$E(X_{n \wedge T})$ $\quad (\Omega, \mathcal{F}, P)$

<u>Άσκηση</u> $(X_n)_{n \geq 0}$ τ.μ.

$T: \Omega \to \mathbb{N} \cup \{\infty\}$ τ.μ.

Τότε για κάθε $n \in \mathbb{N}$, η

$X_{n \wedge T}$ είναι τ.μ.

Λύση

$$\omega \mapsto X_{n \wedge T(\omega)}(\omega)$$

$A \in \mathcal{B}(\mathbb{R})$

$$C = \{X_{n \wedge T} \in A\} = \bigcup_{k=0}^{n-1} \left(C \cap \{T = k\} \right)$$

$$\cup \left(C \cap \{T \geq n\} \right) = \bigcup_{k=0}^{n-1} \left(\{X_k \in A\} \cap \{T = k\} \right)$$

$$\cup \left(\{X_n \in A\} \cap \underbrace{\{T \geq n\}}_{\in \mathcal{F}} \right) \in \mathcal{F}$$

(X_n) martingale

T χρόνος διακοπής ώστε i, ii, ή iii

τότε $E(X_T) = E(X_0)$

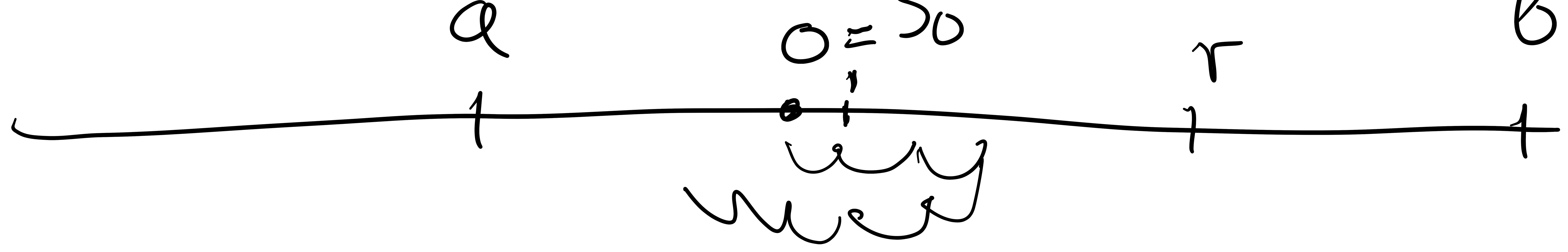
//

<u>Εφαρμογή</u> (το πρόβλημα εξόδου για τον ΣΑΤΠ. Πτώχευση παίκτη τυχερών παιχνιδιών)

έστω $(S_n)_{n \geq 0}$ ο ΣΑΤΠ

$\left(S_n = X_1 + \dots + X_n,\ \mathcal{F}_n = \sigma(X_1, \dots, X_n)\right)$

και $a, b \in \mathbb{Z}$ $\quad a < 0 < b$



$\forall r \in \mathbb{Z}$, έστω

$$T_r = \inf\{k \in \mathbb{N} : S_k = r\}$$

Ο T_r είναι χρόνος διακοπής.

