# Harmonic Analysis Notes

Lin, Fall 2009

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References


1. E. Stein, *Singular Integrals and Differentiable Properties of Functions*
2. E. Stein, (G. Weiss) *Fourier Analysis on* $\mathbb{R}^n$
3. E. Stein, *Harmonic Analysis*
4. Dym, McKean, *Fourier Series and Integrals*


# Week 1 (9/9/2009)

## Hardy-Littlewood Maximal Functions

Notation:

1. $|E|$ denotes the measure of the set. Which measure depends on context. For instance, $|V|$ could be Lebesgue measure, and $|\partial V|$ could be the $\mathcal{H}^{n-1}$-measure of $\partial V$. (related to trace, probably)

Let $f \in L^1(\mathbb{R}^n)$, and define

$$M_f(x) = \sup_{B_r(x), r>0} \frac{\int_{B_r(x)} |f(y)|dy}{|B_r(x)|}$$

and

$$\lambda_f(\alpha) = |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}|$$

Two tools:

1. (Fubini type result) If $f \in L^p(\mathbb{R}^n)$, note that

$$\int_{\mathbb{R}^n} |f(x)|^p\,dx = -\int \alpha^p\,d\lambda(\alpha) = p\int_0^\infty \alpha^{p-1}\lambda(\alpha)\,d\alpha$$

**Proof.** Fubini's Theorem gives

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)|^p\,dx &= \int_{\mathbb{R}^n}\int_0^{|f(x)|} p\alpha^{p-1}\,d\alpha\,dx \\ &= p\int_0^\infty \alpha^{p-1}\int_{\{|f(x)|>\alpha\}} dx\,d\alpha \\ &= p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)\,d\alpha\end{aligned}$$

□

2. (Chebyshev) If $f\in L^1(\mathbb{R}^n)$, then

$$\lambda_f(\alpha)\le\frac{\int_{\mathbb{R}^n}|f(x)|dx}{\alpha}$$

**Proof.**

$$\int_{\mathbb{R}^n}|f(x)|dx\ge\int_{\{|f|>\alpha\}}|f(x)|dx\ge\alpha|\{|f|>\alpha\}|=\alpha\lambda_f(\alpha)$$

□

The main goal is to prove the following properties of the Hardy Maximal function $M_f$:

**Theorem 1.**

a) *If $f\in L^p(\mathbb{R}^n)$, $1\le p\le\infty$, then $Mf$ is finite a.e.*

b) *(Weak (1,1) estimate) If $f\in L^1(\mathbb{R}^n)$, then*

$$|\{x: M_f(x)>\alpha\}|\le\frac{A}{\alpha}\int_{\mathbb{R}^n}|f(x)|dx$$

*for some constant $A$ depending on the dimension $n$.*

c) *If $f\in L^p(\mathbb{R}^n)$, $1<p\le\infty$, then $M_f\in L^p(\mathbb{R}^n)$, and*

$$\|M_f\|_{L^p}\le C\|f\|_{L^p}$$

*with a constant $C$ depending on the dimension $n$ and $p$.*

**Remarks.**

1. If $f\in L^1(\mathbb{R}^n)$ and $f$ is not identically zero, then $M_f\notin L^1(\mathbb{R}^n)$. In fact,

$$M_f(x)\ge\frac{C_0}{|x|^n}$$

in $\mathbb{R}^n$ for $|x|\ge 1$.

For example, consider $\delta_0(x)$, which is approximated by $C_0^\infty$ (smooth compactly supported) functions.

Note

$$M_{\delta_0}(x)=\frac{1}{\omega_n|x|^n}\notin L^1(\mathbb{R}^n)$$

where $|B_{|x|}(0)|=\omega_n|x|^n$.

**Proof.** First we show this for compactly supported $f$, with $\|f\|_{L^1}>0$ (i.e. not identically zero), and $\mathrm{supp}(f)\subset B_R$ for some $R>0$. Note that for $|x|>R$,

$$M_f(x)\ge\frac{1}{|B_{|x|}|}\int_{B_{|x|}}|f(y)|dy=\frac{C\|f\|_{L^1}}{(|x|+R)^n}\notin L^1(\mathbb{R}^n)$$

In general, we can approximate $f$ by $f\mathbf{1}_{\{|x|>R\}}$ and there exists $R$ such that

$$\|f - f\mathbf{1}_{\{|x|>R\}}\|_{L^1} \leq \varepsilon$$

Then

$$M_f(x) \geq \frac{1}{|B_{|x|}|}\int_{B_{|x|}} |f(y)|\,dy \geq \frac{C\|f\mathbf{1}_{\{|x|>R\}}\|_{L^1} - \varepsilon}{(|x|+R)^n} \notin L^1(\mathbb{R}^n)$$

□

2. If $\operatorname{supp}(f) \subseteq B_1$, then $M_f \in L^1(B_1)$ if $|f|\log(1+|f|) \in L^1(B_1)$. (The Shannon Entropy)

   **Proof.** We consider the integral of $M_f(x)$:

$$\begin{aligned} 0 &\leq \int_{B_1} M_f(x)\,dx \\ &= \int_{B_1 \cap \{M_f \leq 1\}} M_f(x)\,dx + \int_{B_1 \cap \{M_f \geq 1\}} M_f(x)\,dx \\ &\leq |B_1| + \lambda_{M_f}(1) + \int_1^\infty \lambda_{M_f}(\alpha)\,d\alpha \end{aligned}$$

   where we have used the Fubini result earlier (slightly modified to account for integration over the smaller set $\{M_f \geq 1\}$.

   Now we apply a trick, which we justify later:

$$\lambda_{M_f}(\alpha) \leq \frac{2A}{\alpha}\int_{\{|f|>\alpha/2\}} |f|\,dx \qquad (\star)$$

   Given this, we apply Fubini:

$$\begin{aligned} \int_1^\infty \lambda_{M_f}(\alpha)\,d\alpha &\leq \int_1^\infty \frac{2A}{\alpha}\int_{|f|>\alpha/2} |f|\,dx\,d\alpha \\ &= 2A\int_{B_1 \cap \{|f|>1/2\}} |f(x)| \int_1^{2|f(x)|} \frac{1}{\alpha}\,d\alpha\,dx \\ &= 2A\int_{B_1 \cap \{|f|>1/2\}} |f(x)|\log|f(x)|\,dx \end{aligned}$$

   which is finite since we the integral avoids possible singularities and $|f(x)|\log|f(x)|$ behaves asymptotically like $|f(x)|\log(1+|f(x)|)$. □

The second remark in particular will be helpful in proving the theorem; a similar trick will be used. The main ingredient of the proof, however, lies in covering lemmas.

## Covering Lemmas

**Lemma 2. (Vitali Covering Lemma)** *Let $E$ be a measurable set in $\mathbb{R}^n$ covered by a family of balls $\mathcal{F} = \{\mathcal{B}_\alpha\}$ of bounded diameter. Then there is a countable disjoint subfamily $\mathcal{F}'$ of balls in $\mathcal{F}$, i.e. $\mathcal{F}' = \{B_i\}_{i=1}^\infty$ with $B_i \in \mathcal{F}$, such that*

$$E \subset \bigcup_{i=1}^\infty \widehat{B_i}$$

*where $\widehat{B_i} = 5B_i$.*

**Lemma 3. (Besicovitch Covering Lemma)** *Let $E$ be a set in $\mathbb{R}^n$ and let $\mathcal{F}$ be a family of balls with centers at $x \in E$, $\mathcal{F} = \{B_{r_x}(x): x \in E\}$. Then there is some integer $N > 0$, depending only on the dimension, and $N$ subfamilies of $\mathcal{F}$, $\mathcal{F}_1, \ldots \mathcal{F}_N$, such that*

$$E \subset \bigcup_{i=1}^{N(n)} \bigcup_{B \in \mathcal{F}_i} B$$

*where balls in each $\mathcal{F}_i$ are disjoint.*

**Remarks.**

1. The Vitali covering lemma only requires metric space structure, independent of dimension, etc. However, the lemma is useless unless the measure has some structure, i.e. "doubling":

   $$\mu(2B) \leq C_0 \mu(B)$$

   for some constant $C_0$. In Euclidean space $\mathbb{R}^n$ this constant is $2^n$. Note that the subcollection is disjoint, and to cover we then need to enlarge the sets.

2. The Besicovitch covering lemma depends on Euclidean structure, but is useful for arbitrary measures. The difference between the two covering lemmas is that this lemma allows for a bounded number of overlaps when covering $E$, and that the sets in the cover do not need to be enlarged.

**Proof. (Vitali Covering)** Let

$$\mathcal{F}_j = \left\{ B \in \mathcal{F}: \frac{R}{2^{j+1}} \leq \operatorname{diam}(B) \leq \frac{R}{2^j} \right\}$$

for $j = 0, 1, 2, \ldots$ and $R = \sup_{B \in \mathcal{F}} \operatorname{diam}(B)$. Note if the diameter is arbitrarily large (i.e. $R = \infty$), then we can just cover the whole space easily, taking a sequence of $B_n$ whose diameters go to $\infty$. Thus we assume $R < \infty$. We then construct a sequence as follows.

First let $\beta_0$ be the maximal (countable) subcollection of disjoint balls from $\mathcal{F}_0$ (existence is guaranteed by Hausdorff Maximal Principle). Then let $\beta_j$ be the maximal subcollection of disjoint balls from $\mathcal{F}_j$ such that they are also disjoint from the previously chosen sets. Then letting $\mathcal{F}' = \bigcup_{j=0}^{\infty} \beta_j$ gives the desired collection.

To prove this, suppose $B \in \mathcal{F}$. Then $B \in \mathcal{F}_k$ for some $k$, and $B \cap \left[ \bigcup_{j=0}^{k} \beta_j \right] \neq \emptyset$, otherwise this contradicts the maximality of $\beta_k$ (can fit in one more set). Thus there exists $S \in \bigcup_{j=0}^{k} \beta_j$ such that $B \cap S \neq \emptyset$. Note $\operatorname{diam}(B) \leq 2 \operatorname{diam}(S)$ by construction. (Worst case $S \in \beta_k$, where $2 \operatorname{diam}(S) \geq \frac{R}{2^j} \geq \operatorname{diam}(B)$). Thus $B \subset \hat{S} = 5S$ (draw a picture). This implies that for any $B \in \mathcal{F}$, $B \subset \hat{S}$ for some $S \in \mathcal{F}'$, and so $E \subset \bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{F}'} B$. □

The proof of the other covering lemma is to be looked up somewhere else... It may be useful someday.

Now we return to the proof of the theorem.

**Proof. (Theorem 1)** Let $E_\alpha = \{x: M_f > \alpha\}$, $\alpha \geq 0$. Then for all $x \in E_\alpha$, by definition of $M_f$ we have that there exists some $r_x > 0$ such that

$$\frac{1}{|B_{r_x}(x)|} \int_{B_{r_x}(x)} |f(y)| dy > \alpha$$

Then $\{B_{r_x}(x) \colon x \in E_\alpha\}$ covers $E_\alpha$, and we can apply the Vitali covering lemma to find a countable subcover. There exists $\{x_i\}_{i=1}^\infty$ and corresponding $r_{x_i}$ such that $E_\alpha \subset \bigcup_{i=1}^\infty \widehat{B_i}$, where $B_i = B_{r_{x_i}}(x_i)$, mutually disjoint. Then

$$|E_\alpha| \leq \left| \bigcup_{i=1}^\infty \widehat{B_i} \right| \leq \sum_{i=1}^\infty \left| \widehat{B_i} \right| = 5^n \sum_{i=1}^\infty |B_i|$$

From the above we have $|B_i| < \frac{1}{\alpha} \int_{B_i} |f(y)| dy$, so that

$$|E_\alpha| \leq \frac{5^n}{\alpha} \sum_{i=1}^\infty \int_{B_i} |f(y)| dy = \frac{5^n}{\alpha} \int_{\bigcup B_i} |f(y)| dy \leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f(y)| dy$$

where the equality follows from $B_i$ being disjoint. This proves part (b) and taking $\alpha \to \infty$ proves part (a) in the case $p = 1$. For $p > 1$, we now prove the trick $(\star)$ above:

**Lemma.**

$$\lambda_{M_f}(\alpha) \leq \frac{2A}{\alpha} \int_{\{|f| > \alpha/2\}} |f| dx$$

**Proof.** Note that $|f(x)| \leq \frac{\alpha}{2} + \mathbf{1}_{\{|f| > \alpha/2\}} |f|$ Calling $f_1 = |f| \mathbf{1}_{\{|f| > \alpha/2\}}$, we have

$$M_f(x) \leq \frac{\alpha}{2} + M_{f_1}(x)$$

and note that if $M_f > \alpha$, then $M_{f_1} > \alpha/2$, so that

$$\begin{aligned} \lambda_{M_f}(\alpha) &= |E_\alpha| \\ &= |\{x \colon M_f > \alpha\}| \\ &\leq |\{x \colon M_{f_1} > \alpha/2\}| \end{aligned}$$

Now using the result we just proved,

$$|\{x \colon M_{f_1} > \alpha/2\}| \leq \frac{2A}{\alpha} \|f_1\|_{L^1} = \frac{2A}{\alpha} \int_{|f| > \alpha/2} |f| dx$$

$\square$

Now we can prove (b) of the theorem for $1 < p < \infty$. Let $g = M_f$.

$$\begin{aligned} \int_{\mathbb{R}^n} g^p \, dx &= p \int_0^\infty \alpha^{p-1} \lambda_g(\alpha) \, d\alpha \\ &= p \int_0^\infty \alpha^{p-1} \frac{2A}{\alpha} \int_{|f| > \alpha/2} |f(x)| \, dx \, d\alpha \\ &= 2Ap \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f|} \alpha^{p-2} d\alpha \, dx \\ &= 2Ap \int_{\mathbb{R}^n} |f(x)|^{p-1} \frac{2^{p-1}}{p-1} \, dx \end{aligned}$$

Taking $p$-th roots, we conclude

$$\|M_f(x)\|_{L^p} \leq 2A^{1/p} \left( \frac{p}{p-1} \right)^{1/p} \|f\|_{L^p}$$

where the constant $A$ from Vitali, affected by the doubling constant. This proves part (c) for $1 < p < \infty$, and taking For $p = \infty$, the theorem (parts (a) and (c)) follows from the trivial integral estimate

$$M_f(x) \leq \|f\|_{L^\infty}$$

Finally, for $1 < p < \infty$, $(c) \Longrightarrow (a)$ follows from the first tool above,

$$\|M_f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} \lambda_{M_f}(\alpha)\, d\alpha$$

where the finiteness of $\|M_f\|_{L^p}$ implies the integrability of $\alpha^{p-1}\lambda_{M_f}(\alpha)$. This implies that

$$\lim_{\alpha \to \infty} \alpha^{p-1} \lambda_{M_f}(\alpha) = 0$$

from which we conclude $\lambda_{M_f}(\alpha) = |\{|M_f| > \alpha\}| \to 0$, i.e. $M_f$ is finite almost everywhere. □

As a Corollary to the Theorem, we can show that

$$|f| \leq |M_f|$$

for almost every $x \in \mathbb{R}^n$.

