Prüfer sequence

In combinatorial mathematics, the **Prüfer sequence** (also **Prüfer code** or **Prüfer numbers**) of a labeled tree is a unique sequence associated with the tree. The sequence for a tree on n vertices has length $n – 2$, and can be generated by a simple iterative algorithm. Prüfer sequences were first used by Heinz Prüfer to prove Cayley's formula in 1918.

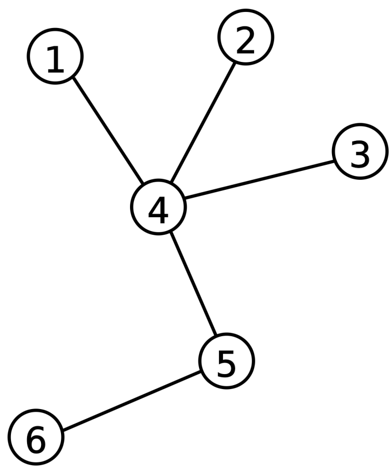
Algorithm to convert a tree into a Prüfer sequence

One can generate a labeled tree's Prüfer sequence by iteratively removing vertices from the tree until only two vertices remain. Specifically, consider a labeled tree T with vertices $\{1, 2, ..., n\}$. At step i, remove the leaf with the smallest label and set the ith element of the Prüfer sequence to be the label of this leaf's neighbour.

The Prüfer sequence of a labeled tree is unique and has length $n – 2$.

Example

Consider the above algorithm run on the tree shown to the right. Initially, vertex 1 is the leaf with the smallest label, so it is removed first and 4 is put in the Prüfer sequence. Vertices 2 and 3 are removed next, so 4 is added twice more. Vertex 4 is now a leaf and has the smallest label, so it is removed and we append 5 to the sequence. We are left with only two vertices, so we stop. The tree's sequence is {4,4,4,5}.



A labeled tree with Prüfer sequence {4,4,4,5}.

Algorithm to convert a Prüfer sequence into a tree

Let `{a[1], a[2], ..., a[n]}` be a Prüfer sequence:

The tree will have `n+2` nodes, numbered from `1` to `n+2`. For each node set its degree to the number of times it appears in the sequence plus 1. For instance, in pseudo-code:

```
Convert-Prüfer-to-Tree(a)
n ← length[a]
T ← a graph with n + 2 isolated nodes, numbered 1 to n + 2
degree ← an array of integers
for each node i in T
    do degree[i] ← 1
for each value i in a
    do degree[i] ← degree[i] + 1
```

Next, for each number in the sequence `a[i]`, find the first (lowest-numbered) node, `j`, with degree equal to 1, add the edge `(j, a[i])` to the tree, and decrement the degrees of `j` and `a[i]`. In pseudo-code:

```
for each value i in a
    for each node j in T
        if degree[j] = 1
           then Insert edge[i, j] into T
```

```
                    degree[i] ← degree[i] - 1
                    degree[j] ← degree[j] - 1
                    break
```

At the end of this loop two nodes with degree 1 will remain (call them u, v). Lastly, add the edge (u,v) to the tree.

```
u ← v ← 0
for each node i in T
    if degree[i] = 1
        then if u = 0
            then u ← i
            else v ← i
                 break
Insert edge[u, v] into T
degree[u] ← degree[u] - 1
degree[v] ← degree[v] - 1
return T
```

Cayley's formula

The Prüfer sequence of a labeled tree on n vertices is a unique sequence of length $n – 2$ on the labels 1 to n — this much is clear. Somewhat less obvious is the fact that for a given sequence S of length $n–2$ on the labels 1 to n, **there is a *unique* labeled tree whose Prüfer sequence is *S***.

The immediate consequence is that Prüfer sequences provide a bijection between the set of labeled trees on n vertices and the set of sequences of length $n–2$ on the labels 1 to n. The latter set has size n^{n-2}, so the existence of this bijection proves Cayley's formula, i.e. that there are n^{n-2} labeled trees on n vertices.

Other applications

- Cayley's formula can be strengthened to prove the following claim:

  The number of spanning trees in a complete graph K_n with degrees $d_1, d_2, ..., d_n$ is equal to the multinomial coefficient

  $${n-2 \choose d_1-1,\, d_2-1,\, \ldots,\, d_n-1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}.$$

  The proof follows by observing that in the Prüfer sequence number i appears exactly $(d_i - 1)$ times.
- Cayley's formula can be generalized: a labeled tree is in fact a spanning tree of the labeled complete graph. By placing restrictions on the enumerated Prüfer sequences, similar methods can give the number of spanning trees of a complete bipartite graph. If G is the complete bipartite graph with vertices 1 to n_1 in one partition and vertices $n_1 + 1$ to n in the other partition, the number of labeled spanning trees of G is $n_1^{n_2-1} n_2^{n_1-1}$, where $n_2 = n - n_1$.
- Generating uniformly distributed random Prüfer sequences and converting them into the corresponding trees is a straightforward method of generating uniformly distributed random labelled trees.

References

External links

- Prüfer code (http://mathworld.wolfram.com/PrueferCode.html) – from MathWorld

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