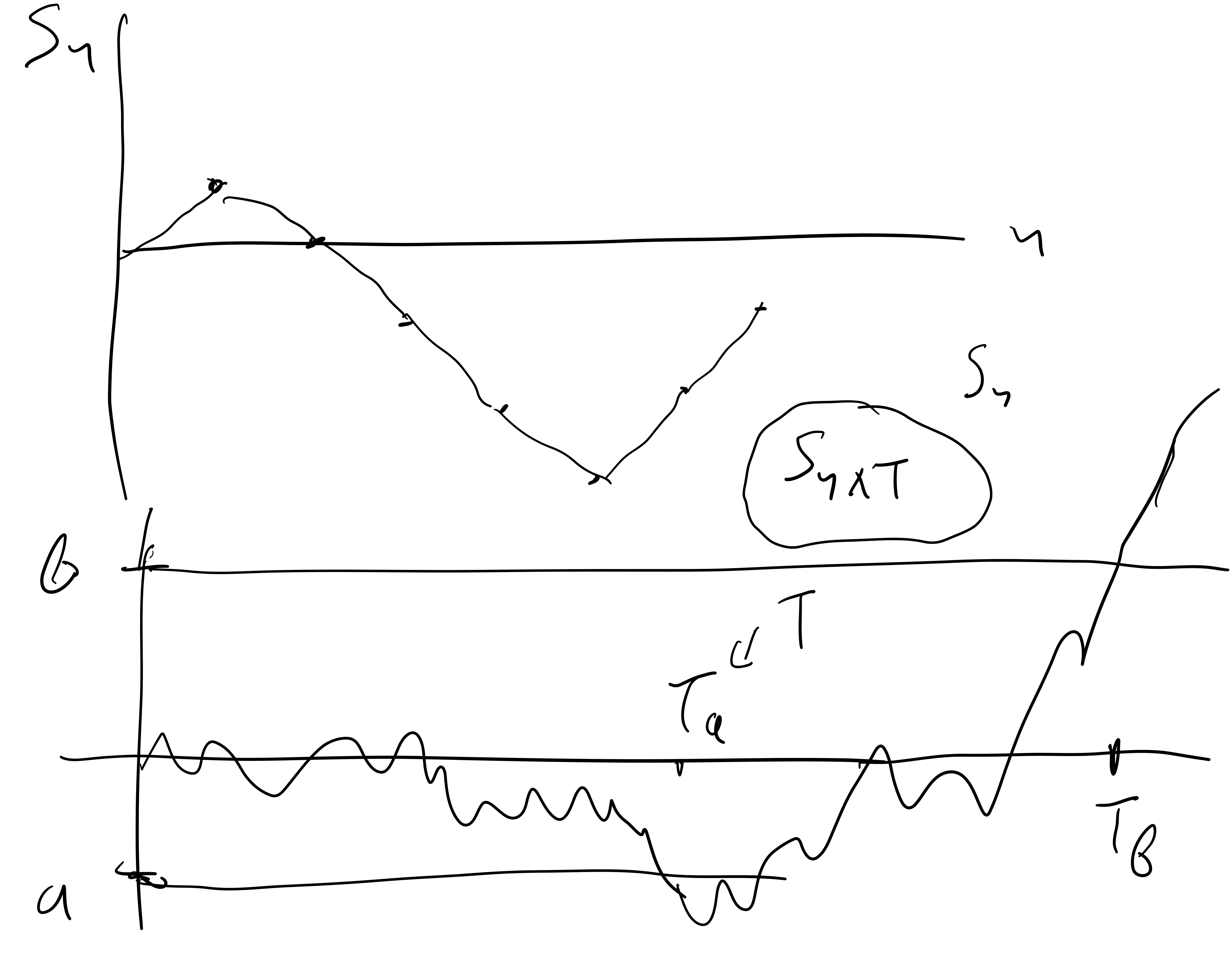
Έστω $T = T_a \wedge T_b$.

Ισχυρισμός

i) $ET < \infty$

ii) $P(T_b < T_a) = \dfrac{|a|}{|a|+b}$

iii) $ET = |a|b$



i) Το $M_n = S_n^2 - n$ είναι martingale

T χρόνος διακοπής.

Για $n \in \mathbb{N}$, ο $T \wedge n$ είναι και αυτός χρόνος διακοπής φραγμένος.