**Corollary 4.** *If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ (or $L^1_{\text{loc}}$, say). Then*

$$\lim_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy = f(x)$$

*for almost every $x \in \mathbb{R}^n$. (i.e. almost every point is a Lebesgue point of $f$)*

**Proof. (Idea)** First, we note that for continuous functions, the statement follows by continuity. The idea is to approximate by continuous functions. For any $\eta > 0$, we note that there exists $g \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ such that $\|f - g\|_{L^1} < \eta$. Then

$$|\{M(f - g) > \varepsilon\}| \leq \frac{A}{\varepsilon} \|f - g\|_{L^1} = \frac{A\eta}{\varepsilon}$$

Since $\frac{1}{|B_r(x)|} \int_{B_r(x)} f(x)\, dy = f(x)$, it suffices to prove that

$$\limsup_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy = 0$$

Now examining the decomposition $f(y) = g(y) + (f - g)(y)$, we study the set

$$\left\{ x\colon \limsup_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f(x)|\, dy > \varepsilon \right\}$$

in the hopes of bounding the measure by something that vanishes as $\varepsilon \to 0$.

By the triangle inequality, the above set is contained in the set where

$$\left\{ \limsup_{r \searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - g(x)|\, dy + M_{f-g}(x) + |f(x) - g(x)| > \varepsilon \right\}$$

by the triangle inequality. Note that $\limsup_{r\searrow 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - g(x)| dy = 0$ by continuity. Moreover, this set is contained in

$$\left\{M_{f-g}(x) > \frac{\varepsilon}{2}\right\} \bigcup \left\{|f(x) - g(x)| > \frac{\varepsilon}{2}\right\}$$

noting that if $A + B > \varepsilon$, then either $A$ or $B$ must be larger than $\frac{\varepsilon}{2}$ (or else $A + B < \varepsilon$).

Now, since $\|f - g\|_{L^1} < \eta$, by Chebyshev inequality $\left|\left\{f(x) - g(x) > \frac{\varepsilon}{2}\right\}\right| \leq \frac{2\eta}{\varepsilon}$. Thus we have that by monotonicity of measure ($A \subset B \Longrightarrow |A| \leq |B|$),

$$\begin{aligned}
\left|\left\{x\colon \limsup_{r\searrow 0} \frac{1}{|B_r(x)|} \int |f(y) - f(x)| dy > \varepsilon\right\}\right| &\leq \left|\left\{M_{f-g}(x) > \frac{\varepsilon}{2}\right\} \bigcup \left\{|f(x) - g(x)| > \frac{\varepsilon}{2}\right\}\right| \\
&\leq \left|\left\{M_{f-g}(x) > \frac{\varepsilon}{2}\right\}\right| + \left|\left\{|f(x) - g(x)| > \frac{\varepsilon}{2}\right\}\right| \\
&\leq \frac{2A\eta}{\varepsilon} + \frac{2\eta}{\varepsilon} \\
&= (2A + 2)\frac{\eta}{\varepsilon}
\end{aligned}$$

Now for any $\varepsilon > 0$ given, we can choose $\eta = \varepsilon^2$ to bound the set by $(2A + 2)\varepsilon$, and taking $\varepsilon \to 0$ gives the result. □

**Remark 5.** Note that denoting $M_r f(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy$, the result tells us that $\lim_{r\to 0} M_r f(x)$ exists a.e. and equals $f(x)$ a.e. The general tool that gives us convergence a.e. is the weak $L^p$ estimate on $Mf$, the maximal function. This will be a common theme when proving that limits exist a.e.

**Remark.** Above, the results of the theorem hold if we replace balls by other shapes. Replacing balls by squares works (squares can be bounded between two circles). Replacing balls by rectangles also works, if the sides are parallel to the axes. Even weaker, replacing balls by the dilation/translation family $\{\lambda O, \lambda > 0\}$ where $O$ is a fixed convex set containing the origin.

Why do we care about different shapes? Depends on the structure of the problem, some shapes appear more naturally. A story about Fritz-John, who proved that the affine transform of a convex set is bounded between two balls.

# Week 2 (9/16/2009)

## Decomposition of Sets and Functions

**Theorem 6. (Whitney's Decomposition)** *Let $F$ be a closed subset of $\mathbb{R}^n$ and let $\Omega = \mathbb{R}^n \backslash F$. Then we can find a sequence of cubes $Q_j$ with sides parallel to the axis and whose interior are mutually disjoint, such that $\Omega = \bigcup_{j=1}^{\infty} Q_j$ and whose side lengths $l(Q_j)$ are comparable with $d(Q_j, F)$, i.e.*

$$c_1 \operatorname{diam} Q_j \leq d(Q_j, F) \leq c_2 \operatorname{diam}(Q_j)$$

*for all $j$. Here $d(Q_j, F) = \inf_{x\in Q_j, y\in F} d(x, y)$. In Stein's proof, $c_1 = 1$ and $c_2 = 4$.*

**Remark.** This allows us to obtain a partition of unity for arbitrary open sets in $\mathbb{R}^n$. This will be useful in extending functions defined on closed subsets $F$ to the entire space. The proof is quite difficult. See Stein's book.

**Theorem 7. (Calderon-Zygmund Decomposition)** *Given $0 \leq f(x) \in L^1(\mathbb{R}^n)$, $\alpha > 0$, we can decompose $\mathbb{R}^n$ in the following way:*

*a)* $\mathbb{R}^n = F \cup \Omega$ *where $F$ and $\Omega$ are disjoint (or overlap on a set of measure zero)*

*b)* $f(x) \leq \alpha$ *almost everywhere on $F$*

*c)* $\Omega = \bigcup_{i=1}^{\infty} Q_j$ *where $Q_j$ are cubes with mutually disjoint interiors with the* ***additional*** *constraint that*

$$\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx \leq 2^n \alpha$$

This decomposes the space into a region $F$ where $f$ is small, and another region where $f$ may be large, but with some control on the $L^1$ norm of $f$.

Later, we will see other decompositions. A refinement of this decomposition is the *atomic decomposition of Hardy Spaces*, proved by Fefferman and Stein, involving the replacement of $L^1$ and $L^\infty$ by $H^1$ and BMO... and L. Carleson's *decomposition of $\mathcal{N}$-space*, involving the dual to Carleson measures... and as an aside, Carleson proved a.e. convergence of Fourier series of $L^2$ functions. We will also see this later.

**Proof. (of C-Z Decomposition)** Given $0 \leq f \leq L^1(\mathbb{R}^n)$ and $\alpha > 0$, there exists a decomposition of $\mathbb{R}^n$ into large cubes of size $\gg 1$ such that for each cube $Q_0$ we have that

$$\frac{1}{|Q_0|} \int_{Q_0} f(x)\, dx < \alpha$$

This follows simply because

$$\frac{1}{|Q_0|} \int_{Q_0} f(x)\, dx \leq \frac{\|f\|_{L^1}}{|Q_0|}$$

so that taking $|Q_0|$ arbitrarily large allows us to make the average arbitrarily small, and in particular, less than $\alpha$. Then we perform a simple iterative process. Decompose each $Q_0$ into $2^n$ smaller cubes of equal sizes. There are then two possibilities for each $Q_1$ of smaller size:

a) $\frac{1}{|Q_1|} \int_{Q_1} f(x)\, dx < \alpha$

b) $\frac{1}{|Q_1|} \int_{Q_1} f(x)\, dx \geq \alpha$

In case (b), we take $Q_1$ to be one of the cubes in the C-Z decomposition. Otherwise, so long as there is any cube $Q$ in case (a), we decompose $Q$ the same way as $Q_0$, which is a process that produces at most countably many cubes for the C-Z decomposition.

Let $Q_j$ be a list of all the cubes produced by the above process. Then we use $\Omega = \bigcup_j Q_j$ and $F = \mathbb{R}^n \backslash \Omega$. For each $Q_j$, note that

$$\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx$$

and that $Q_j$ has a precessor $\hat{Q}_j$ in the process above such that $\frac{1}{|\hat{Q}_j|} \int_{\hat{Q}_j} f(x)\, dx < \alpha$. This implies that

$$\frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx \leq \frac{2^n}{|\hat{Q}_j|} \int_{\hat{Q}_j} f(x)\, dx \leq 2^n \alpha$$

This gives the third condition of C-Z. Now for any $x_0 \in F$, we note that there is a sequence of cubes $\{Q_j'\}$ such that $x_0 \in Q_j$ and that $\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx < \alpha$. Then by the Lebesgue differentiation theorem, $f(x_0) \leq \alpha$ for almost all $x_0 \in F$. This is because for almost every $x_0 \in F$,

$$f(x_0) = \lim_{r \to 0} \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} f(x)\,dx = \lim_{j \to \infty} \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \leq \alpha$$

Replacing balls with cubes is a technical point, but intuitively we can find a sequence of balls which nest nicely with the cubes, and in the limit the difference of the average integrals tends to zero. This proves the second condition.

This second condition can also be proved in a different manner. Assume the measure of the set $\{x \in F\colon f(x) > \alpha\}$ has positive measure... Then something about Lebesgue points and then can find a cube for which the integral is larger than $\alpha$, a contradiction... (to figure out later) $\square$

**Note.** I do not know if the $Q_j$ in the above construction satisfy the comparable length condition in Whitney's decomposition. Most likely not...

**Corollary 8.** *Suppose $f \geq 0$, $f \in L^1(\mathbb{R}^n)$, $\alpha > 0$ and $F, \Omega, Q_j$ has in the C-Z Decomposition. Then there are two constants $A, B > 0$ (dependent on dimension) such that*

*a)* $|\Omega| \leq \frac{A}{\alpha}\|f\|_{L^1}$ *($A = 1$ in proof)*

*b)* $\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \leq B\alpha$ *($B = 2^n$ in proof)*

Sometimes it is required that $\Omega$ be open, and then we need a stronger decomposition than C-Z and a different approach using Whitney's Decomposition and the maximal function.

**Proof. (of C-Z with closed $F$)** For $0 \leq f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ given, define

$$F = \{x \in \mathbb{R}^n\colon Mf \leq \alpha\}$$

note that $F$ is closed by definition. This implies $\Omega = \{x \in \mathbb{R}^n\colon Mf > \alpha\} = F^c$ is open. Recall the weak (1,1) estimate from before:

$$|\Omega| \leq \frac{A}{\alpha}\|f\|_{L^1}$$

(which followed from Vitali, with $A = 5^n$). If we use Whitney's decomposition of $\Omega = \bigcup_j Q_j$ with $Q_j$ cubes with mutually disjoint interior and

$$c_1 \operatorname{diam} Q_j \leq d(Q_j, F) \leq c_2 \operatorname{diam} Q_j$$

Now let $p_j \in F$ such that $d(p_j, Q_j) = d(Q_j, F)$.

We can then find a ball $B_j$ centered at $p_j$ such that diam $B_j \approx l(Q_j)$ (comparable lengths) and $Q_j \subset B_j$. Also since $p_j \in F$, $Mf(p_j) \leq \alpha$, which implies that

$$\alpha \geq \frac{1}{|B_j|}\int_{B_j} f(x)\,dx \geq \frac{c}{|Q_j|}\int_{Q_j} f(x)\,dx$$

($\int_{Q_j} f \leq \int_{B_j} f$ and $|B_j| \leq c|Q_j|$) so that $\alpha \leq \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \leq C\alpha$ (the first inequality follows from $Q_j \subset \Omega = \{x\colon Mf > \alpha\}$). $\square$

**Note.** A tiny note on convention, wherever possible capital constants $C$ will usually denote large quantities and lower case constants $c$ will denote small quantities.

To draw complete connections between the different results of the past two lectures, we will show that the weak (1,1) estimate

$$|\{Mf > \alpha\}| \leq \frac{A}{\alpha}\|f\|_{L^1}$$

can be derived from the C-Z decomposition. Note that the C-Z decomposition itself did not need the maximal function to prove, although using the maximal function with Whitney's decomposition gave a C-Z decomposition with $F$ closed.

**Proof. (Weak (1,1) Estimate from C-Z decomposition)**

Let $f \in L^1(\mathbb{R}^n)$, $f \geq 0$ and $\alpha > 0$. The C-Z Decomposition gives us $F, \Omega, Q_j$ with $\Omega = \bigcup_{j=1}^{\infty} Q_j$ and $F = \Omega^c$, with

$$|\Omega| \leq \frac{1}{\alpha}\|f\|_{L^1}$$

since $\frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \geq \alpha$. Let $\hat{Q}_j = 3Q_j$ and $\hat{\Omega} = \bigcup_{j=1}^{\infty} \hat{Q}_j$. Then

$$|\hat{\Omega}| \leq 3^n|\Omega| \leq \frac{3^n}{\alpha}\|f\|_{L^1}$$

The claim is that if $x \notin \hat{\Omega}$, then $Mf(x) \leq C\alpha$ for some constant $C$ not dependent on $\alpha$. First note that there is a sequence of dyadic cubes $R_j$ with $|R_{j+1}| = \frac{1}{2^n}|R_j|$ converging to $x$, with

$$\frac{1}{|R_j|}\int_{R_j} f(x)\,dx \leq \alpha$$

The above condition guarantees that all the adjacent cubes of the same size as $R_j$ are also contained in $F$, or in other words, $\hat{R}_j = 3R_j$ satisfies $\hat{R}_j \subset F$. Note that this implies

$$\frac{1}{\left|\hat{R}_j\right|}\int_{\hat{R}_j} f(x)\,dx \leq \alpha$$

as well. Therefore $Mf(x) \leq C\alpha$ follows from

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} f(x)\,dx \leq \frac{C}{|\hat{R}_j|}\int_{\hat{R}_j} f(x)\,dx \leq C\alpha$$

where given any ball $B_r(x)$, we can find $\hat{R}_j$ such that $|\hat{R}_j| \leq C|B_r(x)|$ (if we use $L^\infty$ metric on $\mathbb{R}^n$, where the balls become cubes, then this constant is $C = 2^n$). Note that without $\hat{R}_j$, i.e. if we used just $R_j$, we may not be able to embed the balls in this fashion, since $x$ may be on the corner of all the $R_j$'s. Finally, this implies that

$$|\{x: Mf > C\alpha\}| \leq |\hat{\Omega}| \leq 3^n|\Omega| \leq \frac{3^n}{\alpha}\|f\|_{L^1}$$

which proves the weak (1,1) estimate. □

Now recall an observation from last lecture:

$$|\{x\colon Mf>\alpha\}|\leq\frac{2A}{\alpha}\int_{\{|f|>\alpha/2\}}|f(x)|dx$$

which was used to prove a sufficient condition for $Mf\in L^1(B)$ for bounded set $B$. On the other hand, we can now show that

$$|\{x\colon Mf>\alpha\}|\geq\frac{c'}{\alpha}\int_{\{|f|>\alpha/2\}}|f(x)|dx$$

for some constant $c'>0$, so that $|\{x\colon Mf>\alpha\}|$ is comparable to $\frac{1}{\alpha}\int_{\{|f|>\alpha/2\}}|f(x)|dx$.

