\forall i \in I$

$\Leftarrow$ Αν $B \in \mathcal{B}(\mathbb{R})$

$$f^{-1}(B) = \bigcup_{a_i \in B} A_i \in \sigma(\mathcal{E})$$

$\Rightarrow$ Έστω $i_0 \in I$.

Ας υποθ. ότι $f|A_{i_0}$ όχι σταθερή

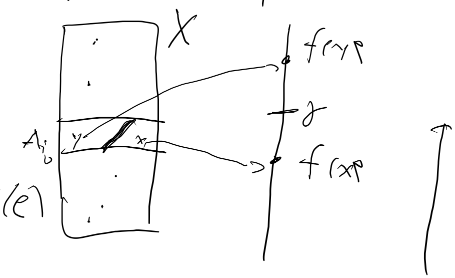
$\exists\, x, y \in A_{i_0}$ με $f(x) < f(y)$

έστω $f(x) < \gamma < f(y)$

$f^{-1}((-\infty, \gamma]) \in \sigma(\mathcal{E})$

$\Gamma = f^{-1}((-\infty,\gamma]) \cap A_{i_0} \in \sigma(\mathcal{E})$

$\emptyset \neq \Gamma \subsetneq A_{i_0}$



$$f : X \longrightarrow \mathbb{R}$$

$$\mathcal{B}(\mathbb{R})$$

$$\sigma(f) = \{ f^{-1}(B) : B \in \mathcal{B}(\mathbb{R}) \}$$

Σχ

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = 3 \qquad \forall x \in \mathbb{R}$$

$$\sigma(f) = \{ \emptyset, \mathbb{R} \}$$

$$f^{-1}(B) = \begin{cases} \emptyset & \text{αν } 3 \notin B \\ \mathbb{R} & \text{αν } 3 \in B \end{cases}$$