Άρα $0 = E(M_0) = E(M_{n \wedge T}) =$

$$= E(S_{n \wedge T}^2) - E(n \wedge T)$$

$$\Rightarrow \underline{E(n \wedge T)} = E(S_{n \wedge T}^2) \quad (*)$$

Θ. Μον. Σύγκλ.
$$\leq \max\{a^2, b^2\}$$

$$\Rightarrow \quad E(T) \leq a^2 \vee b^2 < \infty$$

(ii) $(S_n)_{n \geq 0}$ martingale, T χρόνος διακοπής άρα

$$0 = E(S_0) = E(S_{n \wedge T}) \quad (**)$$

$$\lim_{n \to \infty} S_{n \wedge T} = \begin{cases} a & \text{στο } \{T_a < T_b\} \\ b & \text{στο } \{T_b < T_a\} \end{cases}$$

$$1 = P(T<\infty) = P(\{T_a<T_b\} \cup \{T_b<T_a\})$$

Άρα με πιθ. 1

$$\lim_{n\to\infty} S_{n\wedge T} = a\, 1_{T_a<T_b} + b\, 1_{T_b<T_a} = S_T$$

Η (**) μέσω Θ. δειγμ. αφηλ.

$$0 = a\, P(T_a<T_b) + b\, P(T_b<T_a)$$

$$= a\,(1-P(T_b<T_a)) + b\,\underbrace{P(T_b<T_a)}_{x}$$

$$0 = a - ax + bx$$

$$\Rightarrow \quad x = \frac{-a}{b-a} = \frac{|a|}{b+|a|}$$

(iii) Παίρνοντας $n\to\infty$ στην (*).

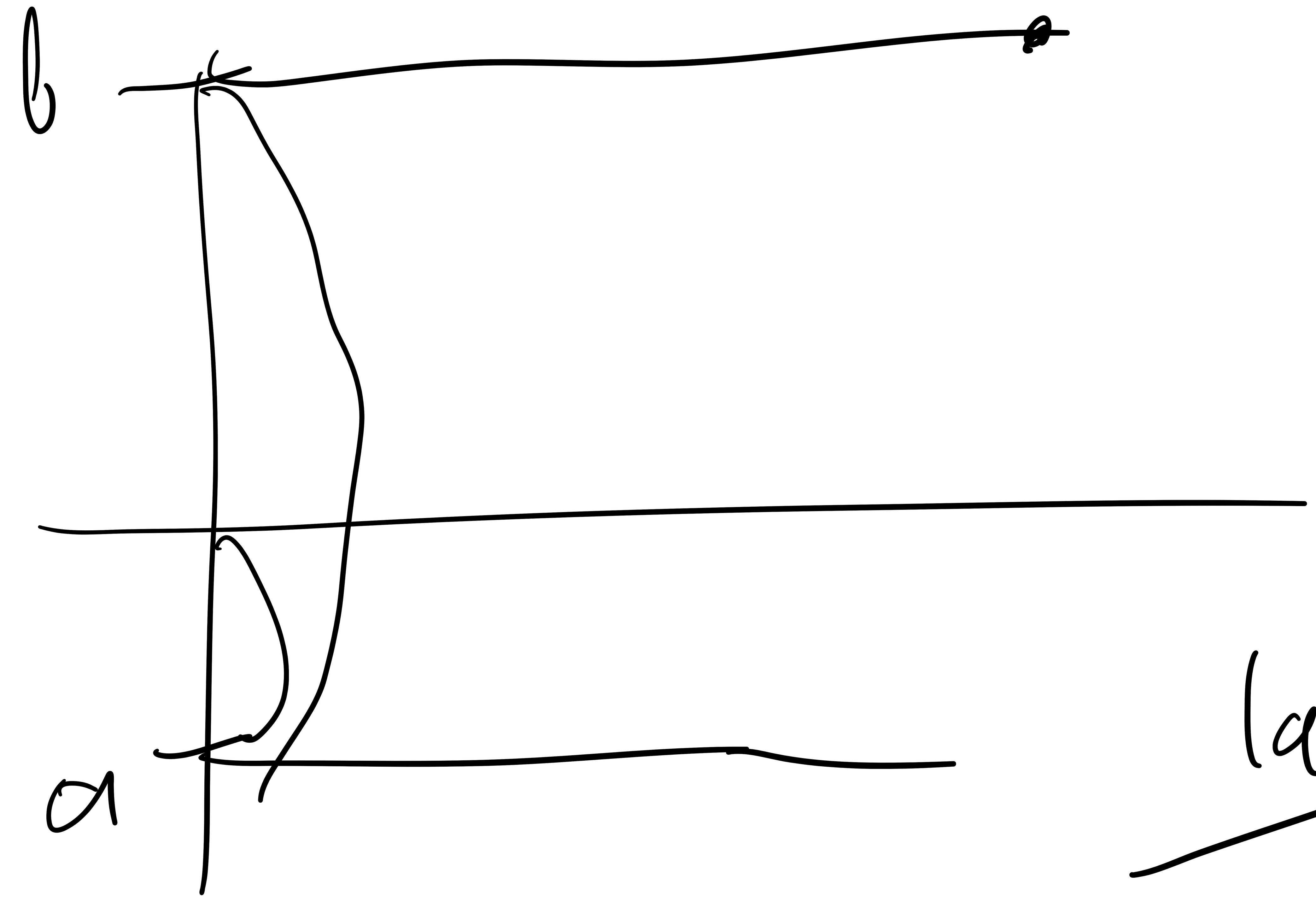
$$\lim_{n\to\infty} S^2_{n\wedge T} = a^2\, 1_{T_a<T_b} + b^2\, 1_{T_b<T_a}$$