**Proposition 9.** *For $f\in L^1(\mathbb{R}^n)$,*

$$|\{x\colon Mf>\alpha\}|\geq\frac{c'}{\alpha}\int_{\{|f|>\alpha/2\}}|f(x)|dx$$

*for some constant $c'>0$.*

**Proof.** Note that from the third property of the C-Z decomposition,

$$2^n\alpha\geq\frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx\geq\alpha$$

Then if $x\in Q_j$, $Mf(x)\geq c\alpha$. We can see this by noting that if $x$ lies in the center of the cube, then we already have that

$$Mf(x)\geq\frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx\geq\alpha$$

using the definition of $Mf$ with cubes instead of balls ($L^\infty$ metric). Otherwise, a worst case scenario is that if $x$ lies in the corner, and the adjacent cubes that share $x$ as a corner are all contained in $F$ (where $0\leq|f|\leq\alpha$). Let the union of all the cubes sharing $x$ as a corner be $Q$. We can bound $Mf$ below by

$$\frac{1}{|Q|}\int_Q|f(x)|dx\geq\frac{1}{2^n|Q_j|}\int_{Q_j}|f(x)|dx\geq 2^{-n}\alpha$$

The other cases lie in between, and so $c=2^{-n}$. Note that

$$|\{|f|\geq\alpha\}|\leq|\Omega|\leq|\{Mf>c\alpha\}|$$

since $|f|\leq\alpha$ a.e. on $F=\Omega^c$ and for every $x\in\Omega$, $x\in Q_j$ and so $Mf(x)>c\alpha$. Finally, using the fact that $2^n\alpha\geq\frac{1}{|Q_j|}\int_{Q_j}|f(x)|dx$ (C-Z property), $|Q_j|\geq\frac{2^{-n}}{\alpha}\int_{Q_j}|f(x)|dx$ and

$$|\{Mf>c\alpha\}|\geq|\Omega|\geq\frac{2^{-n}}{\alpha}\int_\Omega|f(x)|dx\geq\frac{2^{-n}}{\alpha}\int_{|f|\geq\alpha}|f(x)|dx$$

Now replacing $c\alpha$ with $\alpha$, we have

$$|\{Mf>\alpha\}|\geq\frac{c2^{-n}}{\alpha}\int_{|f|\geq\alpha/c}|f(x)|dx\geq\frac{c'}{\alpha}\int_{|f|\geq\alpha/2}|f(x)|dx$$

noting that $c$ is small ($c = 2^{-n}$). $\square$

**Remark.** For the next proof, we will use the form of the inequality

$$|\{Mf > \alpha\}| \geq \frac{c'}{\alpha} \int_{|f| \geq \alpha} |f(x)|\, dx$$

(in the proof above the $\alpha/2$ is arbitrary, and can be replaced by anything larger than $\alpha/c$).

**Corollary 10.** *For a function $f$ supported in $B$ where $f \in L^1(B)$, we have that*

$$f \log(1 + |f|) \in L^1(B) \Longleftrightarrow Mf \in L^1(B)$$

**Proof.** The forward direction ( $\Longrightarrow$ ) was proved last time with the trick discussed above. The converse direction is proved similarly with the inequality we just proved.

$$\begin{aligned}
\|Mf\|_{L^1} &\geq \int_1^\infty |\{x : Mf(x) \geq \alpha\}|\, d\alpha \\
&\geq c' \int_1^\infty \frac{1}{\alpha} \int_{\{|f| > \alpha\}} |f(x)|\, dx\, d\alpha \\
&= c' \int_{|f| > 1} |f(x)| \int_1^{|f|} \frac{d\alpha}{\alpha} \\
&= c' \int_{|f| > 1} |f(x)| \log(|f(x)|)\, dx \\
&\approx \int_B |f(x)| \log(1 + |f(x)|\, dx
\end{aligned}$$

since $\log(|f(x)|)$ behaves asymptotically like $\log(1 + |f(x)|)$ for large $|f(x)|$. This proves the result. $\square$

**Remark.** This result gives us a way to characterize the Hardy space, defined to be the space of functions $f$ such that $Mf \in L^1(B)$.

## An Aside About Theorems in Analysis

First, define Hausdorff metric $\mathrm{HD}(A, B)$ for sets $A, B$ contained in a metric space $(X, d)$.

$$\mathrm{HD}(A, B) = \max\left(\sup_{a \in A} \inf_{b \in B} d(x, y), \sup_{b \in B} \inf_{a \in A} d(x, y)\right)$$

or equivalently,

$$\mathrm{HD}(A, B) \leq \varepsilon \Longleftrightarrow A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon$$

Intuitively, consider the following game: your friend picks an arbitrary point from either $A$ or $B$, and you must travel to the other set. $\mathrm{HD}(A,\, B)$ is then the longest distance you are forced to travel, where your friend picks the worst possible point, and you take the shortest path from this point to the other set.

Also, let $\rho_A(x) = \rho(x, A)$. Then

$$\|\rho_A - \rho_B\|_{L^\infty} = \mathrm{HD}(A, B)$$

In particular, the notion of convergence in HD metric can be translated to convergence of $\rho_A$ in $L^\infty$ norm.

**Remark.** Let $\mathcal{X} = \{C\colon C \text{ is a nonempty closed subset of } \mathbb{R}^n\}$. Then $(\mathcal{X}, \mathrm{HD})$ is a complete metric space. This can be justified by using the identification $A \longleftrightarrow \rho_A$ (metric preserving isomorphism).

**Proof.** Letting $E = \{\rho_A, A \in \mathcal{X}\}$, we note that $(E, L^\infty)$ is a metric space, and it suffices to show that it is complete. It then suffices to show that $E$ is closed, since $(C(\mathbb{R}^n), L^\infty)$ is a complete metric space. Suppose $\rho_{A_i} \to f$ in $L^\infty$ (i.e. uniformly). We want to show the existence of $A \in \mathcal{X}$ such that $\rho_A = f$. We note that for a given $\varepsilon > 0$ and $n$ sufficiently large, $|f(x) - \rho_{A_n}(x)| \leq \varepsilon$ for all $x$. Note that

$$A_n = \{x\colon \rho_{A_n}(x) = 0\} \subset \{x\colon f(x) \leq \varepsilon\} \subset \{x\colon d(x, A_n) \leq 2\varepsilon\} = (A_n)_{2\varepsilon}$$

This implies that

$$d(x, A_n) \leq d(x, \{f \geq \varepsilon\}) \leq d(x, (A_n)_{2\varepsilon})$$

(by an argument involving triangle inequality). Furthermore, since

$$|d(x, A_n) - d(x, (A_n)_{2\varepsilon}| \leq 2\varepsilon$$

we have that

$$f(x) - \varepsilon \leq d(x, A_n) \leq d(x, \{f \geq \varepsilon\}) \leq d(x, A_n) + 2\varepsilon \leq f(x) + 3\varepsilon$$

and letting $\varepsilon \to 0$ we have that $f(x) = d(x, \{f = 0\})$. Letting $A = \{x\colon f(x) = 0\}$ gives the result. □

Many theorems in analysis come in three forms: one that is set-theoretic, function-theoretic, and measure-theoretic. One form of a theorem can usually be transferred to and from other settings by setting up correspondences. Here we have transferred a set-theoretic property of HD to function spaces.

Another example, for instance, are forms of Bolzano Weierstrass (extracting a convergent subsequence from a bounded sequence). Arzela-Ascoli's Theorem, which is function theoretic. Helly's Selection Theorem is an analogue for measures, and for HD there is a corresponding theorem Blaschke Selection Theorem.

**Coming next...**

1. For sublinear operators $T$, where $\|T(x+y)\| \leq \|Tx\| + \|Ty\|$, for example the maximal operator $f \mapsto Mf$, we will derive some interpolating theorems that allow us to determine if it is a bounded operator on $L^p$ spaces. The Marcinkiewicz interpolation theorem is the relevant result, and it will tell us that the Hardy Littlewood Maximal Operator $M$ is bounded from $L^p \to L^p$ for $1 < p \leq \infty$

2. The Riesz-Thorin Interpolation Theorem for linear operators tells us that if $T$ is a bounded linear operator from $L^p \to L^p$ and from $L^q \to L^q$, then it is a bounded operator from $L^r \to L^r$ for $p \leq r \leq q$. The proof involves Hadamard's 3-circle theorem, with $C_r = C_p^t C_q^{1-t}$ and $\frac{1}{r} = \frac{t}{p} + \frac{1-t}{q}$.

## Week 3 (9/23/2009)

### Interpolation Theorems

We will finish developing analysis tools. The following interpolation theorems allow us to conclude the boundedness of operators $T$ from $L^p \to L^p$ given bounds on the operator taken as a mapping from $L^{p_i} \to L^{p_i}$.

**Theorem 11. (Marcinkiewicz Interpolation)** *Let $T$ be a sublinear map, e.g.*

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|$$

*and let $T$ be defined on $L^{p_1} \cap L^{p_2}$ with $1 \leq p_1 < p_2 \leq \infty$ satisfying*

$$|\{x \colon |Tf(x)| \geq \alpha\}| \leq c_i \frac{\|f\|_{p_i}^{p_i}}{\alpha^{p_i}}, \quad i = 1, 2$$

*(i.e. "$T$ is weak-type $(p_1, p_1)$ and weak-type $(p_2, p_2)$")*

*Then $T$ is a bounded operator from $L^p \to L^p$ for $p_1 < p < p_2$, (strong-type $(p, p)$), i.e.*

$$\|Tf\|_p \leq c_p \|f\|_p$$

*where the constant $c_p$ depends only on $c_1, c_2$.*

As a quick remark, in the special case where $Tf = Mf$, the maximal operator, we already proved this result. We showed that $M$ is weak-type $(1,1)$ and strong-type $(\infty, \infty)$ (which implies weak-type $(\infty, \infty)$), and if we use the above result then we are done. The proof of the Marcinkiewics Interpolation is very similar to the steps of the proof of the maximal inequality from the first lecture.

**Proof.** The idea of the proof is to exploit the bounds for $L^{p_i}$. Given any $f \in L^p$, we can decompose $f$ as a sum of a function in $L^{p_1}$ and a function in $L^{p_2}$. The intuition is that near $\infty$, where decay determines integrability, higher powers $L^p$ cause faster decay, so for functions bounded above (in absolute value), $f \in L^p \Rightarrow f \in L^{p_2}$. Locally, lower powers of $L^p$ behave nicer (higher powers cause large values to blow up faster), and so for functions bounded below (in absolute value), $f \in L^p \implies f \in L^{p_1}$. Thus for any $\alpha$, we can decompose $f = f_1 + f_2$, where $f_1 = f\mathbf{1}_{\{|f| \geq \alpha\}} \in L^{p_1}$ and $f_2 = f\mathbf{1}_{\{|f| < \alpha\}} \in L^{p_2}$. To show this quickly,

$$\begin{aligned}
\int_{|f| \geq \alpha} |f(x)|^{p_1}\, dx &= \int_{\alpha \leq |f| \leq 1} |f(x)|^{p_1}\, dx + \int_{|f| \geq \max(\alpha, 1)} |f(x)|^{p_1}\, dx \\
&\leq |\{\alpha \leq |f| \leq 1\}| + \int_{|f| \geq \max(\alpha, 1)} |f(x)|^p\, dx \\
&\leq |\{\alpha \leq |f| \leq 1\}| + \|f\|_p^p \\
&< \infty
\end{aligned}$$

where we note that $|f(x)|^{p_1} \leq |f(x)|^p$ for $|f(x)| \geq 1$, and $|\{\alpha \leq |f| \leq 1\}| < \infty$ since

$$\infty > \int |f(x)|^p\, dx \geq \int_{\alpha \leq |f| \leq 1} |f(x)|^p\, dx \geq \alpha^p\, |\{\alpha \leq |f| \leq 1\}|$$

Thus $f_1 \in L^{p_1}$. An analogous proof holds to show $f_2 \in L^{p_2}$. For sanity, we do it anyway:

$$\begin{aligned}
\int_{|f| \leq \alpha} |f(x)|^{p_2}\, dx &= \int_{1 \leq |f| \leq \alpha} |f(x)|^{p_2}\, dx + \int_{|f| \leq \min(\alpha, 1)} |f(x)|^{p_2}\, dx \\
&\leq \alpha^{p_2} |\{1 \leq |f| \leq \alpha\}| + \|f\|_p^p \\
&< \infty
\end{aligned}$$