Θ. δειγμ. αφηλίων. Δίνει ότι $E(T) =$

$$= \lim_{n\to\infty} E(S^2_{n\wedge T}) = a^2 P(T_a < T_b) + b^2 P(T_b < T_a) =$$

$$a^2 \frac{b}{|a|+b} + b^2 \frac{|a|}{|a|+b} = \frac{|a|b(|a|+b)}{|a|+b}$$

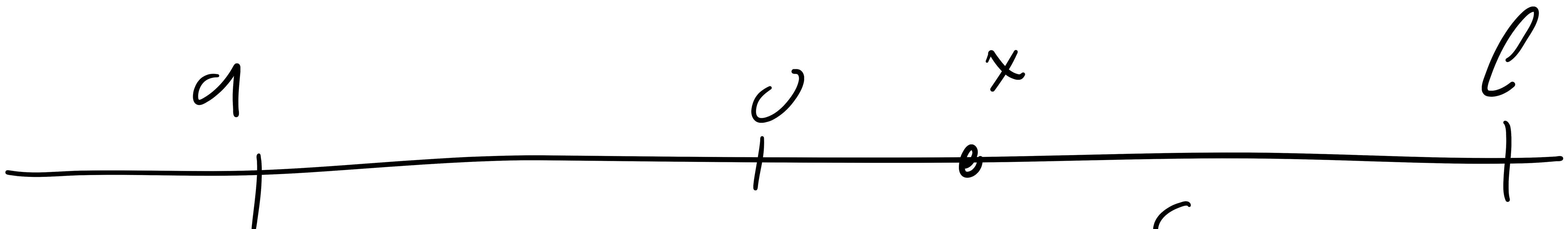
$$= |a| b$$



$$\frac{|a|}{|a|+b}$$

$P(T_a < T_b)$ $E(S_T) = 0$

$E(T)$ $E(S_T^2 - T) = 0$



$S_n = x + X_1 + X_2 + \dots + X_n$

$$f(x) = P(T_\alpha < T_\beta \mid S_0 = x)$$

$$f(\alpha) = 1 \qquad f(x) = \frac{1}{2} f(x-1) + \frac{1}{2} f(x+1)$$

$$f(\beta) = 0$$

$$f(x) = cx + \delta \qquad f(x) = \frac{\beta - x}{|\alpha| + \beta}$$

$$f(0) = \frac{\beta}{|\alpha| + \beta}$$

$$h(x) = E(T \mid S_0 = x)$$