Now that we have this decomposition, note that

$$\begin{aligned}
|\{x \colon |Tf| > 2\alpha\}| &\leq |\{x \colon |Tf_1| > \alpha\}| + |\{x \colon |Tf_2| > \alpha\}| \\
&\leq \frac{c_1 \|f_1\|_{p_1}^{p_1}}{\alpha^{p_1}} + \frac{c_2 \|f_2\|_{p_2}^{p_2}}{\alpha^{p_2}} \\
&= \frac{c_1}{\alpha^{p_1}} \int_{|f| \geq \alpha} |f(x)|^{p_1}\, dx + \frac{c_2}{\alpha^{p_2}} \int_{|f| < \alpha} |f(x)|^{p_2}\, dx
\end{aligned}$$

since if $2\alpha < |Tf| = |Tf_1 + Tf_2| \leq |Tf_1| + |Tf_2|$, either $|Tf_1| > \alpha$ or $|Tf_2| > \alpha$ (otherwise the sum is $\leq 2\alpha$, a contradiction). Now we use the Fubini type result to compute $\|Tf\|_p^p$:

$$\begin{aligned}
\|Tf\|_p^p &\simeq \int_0^\infty \alpha^{p-1} |\{|Tf| > 2a\}|\, d\alpha \\
&\leq \int_0^\infty c_1 \alpha^{p - p_1 - 1} \int_{|f| \geq \alpha} |f(x)|^{p_1}\, dx\, d\alpha \\
&\quad + \int_0^\infty c_2 \alpha^{p - p_2 - 1} \int_{|f| < \alpha} |f(x)|^{p_2}\, dx\, d\alpha \\
&= \int |f(x)|^{p_1} \int_0^{|f(x)|} c_1 \alpha^{p - p_1 - 1}\, d\alpha\, dx & (p > p_1) \\
&\quad + \int |f(x)|^{p_2} \int_{|f(x)|}^\infty c_2 \alpha^{p - p_2 - 1}\, d\alpha\, dx & (p < p_2) \\
&= \frac{c_1}{p - p_1} \int |f(x)|^p\, dx + \frac{c_2}{p_2 - p} \int |f(x)|^p \\
&= \tilde{c}\, \|f\|_p^p \\
&< \infty
\end{aligned}$$

where $\tilde{c} = \frac{c_1}{p - p_1} + \frac{c_2}{p_2 - p}$. This proves that $T$ is a bounded operator from $L^p \to L^p$ with norm bounded by $\frac{c_1}{p - p_1} + \frac{c_2}{p_2 - p}$. □

**Remark.** Note that we do not have good control over the operator norm of $T$ as a bounded operator from $L^p \to L^p$, since as $p \to p_1$, our bound on the norm tends to $\infty$. With a slightly stronger assumption, we can get good control over the norm:

**Theorem 12. (Riesz-Thorin)** *If $T$ is a bounded* ***linear*** *operator from $L^{p_i} \to L^{p_i}$ for $i = 1, 2$ with norms $c_i$, i.e. $|Tf|_{L^{p_i}} \leq c_i |f|_{L^{p_i}}$, then $T$ is a bounded linear operator from $L^p \to L^p$ with $p_1 \leq p \leq p_2$ with operator norm bounded by $c_1^t c_2^{1-t}$ where $\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}$.*

**Remark.** Before the proof, note that by Marcinkiewicz interpolation, since bounded $L^{p_i} \to L^{p_i}$ implies weak-type $(p_i, p_i)$, we already know that $T$ is a bounded operator from $L^p \to L^p$. This theorem gives good control on the operator norm (as claimed in the previous remark).

First we will need a lemma from complex analysis:

**Lemma 13. (Hadamard 3-lines Theorem)** *Let $f(z)$ be holomorphic in a strip $a < \mathrm{Re}(z) < b$ and let $M(x) = \max_{\mathrm{Re}\, z = x} |f(x)|$. Then $\log M(x)$ is convex in $x$.*

**Proof.** Consider $f_\varepsilon(z) = f(z)e^{\varepsilon z^2}$, noting that $\left|e^{\varepsilon z^2}\right| = e^{\varepsilon(x^2-y^2)}$ where $a<x<b$ and so as $|y|\to\infty$, so long as $\log|f(z)| = o\big(y^2\big)$, $f_\varepsilon(z)\to 0$. Thus by the maximum modulus principle, $|f_\varepsilon(z)|$ achieves its maximum along Re $z=a$ or Re $z=b$. Denoting $M_\varepsilon(x) = \max_{\mathrm{Re}\,z=x} |f_\varepsilon(z)|$, this implies that $M_\varepsilon(x)\le\max(M_\varepsilon(a), M_\varepsilon(b))$. Taking $\varepsilon\to 0$ shows that $M(x)\le\max(M(a),M(b))$.

So let's try.... $f(z)e$

To show convexity in $x$, now consider $f_t(z) = f(z)e^{tz}$, noting that $|e^{tz}| = e^{tx}$ which is bounded. The above result applied to $f(z)e^{tz}$, so long as $\log|f(z)| = o(y^2)$ as $y\to\infty$, implies that

$$\begin{aligned} M_t(x) &\le \max(M_t(a), M_t(b)) \\ e^{tx}M(x) &\le \max(e^{ta}M(a), e^{tb}M(b)) \\ M(x) &\le \max(e^{t(a-x)}M(a), e^{t(b-x)}M(b)) \end{aligned}$$

In particular, for any $t$ we have that

$$\log M(x)\le t(a-x)+\log M(a) \quad\text{and}\quad \log M(x)\le t(b-x)+\log M(b)$$

We then pick the values of $t$ such that

$$\log M(x)\le\frac{b-x}{b-a}\log M(a) \quad\text{and}\quad \log M(x)\le\frac{x-a}{b-a}\log M(b)$$

For the first one, $t=\frac{1}{b-a}\log M(a)$ and for the second one, take $t=\frac{-1}{b-a}\log M(b)$. Combining the two gives the convex inequality

$$\log M(x)\le\frac{b-x}{b-a}\log M(a)+\frac{x-a}{b-a}\log M(b)$$

which is valid so long as $M(a), M(b)>1$. Note that with $\lambda=\frac{x-a}{b-a}$ and $x=(1-\lambda)a+\lambda b$, this becomes

$$\log M((1-\lambda)a+\lambda b)\le(1-\lambda)\log M(a)+\lambda\log M(b)$$

Now without loss of generality we can assume $M(a), M(b)>1$, since we can always scale $f(z)$, which affects $\log M(x)$ by a constant term (does not affect convexity). Note that $M(a)\ne 0$, or else $f(z)=0$ on Re $z=a$, from which we conclude that $f$ must be identically zero, in which case the theorem is trivial. Likewise for $M(b)$. □

**Remark.** The conclusion of the lemma above can be written

$$\log M(x) = t\log M(a)+(1-t)\log M(b)$$

or

$$M(x) = M(a)^t M(b)^{1-t}$$

with $x = ta+(1-t)b$.

Using this tool we can then prove the Riesz-Thorin interpolation theorem:

**Proof. (of Riesz-Thorin)** The operator norm of $T$ in $L^p$, $\|T\|_{p,p}$ can be written by duality theory

$$\|T\|_{p,p} = \sup_{\substack{\|f\|_p\le 1\\ \|g\|_q\le 1}}\left|\int gTf\,d\mu\right|$$

For starters, we assume that $T$ is a complex operator, otherwise we can extend $T(f+ig)=Tf+iTg$. Thus $f$, $g$ are assumed to be complex functions. We can make a reduction by factoring $f$ as $f=\varphi|f|$ with $|\varphi|=1$, and likewise for $g=\psi|g|$, so

$$\|T\|_{p,p}=\sup_{\substack{\|f\|_p,\|g\|_q\leq 1\\ f,g\geq 0\\ |\varphi|,|\psi|=1}}\left|\int g\psi T(f\varphi)\,d\mu\right|$$

We can make a further reduction by noting that $f\in L^p\Longleftrightarrow f^p\in L^1$, and thus

$$\|T\|_{p,p}=\sup_{\substack{\|f\|_1,\|g\|_1\leq 1\\ f,g\geq 0\\ |\varphi|,|\psi|=1}}\left|\int g^x\,\psi T(f^{\,1-x}\varphi)\,d\mu\right|$$

with $x=1/q$. Denote $F(f,g,\varphi,\psi,x):=\int g^x\psi T(f^{1-x}\,\varphi)\,d\mu$, and now we extend to complex, replacing $f^{1-x}$, $g^x$ with $f^{1-z}$, $g^z$. Thus $F(f,g,\varphi,\psi,z)$ makes sense, and the claim is that (for a fixed $f,g,\varphi,\psi$) it is holomorphic in the strip $\frac{1}{q_1}\leq \operatorname{Re} z\leq\frac{1}{q_2}$. For one, $p_1\leq p\leq p_2$ implies that $\frac{1}{p_1}\geq\frac{1}{p}\geq\frac{1}{p_2}\Longrightarrow\frac{1}{q_1}\leq\frac{1}{q}\leq\frac{1}{q_2}$ so that in this region $T$ is a bounded operator from $L^{1-1/\operatorname{Re} z}\rightarrow L^{1-1/\operatorname{Re} z}$. Note that

$$g^z=g^{\operatorname{Re} z}e^{i\operatorname{Im} z\log g}$$

where the second term has modulus 1, and hence can be absorbed in the $\psi$ term. The same holds for $f^{1-z}$. Now $g^z(x)$ and $f^{1-z}(x)$ are analytic functions of $z$ for every $x$ such that $g(x)$, $f(x)$ are nonzero (defined by $e^{z\log g(x)}$). Therefore $T(f^{1-z}\,\varphi)$ is a complex function in $L^{1-1/\operatorname{Re} z}$ and by linearity it is analytic as well, noting that if $\phi(z)$ is analytic then $(T\phi)(z)$ is analytic too:

$$(T\phi)'(z)=\lim_{h\rightarrow 0}\frac{T\phi(z+h)-T\phi(z)}{h}=\lim_{h\rightarrow 0}T\left(\frac{\phi(z+h)-\phi(z)}{h}\right)=T(\phi'(z))$$

To handle the potential problems where $f,g$ vanish, we can approximating $f,g$ by functions $f_\varepsilon$, $g_\varepsilon$ that never vanish (add $\varepsilon e^{-\varepsilon x^2}$ for instance). This will cause the integrand to be holomorphic for all $x$, and then taking the integral $F(f_\varepsilon,g_\varepsilon,\varphi,\psi,z)$ is holomorphic. Taking a limit as $\varepsilon\rightarrow 0$ gives $F(f,g,\varphi,\psi,z)$ which we have now expressed as a limit of holomorphic functions. So long as the limit is uniform on every compact subset of the region $\frac{1}{q_1}\leq\operatorname{Re} z\leq\frac{1}{q_2}$, this limit becomes holomorphic. In fact, the convergence is uniform for all $z$ in the region since the real part is bounded.

Then now that we have that $F(f,g,\varphi,\psi,z)$ is holomorphic, we can apply the three lines lemma to get that

$$|F(f,g,\varphi,\psi,x)|\leq\left|F\left(f,g,\varphi,\psi,\frac{1}{q_1}\right)\right|^t\left|F\left(f,g,\varphi,\psi,\frac{1}{q_2}\right)\right|^{1-t}$$

with $\frac{1}{q}=\frac{t}{q_!}+\frac{1-t}{q_2}$and taking the supremum in $f,g,\varphi,\psi$ gives the result

$$\|T\|_{p,p}\leq\|T\|_{p_1,p_1}^t\|T\|_{p_2,p_2}^{1-t}$$

where $1-\frac{1}{q}=t-\frac{t}{q_1}+(1-t)-\frac{1-t}{q_2}$ or $\frac{1}{p}=\frac{t}{p_1}+\frac{1-t}{p_2}$.

□

Note that this technique extends to the case where $T: L^{p_i} \to L^{q_i}$ $i = 0, 1$ to conclude that $T$ is a bounded map from $L^{p_t} \to L^{q_t}$ where $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$ and $\frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$ with norm $M_0^{1-t} M_1^t$. What needs adjustment in the proof above is

$$\|T\|_{p,q} = \sup_{\substack{\|f\|_1, \|g\|_1 \leq 1 \\ f, g \geq 0 \\ |\varphi|, |\psi| = 1}} \left| \int g^{1/q_z'} \psi T(f^{1/p_z} \varphi) \, d\mu \right|$$

and establish that this is a holomorphic from $0 < \operatorname{Re} z < 1$.

## Fourier Transforms

The starting point for discussion about Fourier Transforms are integrable functions $f \in L^1(\mathbb{R}^n)$. Our definition of the Fourier Transform that we will be working with (other definitions differ in how the constants are distributed between the transform and its inverse) is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$$

with inverse

$$\check{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$$

**Note: This is a departure from class, inserting the $2\pi$ in the definition here saves a lot of headaches!**

Consider two simple operations:

- Translation by $h \in \mathbb{R}^n$:

  $$\tau_h f(x) := f(x + h)$$

  with Fourier transform

  $$\widehat{\tau_h f}(\xi) = e^{i \langle h, \xi \rangle} \hat{f}(\xi)$$

- Dilation by $a > 0$:

  $$\delta_a f(x) := f(a x)$$

  with Fourier transform

  $$\widehat{\delta_a f}(\xi) = a^{-n} \hat{f}\left(\frac{\xi}{a}\right)$$

The proofs are quite direct, involving change of variables. Here are a few more useful properties:

1. With the naive integral estimate,

   $$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$$

2. If $f \geq 0$, then

   $$\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \hat{f}(0) = \int_{\mathbb{R}^n} f(x) \, dx$$

just by noting that if $\xi \neq 0$ then the oscillation from the exponential will cause the value of $|\hat{f}(\xi)|$ to be smaller.

3. For one dimension, for indicators $\mathbf{1}_{(a,b)}$ we have

$$\widehat{\mathbf{1}_{(a,b)}}(t) = \frac{e^{-ibt} - e^{-iat}}{it}$$

which is by direct computation.

4. (*Riemann Lebesgue Lemma*) Again for one dimension, for $f \in L^1(\mathbb{R})$, as $t \to \infty$,

$$\hat{f}(t) \longrightarrow 0$$

To prove this we can approximate by $C_0^\infty$ functions, (which we then approximate by simple functions, then indicators).

Denote the Fourier transform operator by $\mathcal{F}(f) := \hat{f}$. We would like to classify the image of the Fourier trasnform on the space of integrable functions, $\mathcal{F}(L^1)$. We can show that

$$\mathcal{F}(L^1) \subseteq C_0(\mathbb{R}^n)$$

where $C_0(\mathbb{R}^n)$ are the continuous functions that vanish at $\infty$. This result can be proved from building up from indicators also (or using the modulus of continuity of $e^{-ix}$, can prove that the fourier transform of an integrable function is uniformly continuous).

However, there exist functions in $C_0(\mathbb{R}^n)$ for which there does not exist a corresponding $L^1$ function in the preimage. Consider for simplicity $n = 1$. We show this by considering an estimate for odd functions $f(-x) = -f(x)$. In this case, we know that

$$\hat{f}(x) = -i \int_{-\infty}^{\infty} \sin(tx) f(t) dt$$

since the integral of $\cos(tx) f(t)$ vanishes since it is odd. Then we note that

$$\begin{aligned} \left| \int_1^\infty \frac{\hat{f}(x)}{x} dx \right| &\leq \left| \int_1^\infty \frac{1}{x} \int_{-\infty}^{\infty} \sin(tx) f(t) dt dx \right| \\ &= \left| \int_{-\infty}^{\infty} f(t) \int_1^\infty \frac{\sin(tx)}{x} dx dt \right| \\ &\leq b \|f\|_{L^1} \end{aligned}$$

noting that $\int_1^\infty \frac{\sin(tx)}{x} dx$ is bounded for all $t$. Using this estimate, we can make the LHS $\infty$ for some choice of $\hat{f}(x) \in C_0(\mathbb{R})$ by considering $g(x) = \frac{1}{\log x}$. Then supposing there existed $f$ such that $\mathcal{F} f = g$, by the above estimate the left hand side is $\int_1^\infty \frac{1}{x \log x} dx = \infty$, and thus $\|f\|_{L^1} \geq \infty$ and $f$ is not integrable. Thus $\mathcal{F}(L^1(\mathbb{R})) \neq C_0(\mathbb{R})$.

As an aside, having $\mathcal{F}(L^1) \subset C_0(\mathbb{R})$, we can make sense of the following interesting proposition:

**Proposition 14.** *If $f, g \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f} g \, d\xi = \int_{\mathbb{R}^n} f \hat{g} \, dx$$

*Note that both sides are well defined, since $\hat{f}, \hat{g} \in C_0(\mathbb{R}^n)$.*

**Proof.** This is just Fubini:

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{f} g \, d\xi &= \int_{\mathbb{R}^n} g(\xi) \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi\rangle} \, dx d\xi \\ &= \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} g(\xi) e^{-2\pi i \langle x, \xi\rangle} \, d\xi dx \\ &= \int_{\mathbb{R}^n} f \hat{g} \, dx\end{aligned}$$

□

## Schwarz Class

On the positive side, there exists a special class of functions for which we can classify the image of the Fourier transform, the Schwarz class $\mathcal{S}$:

$$\mathcal{S} = \left\{ \varphi \in C^\infty(\mathbb{R}^n) \colon \lim_{x \to \infty} |D^k \varphi| (1 + |x|^2)^m \to 0 \; \forall k, \varphi \right\}$$

i.e. smooth functions where all derivatives decay faster than any polynomial.

First note the following additional properties concerning the differentiation operator $D\varphi := \frac{1}{2\pi i} \varphi'$. (the addition of the $1/2\pi i$ term simplifies notation). With integration by parts, and the fact that $\varphi$ decays at $\infty$,

$$\widehat{D\varphi}(\xi) = \int \frac{1}{2\pi i} \varphi'(x) e^{-2\pi i x \xi} \, dx = \xi \int \varphi(x) \, e^{-2\pi i x \xi} \, dx = \xi \, \hat{\varphi}(\xi)$$

and since $(-x) \, \varphi(x) e^{-2\pi i x \xi}$ is integrable,

$$\widehat{(-x) \, \varphi}(\xi) = \int \, (-x) \varphi(x) \, e^{-2\pi i x \xi} \, dx = \frac{1}{2\pi i} \cdot \frac{d}{d\xi} \int \varphi(x) e^{-2\pi i x \xi} \, dx = D \, \hat{\varphi}(\xi)$$

Generalizing to higher dimensions (which involves just taking products of these identities, we have that

$$\widehat{D^\alpha (-x)^\gamma \varphi}(\xi) = \xi^\alpha D^\gamma \, \hat{\varphi}(\xi)$$

where $\alpha, \gamma$ are multi-indices $D^\alpha = D_{\alpha_1} \cdots D_{\alpha_n}$, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$. Therefore, the Fourier transform of any $\varphi \in \mathcal{S}$ is also in $\mathcal{S}$, so $\mathcal{F} \colon \mathcal{S} \to \mathcal{S}$. In fact,

**Theorem 15.** *$\mathcal{F}$ is an isomorphism from $\mathcal{S} \to \mathcal{S}$ with inverse $\mathcal{F}^{-1}$ (given by the inversion formula).*

**Proof.** We showed that $\mathcal{F}$ maps from $\mathcal{S} \to \mathcal{S}$, so all that remains is to verify the inversion formula, i.e. $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{Id}$ on $\mathcal{S}$. This is by a computation involving Fubini's Theorem to swap the double integral. However, the complex exponential poses problems since it is not integrable. The idea is then to multiply by a Gaussian $e^{-\varepsilon |x|^2/2}$ and let $\varepsilon \to 0$. So first we prove a useful identity concerning the Fourier transform of the Gaussian $e^{-\varepsilon |x|^2/2}$, first in one dimension:

$$\begin{aligned}\widehat{\exp(-\varepsilon \pi x^2)} &= \int_{-\infty}^{\infty} e^{-2\pi i x \xi} e^{-\varepsilon \pi x^2} \, dx \\ &= e^{-\pi \xi^2/\varepsilon} \int_{-\infty}^{\infty} e^{-\varepsilon \pi (x - i\xi/\varepsilon)^2} \, dx \\ &= e^{-\pi \xi^2/\varepsilon} \int_{-\infty}^{\infty} e^{-\varepsilon \pi x^2} \, dx && \text{(Contour integration)} \\ &= \sqrt{\frac{1}{\varepsilon}} e^{-\pi \xi^2/\varepsilon}\end{aligned}$$

where the contour integration is of $e^{-\varepsilon z^2/2}$ on the boundary of the region $\{-\xi/\varepsilon < \operatorname{Im} z < 0\}$. Generalizing to products, we get

$$\widehat{\exp(-\varepsilon\pi|x|^2)} = \varepsilon^{-n/2} e^{-\frac{\pi|\xi|^2}{\varepsilon}}$$

Now if $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\mathcal{F}^{-1}(\hat{\varphi})(x) &= \int_{\mathbb{R}^n} e^{2\pi i\langle x,\xi\rangle} \hat{\varphi}(\xi)\, d\xi \\
&= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} e^{2\pi i\langle x,\xi\rangle} e^{-\frac{\varepsilon\pi|\xi|^2}{2}} \int_{\mathbb{R}^n} \varphi(y) e^{-2\pi i\langle y,\xi\rangle}\, dy\, d\xi \\
&\overset{\text{(Fubini)}}{=} \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \varphi(y) \int_{\mathbb{R}^n} e^{2\pi i\langle x-y,\xi\rangle} e^{-\frac{\varepsilon\pi|\xi|^2}{2}}\, d\xi dy \\
&= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \varphi(y) \varepsilon^{-n/2} e^{-\frac{\pi|y-x|^2}{\varepsilon}}\, dy \\
&= \varphi(x)
\end{aligned}$$

where we note that $\psi_\varepsilon(x) = \frac{1}{\varepsilon^{n/2}} e^{-\pi|x/\sqrt{\varepsilon}|^2}$ is an approximate identity, so that

$$\varphi * \psi_\varepsilon \to \varphi$$

as $\varepsilon \to 0$. Thus the inversion formula is valid and $\mathcal{F}$ is an isomorphism. □

**Proof. (Second Proof of Inversion Formula)** This proof (from D. Javier, *Fourier Analysis*) uses the weak Parseval equality $\int \hat{f} g = \int g \hat{f}$ which in this case follows again by Fubini. Note that

$$\int_{\mathbb{R}^n} \lambda^{-n} f(\lambda^{-1}x) \hat{g}(x) dx = \int_{\mathbb{R}^n} f(x)\, \hat{g}(\lambda x) dx = \int_{\mathbb{R}^n} \hat{f}(x) \lambda^{-n} g(\lambda^{-1}x) dx$$

and so

$$\int_{\mathbb{R}^n} f(\lambda^{-1}x) \hat{g}(x) dx = \int_{\mathbb{R}^n} \hat{f}(x) g(\lambda^{-1}x) dx$$

taking $\lambda \to \infty$ gives

$$f(0) \int \hat{g}(x)\, dx = g(0) \int \hat{f}(x)\, dx$$

and setting $g(x) = e^{-\pi|x|^2}$ (noting $\hat{g}(x) = e^{-\pi|\xi|^2}$) we get

$$f(0) = \int \hat{f}(x)\, dx$$

and replacing $f$ by $\tau_y f$ gives

$$f(y) = \int \hat{f}(x) e^{2\pi i y\cdot x}\, dx$$

as desired. □

# Week 4 (10/7/2009)

(This week's notes are a little scattered...)

Continuing the discussion about the Fourier transform on the Schwarz' space, we establish further properties of the Fourier transform:

**Proposition 16.** *For all $\varphi, \psi \in \mathcal{S}$ we have the following properties:*

1. $\int_{\mathbb{R}^n} \varphi\bar{\psi} = \int_{\mathbb{R}^n} \hat{\varphi}\hat{\bar{\psi}}$

2. $\widehat{\varphi * \psi} = \hat{\varphi}\hat{\psi}$

3. $\widehat{\varphi\psi} = \hat{\varphi} * \hat{\psi}$

All of these are proved by Fubini. For instance,

$$\begin{aligned}
\widehat{\varphi * \psi} &= \int e^{-2\pi i x\cdot\xi} \int \varphi(y)\psi(x-y)\,dy\,dx \\
&= \int \varphi(y) \int \psi(x-y) e^{-2\pi i x\cdot\xi}\,dx\,dy \\
&= \int \varphi(y) \int \psi(x) e^{-2\pi i (x+y)\cdot\xi}\,dx\,dy \\
&= \int \varphi(y) e^{-2\pi i y\cdot\xi}\,dy \int \psi(x) e^{-2\pi i x\cdot\xi}\,dx \\
&= \hat{\varphi}\hat{\psi}
\end{aligned}$$

By duality we can extend the Fourier transform to the space of tempered distributions, $\mathcal{S}'$, i.e. the space of linear functionals on $\mathcal{S}$. For $u \in \mathcal{S}'$, we define $\hat{u} \in \mathcal{S}'$ by

$$\hat{u}(\varphi) = u(\hat{\varphi})$$

noting $\hat{\varphi} \in \mathcal{S}$. We can easily check that this coincides with the usual definition if $u \in L^1$ (recall functions can be identified with a distribution by

$$u(\varphi) = \int u\varphi$$

and note using the definition for distributions,

$$\hat{u}(\varphi) = \int u\varphi = \int \hat{u}\varphi$$

noting $\hat{u}$ in the integrand is a $C_0$ function which can be identified with the above functional. So $\hat{u}$ in this case is a distribution corresponding to an actual function, and moreover the definitions agree.

Thus,

$$\mathcal{F}\colon \mathcal{S} \to \mathcal{S},\, \mathcal{S}' \to \mathcal{S}'$$

is an isomorphism. The inverse is defined similarly: $\mathcal{F}^{-1}u(\varphi) = u(\mathcal{F}^{-1}\varphi)$ and thus $\mathcal{F} \circ \mathcal{F}^{-1}u(\varphi) = \varphi$.

Now returning to $L^p$ spaces, we know that $\mathcal{F}$: $L^1 \to C_0$. We can extend $\mathcal{F}$ to $L^2$ functions by density, approximating by functions in $L^1 \cap L^2$. We show that for $u \in L^2$, $\hat{u} \in L^2$ as well:

In fact, we will prove that for $u, v \in L^2$,

$$\int u v^* = \int \hat{u}\hat{v}^*$$

(Parseval's equality) from which $\hat{u} \in L^2$ follows (Plancherel).

$$\begin{aligned}
\int \hat{u}(\xi)\hat{v}^*(\xi)d\xi &= \lim_{\varepsilon \to 0^+} \int \hat{u}(\xi)\hat{v}^*(\xi)e^{-\varepsilon\pi|\xi|^2}d\xi \\
&= \lim_{\varepsilon \to 0^+} \int e^{-\varepsilon\pi|\xi|^2} \int u(x)e^{-2\pi i x\cdot\xi}dx \int v^*(y)e^{2\pi i y\cdot\xi}dyd\xi \\
&= \lim_{\varepsilon \to 0^+} \int u(x) \int v^*(y) \int e^{-2\pi i(x-y)\cdot\xi}e^{-\varepsilon\pi|\xi|^2}d\xi dydx \\
&= \lim_{\varepsilon \to 0^+} \int u(x) \int v^*(y)\varepsilon^{-n/2}e^{-\frac{\pi}{\varepsilon}|y-x|^2}dydx \\
&= \int u(x)v^*(x)dx
\end{aligned}$$

using the same technique used in the first proof of the inversion formula above. Thus, we have proved that $\mathcal{F}: L^2 \to L^2$, with $\|\hat{u}\|_{L^2} = \|u\|_{L^2}$. In summary, we have:

- $\mathcal{F}: L^1 \to L^\infty$ with norm 1.
- $\mathcal{F}: L^2 \to L^2$ with norm 1.
- By Riesz-Thorin, this implies that $\mathcal{F}: L^p \to L^{p'}$ with norm 1, and this is for $1 \le p \le 2$.