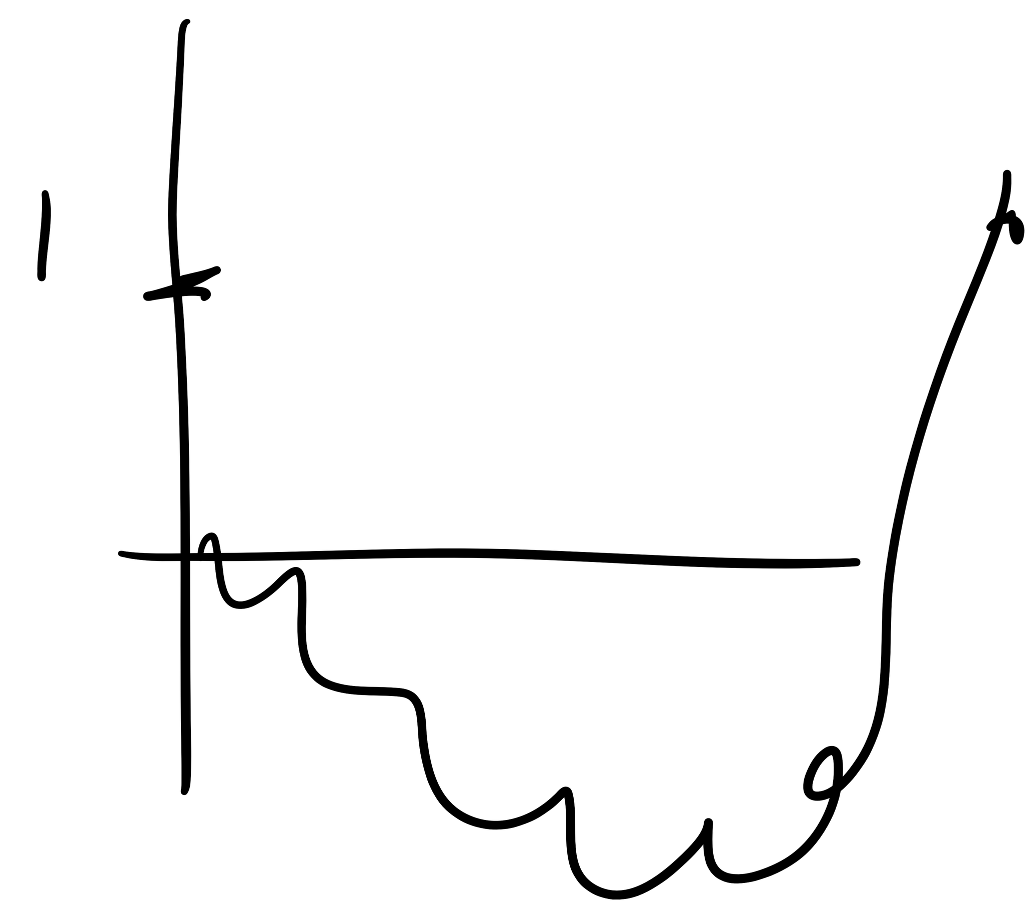
$$h(\alpha) = h(\beta) = 0$$

$$h(x) = 1 + \frac{1}{2}\big(h(x-1) + h(x+1)\big)$$

Εφαρμογή

S όπως πριν.

$$T_1 = \inf\{k \in \mathbb{N} : S_k = 1\}$$



i) $ET_1 = \infty$

ii) $P(T_1 < \infty) = 1$

iii)
$$P(T_1 = n) = \begin{cases} 0 & \text{αν } n \text{ ζυγός} \\ \dfrac{1}{(2k-1)2^{2k}} \dbinom{2k}{k} & \text{αν } n = 2k-1, \; k \in \mathbb{N}^+ \end{cases}$$

Απόδ

i) Αν $E T_1 < \infty$ $\quad E(S_{T_1}) = E(S_0)$

$1 = 0$ άτοπο

ii, iii) $S_n = X_1 + \quad + X_n$

Για $a > 0$ $\quad E(e^{aX_i}) = \frac{1}{2}(e^a + e^{-a})$

$= \cosh(a)$

$$M_n = \frac{e^{aX_1} \, e^{aX_2} \quad e^{aX_n}}{(\cosh(a))^n} \quad , \; n \geq 0$$

είναι martingale ως προς την $\mathcal{F}_n =$

$= \sigma(X_1 \quad , X_n)$

$$M_n = \frac{e^{aS_n}}{(\cosh a)^n} \; , \; n \in \mathbb{N}$$

$\forall n \in \mathbb{N}^+$ ο $T_1 \wedge n$ είναι χρ. διακ.

$$1 = E(M_0) = M(M_{n \wedge T_1}) = E\left(\frac{e^{aS_{n\wedge T_1}}}{(\cosh a)^{n \wedge T_1}}\right)$$

$\cosh a > 1$ για $a > 0$

και $\quad a S_{n \wedge T_1} \leq a \quad \forall n \; \forall \omega$

Σε το $T_1=\infty$ $\quad \lim_{n\to\infty} \frac{e^{aS_{n\wedge T_1}}}{(\cosh a)^{n\wedge T_1}} = 0$

Σε το $T_1<\infty$ $\quad$ " $\quad = \frac{e^{a}}{(\cosh a)^{T_1}}$

οι $\frac{e^{aS_{n\wedge T_1}}}{(\cosh a)^{n\wedge T_1}} \le e^{a}$

Θ. φρ. σύγκλ. $\quad 1 = E\left(1_{T_1<\infty} \frac{e^{a}}{(\cosh a)^{T_1}}\right)$

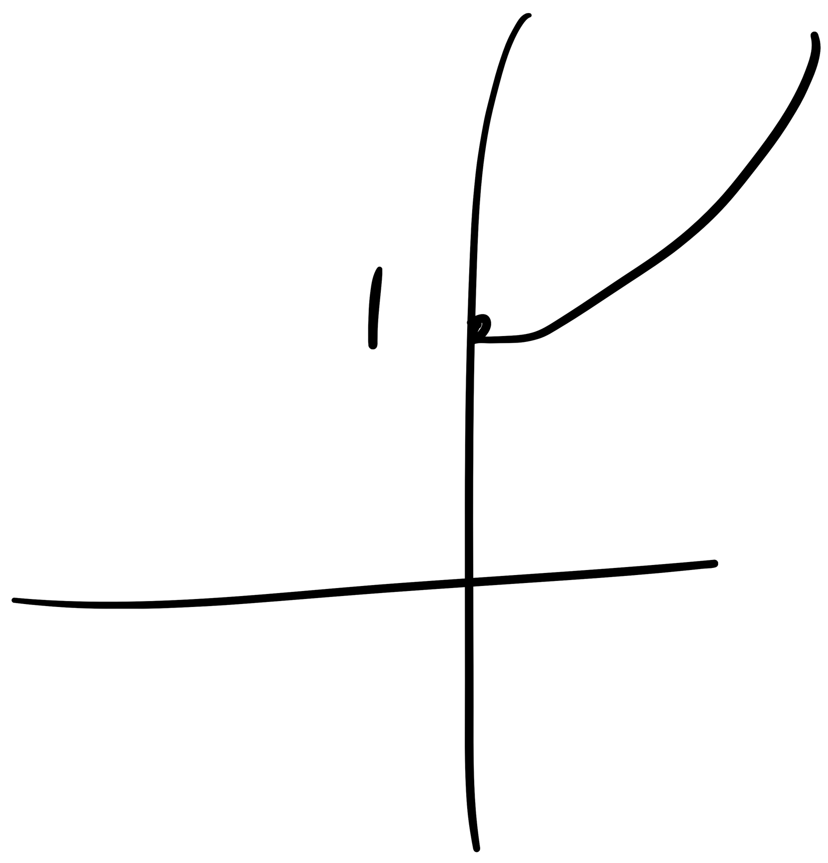
$$e^{-a} = E\left(1_{T_1<\infty}\left(\frac{1}{\cosh a}\right)^{T_1}\right) \quad (***)$$