Now we turn to another property concerning homogeneity of the Fourier trasnform:

**Proposition 17.** *If $u \in \mathcal{S}'$ is homogeneous of degree $\sigma$, then $\hat{u}$ is homogeneous of degree $-n-\sigma$ i.e. for functions, if we use $\delta_t u := u(tx)$, then $u$ is homogeneous of degree $\sigma$ if $\delta_t u(x) = u(tx) = t^\sigma u(x)$, and under this condition, $\hat{u}(t\xi) = t^{-n-\sigma}\hat{u}(\xi)$.*

*For distributions, we define $\delta_t^* u(\varphi) := u(\delta_t\varphi)$, so $u$ is homogeneous of degree $\sigma$ if $\delta_t^* u(\varphi) = t^\sigma u(\varphi)$.*

**Proof.** Using the property of the Fourier transform, we note that for functions,

$$\begin{aligned}
t^\sigma \hat{u} = \widehat{\delta_t u} &= t^{-n}\delta_{1/t}\hat{u}(\xi) = t^{-n}\hat{u}(\xi/t) \\
\hat{u}(\xi/t) &= t^{\sigma+n}\hat{u}(\xi) \\
\hat{u}(t\xi) &= t^{-\sigma-n}\hat{u}(\xi)
\end{aligned}$$

For distributions, we use the result above to get

$$\begin{aligned}
\widehat{\delta_t^* u}(\varphi) &= \delta_t^* u(\hat{\varphi}) \\
&= u(\delta_t\hat{\varphi}) \\
&= u(t^{-n-\sigma}\hat{\varphi}) \\
&= t^{-n-\sigma}u(\hat{\varphi}) \\
&= t^{-n-\sigma}\hat{u}(\varphi)
\end{aligned}$$

□

And now for some convolution properties:

**Properties about Convolution:**

1. (Hausdorff-Young inequality) $\|f * g\|_r \le \|f\|_p \|g\|_q$ for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

2. $\rho_\varepsilon * f \to f$ in $L^p$ with $\rho_\varepsilon$ an approximate identity (converging to the dirac measure $\delta$)

3. A special case of (1) is $\|f * g\|_1 \le \|f\|_1 \|g\|_1$, and using convolution as a product on $L^1$, this turns $L^1$ into a Banach Algebra.

4. $f * g$ inherits the nicer property between $f$, $g$. This manifests itself in the Hausdorff-Young inequality above, the convolution between $L^1$ and $L^p$ is $L^p$.

$$\|f * g\|_p \le \|f\|_p \|g\|_1$$

Also, $C^k$ convolved with something not even continuous is $C^k$. Note

$$(f * g)' = f' * g = f * g'$$

if $f, g$ are differentiable.

## Translation Invariant Operators

Now for some reason we are talking about translation invariant operators (will use in Week 5).

A translation invariant operator $A \in \mathcal{B}(L^p, L^q)$ is a bounded operator satisfying $\tau_h A = A \tau_h$.

**Theorem 18.** *If $A \in \mathcal{B}(L^p, L^q)$ is translation invariant, then*

1. *If $q < p < \infty$, then $A \equiv 0$*
2. *If $q < p = \infty$, then $A|_{L_0^\infty} \equiv 0$.*

**Proof.** Note that $\|Au\|_q \le \|A\| \, \|u\|_p$ for all $u \in L^p$. Then

$$\|Au + \tau_h Au\|_q = \|A(u + \tau_h u)\|_q \le \|A\| \, \|u + \tau_h u\|_p$$

Now let $h \to \infty$. For sufficiently large $h$, $Au$ and $\tau_h Au$ become "essentially" disjoint, since functions in $L^p$ decay at infinity, so that outside a ball of radius $R$ the $L^p$ norm can be bounded by $\varepsilon$. Thus $\|Au + \tau_h Au\|_q \approx 2^{1/q} \|Au\|_q$, noting that

$$\|Au + \tau_h Au\|_q^q \approx 2 \, \|Au\|_q^q$$

Likewise, $\|u + \tau_h u\|_p \approx 2^{1/p} \|u\|_p$, and

$$2^{1/q} \|Au\|_q \lesssim \|A\| \, 2^{1/p} \, \|u\|_p$$

$$\|Au\|_q \lesssim \|A\| \, 2^{1/p - 1/q} \, \|u\|_p < \|A\| \, \|u\|_p$$

for $p > q$ and $\|A\| \neq 0$. Note that the inequality $\|Au\|_q < \|A\| \, \|u\|_p$ can be made strict above since $2^{1/p - 1/q} \le r < 1$. This contradicts the definition of $\|A\|$, so it must be the case that $A \equiv 0$.

The second case is similar, since we restrict $A$ to the space of $L^\infty$ functions that decay at $\infty$. $\square$

Therefore when classifying translation invariant operators, the interesting case is when $p \le q$. In this case we have the following theorem:

**Theorem 19.** *Let $A \in \mathcal{B}(L^p, L^q)$ be translation invariant, with $p \le q$. Then there exists a $T \in \mathcal{S}'$ such that $Au = T * u$.*

**Remark 20.** Note that convolution of a tempered distribution with a function is a tempered distribution, and can be defined by extending the behavior of regular convolutions as distributions. Note if $f * g$ are functions, then as a distribution,

$$\begin{aligned}
(f*g)(\varphi) &= \int (f*g)(x)\,\varphi(x)\,dx \\
&= \int\int \varphi(x)\,f(y)\,g(x-y)\,dx\,dy \\
&= \int (\varphi*\tilde{g})(y)\,f(y)\,dy \\
&= f(\tilde{g}*\varphi) \\
&= \int (\varphi*\tilde{f})(y)\,g(y)\,dy \\
&= g(\tilde{f}*\varphi)
\end{aligned}$$

where $\tilde{f}(x) = f(-x)$.

Thus,

$$(T*u)(\varphi) := T(\tilde{u}*\varphi)$$

and since $\tilde{u}*\varphi \in \mathcal{S}$, $T*u$ is also in $\mathcal{S}'$.

The theorem is proved with the following lemma:

**Lemma 21.** *If $f \in W^{n+1,p}(\mathbb{R}^n)$, for $1 \le p \le \infty$, then $f \in C_0$.*

**Remark.** Recall $W^{k,p}(\mathbb{R}^n)$ is the Sobolev space with norm defined by

$$\sum_{|\alpha| \le k} \|D^\alpha f\|_p$$

(there are other equivalent norms, using for instance an $L^p$ like sum...

**Proof.** Let us consider the case $p=1$ first. The idea is to prove that $\|\hat{f}\|_{L^1} \le C\|f\|_{W^{n+1,1}}$, in which case $f(-x) = \mathcal{F}\hat{f}$ is in $C_0$, since $\mathcal{F}: L^1 \to C_0$. This relies on the inversion formula, which is valid for $f \in L^1$. Note that

$$(1+|x|^2)^{\frac{n+1}{2}} \le c_n \sum_{|\alpha| \le n+1} |x^\alpha|$$

(just expand the LHS, use triangle inequality and bound by the largest constant) Then

$$\begin{aligned}
|\hat{f}(x)| &\le c_n (1+|x|^2)^{-\frac{n+1}{2}} \sum_{|\alpha|\le n+1} |x^\alpha|\,|\hat{f}(x)| \\
&\le c_n (1+|x|^2)^{-\frac{n+1}{2}} \sum_{|\alpha|\le n+1} |\widehat{D^\alpha f(x)}| \\
&\le c_n (1+|x|^2)^{-\frac{n+1}{2}} \sum_{|\alpha|\le n+1} \|\widehat{D^\alpha f}\|_1
\end{aligned}$$

The key is that $(1+|x|^2)^{-\frac{n+1}{2}}$ is integrable in $\mathbb{R}^n$ (behaves like $|x|^{-(n+1)}$), so that

$$\|\hat{f}\|_1 \le C_n \|f\|_{W^{n+1,1}}$$

as desired. Also, we have the bound

$$\|f\|_{\infty} \leq C_n' \|f\|_{W^{n+1,1}}$$

For higher powers of $p$ a different proof seems to be needed... $\square$

**Proof. (of Theorem)** First we claim that $D^{\alpha} A u = A D^{\alpha} u$ for $u \in \mathcal{S}$. This is a straightforward computation. For $h = h_j \mathbf{e}_j$, note

$$D_h A u = \lim_{h_j \to 0} \frac{T_h A u - A u}{h_j} = \lim_{h_j \to 0} A\left(\frac{T_h u - u}{h_j}\right) = A(D_h u)$$

where we have used translation invaraince of $A$ in the second equality above.

Now we show that $Au \in W^{n+1,q}$, in which case $Au \in C_0$ by the lemma. But this follows from

$$D^{\alpha} A u = A D^{\alpha} u \in L^q$$

for any $\alpha$ since $D^{\alpha} u \in L^p$ ($u \in \mathcal{S}$), and in particular, for $|\alpha| \leq n+1$. This shows that $Au \in W^{n+1,q}$, and

$$\|Au\|_{\infty} \leq C_n \|A\| \|u\|_{W^{n+1,p}}$$

This implies that the mapping $u \mapsto Au(0)$ is a linear, continuous functional (linear and bounded) on $W^{n+1,p}$, and thus there exists (by a Riesz representation like theorem) a

$$T(x) = \sum_{|\alpha| \leq n+1} D^{\alpha} T_{\alpha}(-x)$$

with $T_{\alpha} \in L^p$, such that

$$Au(0) = \sum_{|\alpha| \leq n+1} \int D^{\alpha} T_{\alpha}(-y) u(y) = T * u(0)$$

noting that the integration is defined as

$$\sum_{|\alpha| \leq n+1} \int D^{\alpha} T_{\alpha}(-y) u(y) = \sum_{|\alpha| \leq n+1} (-1)^{\alpha} \int T_{\alpha}(-y) D^{\alpha} u(y)$$

(since $D^{\alpha} T_{\alpha}$ in general is a distribution).

From this we show that $Au(x) = T * u(x)$:

$$\begin{aligned}
T * u(x) &= \sum_{|\alpha| \leq n+1} (-1)^{\alpha} \int T_{\alpha}(x-y) D^{\alpha} u(y) \, dy \\
&= \sum_{|\alpha| \leq n+1} (-1)^{\alpha} \int T_{\alpha}(-y) D^{\alpha} u(x+y) \, dy \\
&= \sum_{|\alpha| \leq n+1} (-1)^{\alpha} \int T_{\alpha}(-y) D^{\alpha} \tau_x u(y) \, dy \\
&= T * \tau_x u(0) \\
&= A \tau_x u(0) \\
&= Au(x)
\end{aligned}$$

using the translation invariance of $A$ in the last equality. $\square$

**Remark 22.** The characterization of linear functionals on $W^{k,p}$ as a sum of weak derivatives of classical functions is obtained by... First we identify $W^{k,p} \cong \underbrace{L^p \times \cdots \times L^p}_{k \text{ times}} = (L^p)^k$ by

$$f \longleftrightarrow (f, Df, D^2 f, \ldots, D^k f) \in L^p \times \cdots \times L^p$$

and then the linear functional $T \in (W^{k,p})'$ can be transferred by

$$T(f, Df, D^2 f, \ldots, D^k f) = Tf$$

(apply $T$ to the first coordinate). This can further be extended to $(L^p)^k$ by Hahn Banach. Now by linearity we can write $T = T_0 + \ldots + T_{k+1}$, where

$$T(f_0, \ldots, f_k) = T_0(f_0, 0, \ldots) + \ldots + T_k(\ldots, 0, f_k)$$

which we will denote by $T(f_0, \ldots, f_k) = T_0 f_0 + \ldots + T_k f_k$. Then we can apply Riesz representation to each $T_i$, to get a $g_i \in L^q$ such that

$$T_i f_i = \int f_i g_i$$

and then

$$T(f_0, \ldots, f_k) = \sum_i \int f_i g_i$$

and in particular,

$$T(f, Df, \ldots, D^k f) = \sum_{i=0}^{k} \int D^i f g_i = \sum_{i=0}^{k} \int f \cdot (-1)^i D^i g_i$$

and this gives the characterization,

$$T(f) = \sum_{i=0}^{k} \int (-1)^i D^i g_i f$$

## Fourier Series

**A good reference for this section:** Katznelson. *Introduction to Harmonic Analysis*.

Now we turn our attention to trigonometric series:

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n\theta}$$

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n\theta + b_n \sin 2\pi n\theta$$

A trigonometric series is called a Fourier series if it comes from an integrable function $f$ via the relation

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} \, dx$$

or

$$a_n = 2\int_0^1 f(x) \cos(2\pi n x) \, dx, \; b_n = 2\int_0^1 f(x) \sin(2\pi n x) \, dx$$

**Notation:** If this is the case, we use the notation $f \sim c_n$.

**Goal:** We are building towards the relation between Fourier Series and singular integrals...

**Remark 23.** Not all convergent trigonometric series are Fourier series. Actually, given the Fourier series of $f \in L^1$, $\sum_n c_n e^{2\pi i n\theta}$, the series can be integrated term by term, and the resulting series converges uniformly and is the Fourier series of $\int f$. This follows from the fact that $F(x) = \int_0^x f(t)\,dt$ is an absolutely continuous function and the Fourier series of a absolutely continuous function converges uniformly (need reference here)

Then consider

$$\sum_{n=2}^{\infty} \frac{\sin(2\pi n\theta)}{\log n}$$

which by the Dirichlet test converges everywhere (though not absolutely convergent). Suppose the series is a Fourier series corresponding to $f \in L^1$. Then integrating term by term gives

$$\sum_{n=2}^{\infty} \frac{-\cos(2\pi n\theta)}{2\pi n \log n}$$

which does not converge uniformly, which contradicts the previously stated fact.

See also Katznelson, Corollary to 4.2.

**Remark 24.** There exists a Fourier series which diverges everywhere (pointwise). (There is a proof using Uniform Boundedness Principle and examining the Dirichlet kernel)

**Properties for $L^2$ Functions:**

1. (Besov)

$$\sum_n |c_n|^2 = \int_0^1 e^{-2\pi i n\theta} f(\theta)\,d\theta$$

2. (Riesz-Fisher) If $\sum_n |c_n|^2 < \infty$, there exists $f \in L^2[0,1]$ such that $f \sim c_n$.

3. (Parseval)

$$\|f\|_2^2 = \sum_n |c_n|^2$$

**More General Properties:**

1. If $f \geq 0$, then $|a_n| \leq a_0$, $|b_n| \leq a_0$, and $a_0 = 2\int_0^1 f(x)\,dx$

2. If $f \geq 0$ on $\left[0, \frac{1}{2}\right]$ and $f$ is odd, then $|b_n| \leq 2\pi n b_1$. This uses $|\sin 2\pi n\theta| \leq 2\pi n\,|\sin\theta|$.

3. If $f$ is monotone decreasing on $[0,1]$, then $b_n \geq 0$.

4. If $f$ is convex on $(0,1)$, then $a_n \geq 0$ for $n > 0$.

There is a result from complex analysis that bounds the growth rate of Fourier coefficients:

**Proposition 25.** *Let $F(z) = \sum_{n=1}^{\infty} c_n z^n$ be a holomorphic, one-to-one function on $|z| < 1$ with $c_k \in \mathbb{R}$. Then*

$$|c_n|\gamma^n \leq n r$$

*and in particular, $|c_n| \leq n$.*

## Riemann-Lebesgue and Decay Rates

We know from the Riemann-Lebesgue that $c_n \to 0$ as $n \to \infty$. There are no rates of convergence, even for (uniformly) continuous functions. Given any $\varepsilon_n \to 0$, we can construct a continuous function whose Fourier coefficients decay slower than $\varepsilon_n$. Take a sequence $n_1, n_2, \ldots$ with $k^{-2} \geq \varepsilon_{n_k}$, and construct

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \cos n_k \theta$$

The series converges uniformly, and thus converges to a continuous function $f$ whose Fourier coefficients are $\hat{f}(n_k) = \frac{1}{k^2} \geq \varepsilon_{n_k}$.

Some results in this direction...

**Proposition 26.** *If $f$ is absolutely continuous, and $f \sim c_n$, then $c_n = o\left(\frac{1}{n}\right)$.*

**Proof.** This is an integration by parts:

$$\begin{aligned} c_n &= \int_0^1 f(x) e^{-2\pi i n x} dx \\ &= \frac{1}{2\pi i n} \int_0^1 f'(x) e^{-2\pi i n x} dx \end{aligned}$$

and since $\int_0^1 f'(x) e^{-2\pi i n x} dx \to 0$ by the Riemann Lebesgue Lemma, $c_n = o\left(\frac{1}{n}\right)$. □

**Proposition 27. (F. Riesz)** *There exists a $C_0 \cap \mathrm{BV}$ function $f$ such that $c_n \neq o\left(\frac{1}{n}\right)$.*

*(Note that* $\mathrm{BV} \Longrightarrow c_n = O\left(\frac{1}{n}\right)$*...)*

**Theorem 28. (Wiener)** *If $f \in \mathrm{BV}$, then $f \in C_0$ if and only if $\sum_{|m|<n} |m c_m| = o(n)$.*

**Theorem 29. (Fefferman)** *If $f(x) = \sum a_k e^{2\pi i k x}$ and $|a_k| \leq \frac{1}{k}$, then $f \in \mathrm{BMO}$*

**Theorem 30. (Jouné)** *If $\sum a_k e^{2\pi i k x} \in \mathrm{BMO}$ with $a_k \geq 0$, then $\sum b_k e^{2\pi i k x} \in \mathrm{BMO}$ if $|b_k| < a_k$ (A comparison test-like result)*

# Week 5 (10/14/2009)

Continuing the discussion about Fourier Series, recall that associated to $f \in L^p(0,1)$ we have the Fourier coefficients

$$c_n = \int_0^1 e^{-2\pi i n x} f(x) dx$$

and we write

$$f \sim \sum_n c_n e^{2\pi i n x}$$

Note $|c_n| \leq \int_0^1 |f(x)| dx$, and $a_n \to 0$ as $|n| \to \infty$ for $f \in L^1$ by Riemann Lebesgue.

If $f \in C^k$, $k \geq 1$ then $c_n = o\left(\frac{1}{n^k}\right)$. This is from integrating by parts and applying Riemann Lebesgue.

If $f \in C^k$ for $k \geq 2$, then

$$\left| \sum_n c_n e^{2\pi i n x} \right| \leq \sum_n |c_n| < \infty$$

and so the Fourier series converges uniformly to some $g$ (which may or may not be $f$, though later we will show that it converges to $f$).

Studying the partial sums,

$$\begin{aligned} S_N f(x) &= \sum_{|n| \leq N} c_n e^{2\pi i n x} \\ &= \int_0^1 f(y) \sum_{|n| \leq N} e^{2\pi i n (x-y)} dy \\ &= \int_0^1 f(y) D_N(x-y) dy \\ &= D_N * f(x) \end{aligned}$$

where

$$D_N(y) = \frac{\sin\left[2\pi\left(N + \frac{1}{2}\right) y\right]}{\sin \pi y}$$

(the Dirichlet kernel). Note

$$\frac{1}{2i}\left[e^{\pi i y} - e^{-\pi i y}\right] \sum_{|n| \leq N} e^{2\pi i n y} = \frac{1}{2i}\left[e^{2\pi i (N+1/2) y} - e^{-2\pi i (N+1/2) y}\right]$$

by cancellations. This kernel is signular near $y = 0$, where it behaves roughly like

$$D_N(y) \approx \frac{\sin\left[2\pi\left(N + \frac{1}{2}\right) y\right]}{\pi y}$$

The partial sums of the Fourier series therefore behaves (very) roughly like the Hilbert Transform:

$$H f(x) = \frac{1}{\pi} \mathrm{PV} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy$$

The convergence of $S_N f$ is a very delicate issue since the Dirichlet kernel is highly oscillatory. These can be smoothed out if we consider a different kind of sum (Cesaro sum)

$$\begin{aligned} F_N f(x) &= \frac{1}{N+1} \sum_{n=0}^{N} S_n f(x) \\ &= \int_0^1 f(y) \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i n (x-y)} dy \\ &= \int_0^1 f(y) F_N(x-y) dy \\ &= F_N * f(x) \end{aligned}$$

for some kernel $F_N$ (Fejer kernel). Note

$$F_N(y) = \frac{1}{N+1} \sum_{n=0}^{N} \sum_{m=-n}^{n} e^{2\pi i m y} = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{2\pi i n y}$$

and

$$(\sin \pi y)^2 = -\frac{1}{4}e^{2\pi i y} - \frac{1}{4}e^{-2\pi i y} + \frac{1}{2}$$

multiplying $(\sin \pi y)^2 F_N(y)$ and performing cancellations gives $\frac{1}{N+1}[\sin{(N+1)\pi y}]^2$ so that

$$F_N(y) = \frac{1}{N+1}\left[\frac{\sin(N+1)\pi y}{\sin \pi y}\right]^2$$

The Fejer kernel satisfies three nice properties (of approximate identities)

1. $F_N \geq 0$
2. $\int_0^1 F_N(y)\,dy = 1$
3. $\int_{\delta < y < 1-\delta} F_N(y)\,dy \to 0$ as $N \to \infty$ uniformly

Positivity is obvious. The second property holds from examining the partial Fourier series of 1 (which is just constant $=1$). The third property is a consequence of the simple bound

$$|F_N(y)| \leq \frac{1}{N+1} \cdot \frac{1}{(\sin \pi\delta)^2}$$

for $\delta < y < 1-\delta$.

**Remark 31.** If $f \in C[0,1]$ then $F_N * f \to f$ uniformly as $N \to \infty$. ($F_N$ is an approximate identity)

Thus, above if $f \in C^2$, we have that $S_N f$ and $F_N f$ both converge uniformly to $f$.

**Proof.** If $f \in C^2$, then $S_N f(x) \to g(x)$ and since $F_N f$ are averages of $S_N f$, $F_N f \to g(x)$, and by the remark $F_N f \to f(x)$. Thus $f = g$. □

Even more, we have that

**Proposition 32.** *If $f \in C^\alpha, \alpha > 0$, then $S_N f(x) \to f(x)$.*

**Proof.** Note that

$$f(0) - S_N f(0) = \int_0^1 \frac{\sin\left[2\pi\left(N+\frac{1}{2}\right)y\right]}{\sin \pi y}(f(0) - f(y))\,dy$$

using the fact that $\int D_N(y)\,dy = 1$. Now note that for small $y$,

$$\left|\frac{f(0) - f(y)}{\sin \pi y}\right| \leq \frac{C}{|y|^{1-\alpha}}$$

by Hölder continuity, and thus $\frac{f(0) - f(y)}{\sin \pi y}$ is integrable. Then by the Riemann-Lebesgue lemma,

$$f(0) - S_N f(0) \longrightarrow 0$$

Translating by $x$ shows that $f(x) - S_N f(x) \to 0$ as well. □

In any case, the study of convergence of the partial sums in general requires study of the singular kernel $D_N(y)$. Consider the following operator approach:

$$T_\lambda f(x) = \int_0^1 f(x-y)\,\frac{\sin \lambda y}{\sin \pi y}\,dy$$

noting that $T_{2\pi(N+1/2)} f = S_N f$. Thus we are interested in the behavior as $\lambda \to \infty$. Again for small $y$ we note that

$$T_\lambda f(x) \approx \int_0^1 f(x-y)\,\frac{\sin \lambda y}{\pi y}\,dy$$

which is a singular kernel. Note

- $\int_0^1 \left|\frac{\sin \lambda y}{\pi y}\right| dy = \int_0^\lambda \left|\frac{\sin u}{\pi u}\right| du \sim \log \lambda$, so that as an operator from $C^0 \to C^0$, $T_\lambda$ is not uniformly bounded in $\lambda$. This implies that $T_\lambda f(x) \not\to f(x)$ for some continuous $f$ and some point $x$ by the uniform boundedness principle. (This says that if a family of operators is ptwise bounded, then it is uniformly bounded. Thus, if we do not have uniform bound, then there is a function where $\|T_\lambda f\|_\infty \to \infty$)
- $\left|\frac{\sin \lambda y}{\pi y}\right|_{L^\infty}$ is not uniformly bounded in $\lambda$ (in fact not even finite for any $\lambda$), so that as an operator from $L^1 \to L^1$, $T_\lambda$ is not bounded for any $\lambda$.

Nevertheless, we have that

- $\|T_\lambda f\|_{L^p} \le C_p \|f\|_{L^p}$ for all $\lambda$, $1 < p < \infty$
- $|\{x \colon |T_\lambda f| > \alpha\}| \le \frac{C_1}{\alpha} \|f\|_{L^1}$ for all $\lambda$.

This motivates the study of singular integrals. The Dirichlet kernel is a little more difficult to deal with because of the addition of the highly oscillatory term $\sin(N + \frac{1}{2})y$.

## Singular Integrals

First we consider the Hilbert Transform,

$$Hf(x) := \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}dy := \lim_{\varepsilon \to 0} \frac{1}{\pi}\int_{|y| \ge \varepsilon} \frac{f(x-y)}{y}dy$$

Following Stein's *Singular Integrals...*, beginning of Chapter II, and a few comments from class, we describe a few aspects of the theory of the Hilbert Transform.

1. First, we note $H$ is translation invariant, and naturally we will be dealing with convolutions. In the $L^2$ theory, the tools of Fourier transforms and Plancherel, etc will play a significant role.

2. Classically the $L^p$ theory for proving boundedness of the operator was proved by M. Riesz with complex function theory of $H^p$ spaces (even with $0 < p < 1$). At least for $1 < p < \infty$, the approach is to first look at $1 < p < 2$ first, and then by duality $2 < p < \infty$ is obtained automatically, since the dual of a convolution operator is also a convolution operator. The relation to complex analysis is through the Cauchy integral formula.

3. The Hilbert Transform is not bounded in $L^1$, but a weak (1,1) estimate for the Hilbert Transform was proved by Besicovitch, Titchmarsh, Calderón-Zygmund. This approach is more difficult than (2), but through this approach (2) the $L^p$ boundedness is obtained automatically via Marcinkiewicz interpolation.

   The general situation will be that an operator will not be bounded in $L^1$, and so we generally look for a weak (1,1) estimate.

First, a few remarks about translation invariant operators on $L^1$ and $L^2$.

**Remark 33.** Let $T$ be a bounded linear operator from $L^1 \to L^1$. Then $T$ commutes with translation if and only if $Tf = \mu * f$ where $\mu$ is a measure with $\|\mu\| < \infty$ (i.e. a convolution operator).

From last time we already proved that $Tf$ must be a convolution operator $T * f$ with a distribution $T \in \mathcal{S}'$. Why must $T$ be a measure? To obtain this we can use smoothing and take the limit. Let $T_\varepsilon = \rho_\varepsilon * T$ where $\rho_\varepsilon$ is a smooth, compactly supported (symmetric) approximate identity, so that $T_\varepsilon$ is now smooth. The key is that $T_\varepsilon$ converges to $T$ in distribution $\mathcal{S}'$, since

$$T_\varepsilon(\varphi) = T(\rho_\varepsilon * \varphi) \to T(\varphi)$$

and also that as a convolution operator from $L^1 \to L^1$,

$$\|T_\varepsilon\|_{1,1} = \|\rho_\varepsilon * T\|_{1,1} \leq \|T\|_{1,1}$$

since

$$\|T_\varepsilon * f\|_1 = \|\rho_\varepsilon * (T * f)\|_1 \leq \|T * f\|_1 \leq \|T\|_{1,1} \|f\|_1$$

Also, since $T_\varepsilon \in L^1$, as a convolution operator $\|T_\varepsilon\|_{1,1} = \|T_\varepsilon\|_1$. This follows from the fact that equality can be achieved in the Hausdorff-Young inequality

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

$$\begin{aligned} \int \left| \int f(y)\, g(y - x)\, dy \right| dx &\leq \iint |f(y)| |g(y - x)|\, dy dx \\ &= \int |f(y)| \|g\|_{L^1} dy \\ &= \|f\|_{L^1} \|g\|_{L^1} \end{aligned}$$

with equality when $g$ has the same sign as $f$, so that the integral is positive. This implies that $\|T_\varepsilon\|_{1,1} = \|T_\varepsilon\|_1$.

These facts together should allow us to bring $T_\varepsilon$ to $C_0'$, the space of complex Borel measures. Thus we have that $\|T_\varepsilon\|_{C_0'} = \|T_\varepsilon\|_{L^1} \leq \|T\|_{1,1}$ so that $T_\varepsilon \in C_0'$ for all $\varepsilon$, and since a bounded ball in $C_0'$ is weak* compact, we can extract a subsequence converging to a complex Borel measure. But this implies that this same subsequence converges to a complex Borel measure in $\mathcal{S}'$, and thus $T$ must be a complex Borel measure (the limit in $\mathcal{S}'$ is unique).

**Remark 34.** Let $T$ be a bounded linear operator from $L^2 \to L^2$. Then $T$ commutes with translation if and only if $Tf = g * f$ with $\hat{g} \in L^\infty$.