Για $a \to 0^+$, Θ. μονοτ. σύγκλισης

$$1 = E(1_{T_1<\infty}) \Rightarrow P(T_1<\infty)=1$$

Για $t = \frac{1}{\cosh a}$ η (***) γίνεται

$$e^{-a} = E\left(t^{T_1}\right)$$



$a>0 \iff t\in(0,1)$

$$t = \frac{2}{e^{a}+e^{-a}} \iff e^{a} = \frac{1+\sqrt{1-t^2}}{t}$$

$\forall t \in (0,1) \quad E(t^{T_1}) = \dfrac{t}{1+\sqrt{1-t^2}} =$

$$= \frac{1-\sqrt{1-t^2}}{t} = \frac{1}{t}\left(1-(1-t^2)^{\frac{1}{2}}\right)$$

$$= \frac{1}{t}\left(1 - \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (-t^2)^k\right)$$

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

$$= \sum_{k=1}^{\infty} \binom{1/2}{k} (-1)^{k+1} t^{2k-1}$$

$$E(t^{T_1}) = \sum_{j=0}^{\infty} P(T_1 = j)\, t^j$$

$3-2k$

$1-2k+2$

$$\binom{\frac{1}{2}}{k} = \frac{\left(\frac{1}{2}\right)_k}{k!} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}$$

$$= (-1)^{k-1}\,\frac{1\cdot 1\cdot 3 \quad (2k-3)}{2^k \quad k!}$$

$$P(T_1 = 2k-1) = \frac{1 \cdot 3 \cdots (2k-3)}{2^k \, k!}$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2k-3)\overbrace{(2k-2)(2k-1)\,2k}}{\underline{2^k} \cdot \underline{k!}\; 2^k \; k! \; (2k-1)}$$

$$= \frac{(2k)!}{k!\; k!} \; \frac{1}{2^{2k}} \; \frac{1}{(2k-1)}$$

Ανισότητα Doob

$X = (X_n)_{n \geq 0}$ submartingale

τότε $\forall \lambda > 0$ και $n \in \mathbb{N}$ έχουμε

$$P\left(\sup_{0 \leq k \leq n} X_k \geq \lambda \right) \leq \frac{1}{\lambda} E(X_n^+)$$

$$P(Y \geq a) \leq \frac{1}{a} EY \qquad (Y \geq 0)$$

$$\leq \frac{1}{\lambda} E\left(\sup_{0 \leq k \leq n} X_k \right)$$

$$P\left(\bigcup_{k=0}^{n}\{X_k \geq \lambda\}\right) \leq \sum_{k=0}^{7} P(X_k \geq \lambda)$$
$$\leq \frac{1}{n}\sum_{k=0}^{7} E(X_k)$$

Αποδ.

$\varphi(x) = x^+$ είναι $\nearrow$ κυρτή.

Άρα $(X_n^+)_{n\geq 0}$ submartingale

$$A_k = \{X_0, \ldots, X_{k-1} < \lambda, \; X_k \geq \lambda\} \in \mathcal{F}_k$$

$$\underline{\{\sup_{0\leq k\leq n} X_k \geq \lambda\}} = \bigcup_{k=0}^{n} A_k \quad (\subset \underline{\Omega})$$

$$E(X_n^+) \geq E\left(X_n^+ 1_{\bigcup_{k=0}^{n} A_k}\right)$$
$$= E\left(\sum_{k=0}^{n} X_n^+ 1_{A_k}\right) = \sum_{k=0}^{n} E(X_n^+ 1_{A_k})$$
$$= \sum_{k=0}^{n} \underline{E\left(E(X_n^+ \mid \mathcal{F}_k) 1_{A_k}\right)}$$

$$\overset{\text{subMart}}{\geq} \sum_{k=0}^{n} E\left(X_k^{+} \, 1_{A_k}\right) \geq \sum_{k=0}^{n} E\left(\lambda \, 1_{A_k}\right)$$

$$= \lambda \sum_{k=0}^{n} P(A_k) = \lambda \, P\left(\bigcup_{k=0}^{n} A_k\right)$$

$$= \lambda \, P\left(\left\{ \sup_{0 \leq k \leq n} X_k \geq \lambda \right\}\right)$$

$\Rightarrow$ η ζητούμενη