Again we already know that $Tf = T * f$ for some distribution $T \in \mathcal{S}'$. Applying smoothing, we have that $T_\varepsilon := \rho_\varepsilon * T \in \mathcal{S} \subset L^2$. Note that $T_\varepsilon$ also maps $L^2 \to L^2$:

$$T_\varepsilon f = \rho_\varepsilon * (T * f) \in L^2$$

In this case we can take the Fourier transform so that for $f \in L^2$,

$$\widehat{T_\varepsilon * f} = \hat{T}_\varepsilon \hat{f} \in L^2$$

Thus in the Fourier domain $T_\varepsilon$ is a multiplication operator, it is sufficient for $\hat{T}_\varepsilon \in L^\infty$ by Hölder's inequality. It is also necessary (if $\hat{T}_\varepsilon$ were not bounded, can find $\hat{f} \in L^2$ such that $\hat{T}_\varepsilon \hat{f}$ is not in $L^2$). Like before, $|\hat{T}_\varepsilon|_\infty \leq |T|_{2,2}$, and since $L^\infty$ is the dual of $L^1$, and the bounded ball in $L^\infty$ (as the dual) is weak* compact, as argued above $\hat{T}_\varepsilon$ converges to an $L^\infty$ function.

This characterizes translation-invariant bounded linear operators from $L^p \to L^p$ (we already studied the general $L^p$ case earlier) as convolution operators, which we now study in more detail.

**Theorem 35.** *Let $K \in L^2(\mathbb{R}^n)$ satisfying the following properties*

*a)* $|\hat{K}(\xi)| \leq B$

*b)* $|\nabla K| \leq \frac{B}{|x|^{n+1}}$

*For $f \in L^p \cap L^1$, define $Tf := \int_{\mathbb{R}^n} f(y) K(x-y) dy$. Then for $1 < p < \infty$, we have that*

$$\|Tf\|_p \leq A_{p,B,n} \|f\|_p$$

*i.e. $T$ is a bounded operator from $L^p \to L^p$.*

**Remark.** A quick technical detail is that we define $T$ as an operator on $L^p$ by density of the subspace $L^p \cap L^1$.

**Proof.** We will obtain a strong (2,2) estimate and a weak (1,1) estimate, from which the result follows by Marcinkiewicz interpolation. First, note that since $\hat{K}$ is bounded,

$$\|Tf\|_2 = \|\widehat{Tf}\|_2 = \|\hat{K}\hat{f}\|_2 \leq \|\hat{K}\|_\infty \|\hat{f}\|_2 \leq B\|f\|_2$$

using Plancherel's identity. This establishes a strong (2,2) estimate.

Now we establish the more technical estimate, relying on the Calderón-Zygmund decomposition. We want to show that

$$|\{|Tf| > \alpha\}| \leq \frac{C}{\alpha} \|f\|_1 \text{ for all } f$$

For $f \in L^1(\mathbb{R}^n)$, $\alpha > 0$, we have the Calderón-Zygmund decomposition $\mathbb{R}^n = \Omega \cup F$ disjoint, where $\Omega = \bigcup_j Q_j$ with $Q_j$ dyadic cubes with disjoint interiors with side lengths comparable to the distance from $Q_j$ to $F$, where

1. $|\Omega| \leq \frac{A}{\alpha} \|f\|_1$ where $A$ comes from Vitali covering lemma.
2. On $F$, we have that $|f(x)| \leq \alpha$ almost everywhere.
3. $\alpha \leq \frac{1}{|Q_j|} \int_{Q_j} f(x) dx \leq 2^n \alpha$

Now we write $f(x) = g(x) + b(x)$ ("good" and "bad" functions) where

$$g(x) := \begin{cases} f(x) & x \in F \\ \frac{1}{|Q_j|} \int_{Q_j} f(y) dy & x \in Q_j \end{cases}$$

and so

$$b(x) := \begin{cases} 0 & x \in F \\ f(x) - \frac{1}{|Q_j|} \int_{Q_j} f(y) dy & x \in Q_j \end{cases}$$

In particular, note that $|g(x)| \leq 2^n \alpha$. Now note that

$$|\{|Tf| > \alpha\}| \leq |\{|Tg| > \alpha/2\}| + |\{|Tb| > \alpha/2\}|$$

(If $|Tf|>\alpha$, then either $|Tg|>\alpha/2$ or $|Tb|>\alpha/2$). Also, since $g\in L^1$ and $g\in L^\infty$, $g\in L^2$ with

$$\|g\|_2^2=\int |g|^2\leq 2^n\alpha\int |g|\leq 2^n\alpha\|g\|_1$$

so that $\|Tg\|_2^2\leq B\|g\|_2^2\leq B(2^n\alpha)\|g\|_1$. Thus,

$$\begin{aligned} |\{|Tg|>\alpha/2\}| &\leq \frac{4}{\alpha^2}\|Tg\|_2^2 \\ &\leq \frac{C}{\alpha}\|f\|_1 \end{aligned}$$

This bounds the first part. For the second part, we can write

$$b(x)=\sum_j b_j(x)$$

with $b_j=b$ on $Q_j$ and 0 elsewhere, and by linearity $Tb=\sum_j Tb_j$. Notice that by definition of $b$, $\frac{1}{|Q_j|}\int_{Q_j} b_j=0$. Thus

$$\begin{aligned} Tb_j(x) &= \int_{Q_j} b(y)K(x-y)\,dy \\ &= \int_{Q_j} b(y)\,[K(x-y)-K(x-y_j)]\,dy \end{aligned} \tag{1}$$

where $y_j$ is the center of $Q_j$. Now using the second condition in the theorem,

$$\begin{aligned} |K(x-y)-K(x-y_j)| &\leq |\nabla K(x-\tilde{y}_j)||y-y_j| \\ &\leq \frac{B|y-y_j|}{|x-\tilde{y}_j|^{n+1}} \\ &\leq B\frac{\operatorname{diam}(Q_j)}{|x-\tilde{y}_j|^{n+1}} \end{aligned}$$

where the first line follows from a mean value type result ($\tilde{y_j}$ is on a line segment between $y$ and $y_j$). Now let $x\in F$. Note $|x-\tilde{y}_j|\simeq|x-y|$ since $\operatorname{diam}(Q_j)\simeq d(Q_j,F)$. Likewise, $\operatorname{diam}(Q_j)\simeq d(y,F)$. Then

$$|Tb_j(x)|\lesssim B\frac{d(y_j,F)}{|x-y_j|^{n+1}}\int_{Q_j}|b(y)|\,dy\,,\; x\in F$$

Now note that

$$\int_{Q_j}|b(y)|\,dy\leq\int_{Q_j} f(y)+2^n\alpha\,dy\leq 2^{n+1}\alpha m(Q_j)$$

so that

$$|Tb_j(x)|\lesssim 2^{n+1}\alpha B\frac{d(y_j,F)}{|x-y_j|^{n+1}}m(Q_j)\leq 2^{n+1}\alpha B\int_{Q_j}\frac{d(y,F)}{|x-y|^{n+1}}\,dy$$

for $x\in F$, and summing in $j$, we have

$$|Tb(x)|\lesssim 2^{n+1}\alpha B\int_{\mathbb{R}^n}\frac{d(y,F)}{|x-y|^{n+1}}\,dy$$

So,

$$\int_F|Tb(x)|\,dx \;\lesssim\; 2^{n+1}\alpha B\int_F\int_{\mathbb{R}^n}\frac{d(y,F)}{|x-y|^{n+1}}\,dydx$$

The right hand side is the Marcinkiewicz integral, which we can show is bounded by $cm(F^c)$. This follows Stein's *Singular Integrals...* section 2.4:

$$
\begin{aligned}
\int_F \int_{\mathbb{R}^n} \frac{d(y,F)}{|x-y|^{n+1}}\,dydx &= \int_F \int_{F^c} \frac{d(y,F)}{|x-y|^{n+1}}\,dydx \\
&= \int_{F^c} d(y,F) \int_F \frac{1}{|x-y|^{n+1}}\,dxdy \\
&\leq \int_{F^c} d(y,F) \int_F \frac{1}{|x-y|^{n+1}}\,dxdy \\
&\leq \int_{F^c} d(y,F) \int_{|x|>d(y,F)} \frac{1}{|x|^{n+1}}\,dxdy \\
&\leq \int_{F^c} d(y,F) \int_{r>d(y,F)} \frac{c_n r^{n-1}}{r^{n+1}}\,drdy \\
&\leq \int_{F^c} d(y,F) \frac{c_n}{d(y,F)}\,dy \\
&= c_n m(F^c)
\end{aligned}
$$

Note $F^c = \Omega$. This shows that $\int_F\ |Tb(x)|\,dx \lesssim 2^{n+1}\alpha B c_n m(\Omega) \leq C_{n,B}\|f\|_1$. Thus,

$$
|\{x \in F\colon |Tb(x)| > \alpha/2\}| \leq \frac{2C_{n,B}}{\alpha}\|f\|_1
$$

and since $|\Omega| \leq \frac{C}{\alpha}\|f\|_1$, we have

$$
|\{\ |Tb(x)| > \alpha/2\}| \leq \frac{C_{n,B}}{\alpha}\|f\|_1
$$

which controls the second part. Thus

$$
|\{|Tf| > \alpha\}| \leq \frac{C_{n,B}}{\alpha}\|f\|_1
$$

which is the weak (1,1) estimate. By Marcinkiewicz interpolation, we have boundedness of $L^p$ for $1 < p < 2$, i.e.

$$
\|Tf\|_p \leq A_{n,B,p}\|f\|_p
$$

To extend to $p > 2$, we can use duality. Taking $f \in L^p$, we note that

$$
\|f\|_p = \sup_{\|g\|_q = 1} \int fg\,dx
$$

Now take any $\|g\|_q = 1$, and consider

$$
\begin{aligned}
\int Tfg\,dx &= \int\int K(x-y)\,f(y)\,g(x)\,dxdy \\
&= \int f(y)\left(\int \tilde{K}(y-x)\,g(x)\,dx\right)dy \\
&= \int f(y)\,\tilde{K} * g(y)\,dy
\end{aligned}
$$

where $\tilde{K}(x) = K(-x)$. Then since $g \in L^q$ and $q < 2$, $\tilde{K} * g \in L^q$, so

$$
\|Tg\|_q = \sup_{\|g\|_q = 1} \int Tfg\,dx \leq \|f\|_{L^p}\|\tilde{K} * g\|_q \leq \|f\|_{L^p} A_{n,B,q}
$$

as desired. $\square$

# Week 6 (10/21/2009)

We will now weaken the conditions of Theorem 35. We start by replacing the gradient condition $|\nabla K| \leq \frac{B}{|x|^{n+1}}$ with

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, dx \leq B^*$$

Note that the gradient condition implies this weaker condition:

$$\begin{aligned}
\int_{|x| \geq 2|y|} |K(x-y) - K(x)|\, dx &= \int_{|x| \geq 2|y|} |\nabla K(x - \tilde{y})|\,|y|\, dx \\
&\leq \int_{|x| \geq 2|y|} \frac{B\,|y|}{|x - \tilde{y}|^{n+1}}\, dx \\
&\leq \int_{|x| \geq 2|y|} \frac{B\,|y|}{(|x| - |y|)^{n+1}}\, dx \\
&= B|y| \int_{|y|}^{\infty} \frac{c r^{n-1}}{r^{n+1}}\, dr \\
&= Bc
\end{aligned}$$

Using this new condition, we can prove the weak (1,1) estimate again in a different manner. As in the proof we use Whitney's decomposition, and then split $f = g + b$ as before. This time, however, we consider $\hat{Q}_j = 3Q_j$, $\hat{\Omega} = \bigcup \hat{Q}_j$ and $\hat{F} = \mathbb{R}^n \backslash \hat{\Omega}$. We still have a favorable bound

$$|\hat{\Omega}| \leq \frac{\hat{A}}{\alpha} \|f\|_1$$

and $\hat{F} \subset F$. The difference now is that we estimate $\int |Tb|$ on $\hat{F}$, where we can now use the new condition with $|x| > 2|y|$ since $\hat{F}$ is separated from $\hat{\Omega}$:

$$\begin{aligned}
\int_{\hat{F}} |Tb(x)|\, dx &\leq \int_{\hat{F}} \sum_{j=1}^{\infty} |Tb_j(x)|\, dx \\
&= \sum_{j=1}^{\infty} \int_{\hat{F}} |Tb_j(x)|\, dx \\
&= \sum_{j=1}^{\infty} \int_{\hat{F}} \left| \int_{Q_j} [K(x-y) - K(x - y_j)]\, b(y)\, dy \right| dx
\end{aligned}$$

using (1) (near the definition of $Tb_j$ in the previous section). Continuing, we have

$$\begin{aligned}
&\leq \sum_{j=1}^{\infty} \int_{(\hat{Q}_j)^c} \int_{Q_j} |K(x-y) - K(x-y_j)|\,|b(y)|\, dy dx \\
&= \sum_{j=1}^{\infty} \int_{Q_j} |b(y)| \int_{(\hat{Q}_j)^c} |K(x-y) - K(x-y_j)|\, dx dy \\
&\leq \sum_{j=1}^{\infty} \int_{Q_j} |b(y)| \int_{|x'| \geq 2|y'|} |K(x'-y') - K(x')|\, dx dy \\
&\leq \sum_{j=1}^{\infty} \int_{Q_j} |b(y)| B^*\, dy \\
&\leq B^* \|f\|_{L^1}
\end{aligned}$$

noting that the third inequality follows from the fact that looking at $y' = y - y_j$ and $x' = x - y_j$, for any $x \in \hat{F}$, $|x - y_j|$ is by construction larger than $2\,\mathrm{diam}(Q_j) \geq 2y'$, and this is how we can incorporate the new condition. The rest of the proof is then the same after obtaining the estimate on $Tb$. Thus we have proved

**Theorem 36.** *Suppose $K \in L^2$ satisfies*

*a)* $|\hat{K}(\xi)| \leq B$

*b)* $\int_{|x| \geq 2|y|} |K(x - y) - K(x)|\, dx \leq B$

*Then with* $Tf := \int_{\mathbb{R}^n} f(y) K(x - y)\, dy$ *as before, $T$ is a bounded operator from $L^p \to L^p$, with* $1 < p < \infty$.

We can weaken the conditions even further. As in Stein's *Singular Integrals*, p35, assuming that $K \in L^2$ is too strong and rules out principal-value singular integrals (integrals relying on cancellations between positive and negative values).

**Theorem 37.** *Suppose $K$ satisfies*

*1.* $|K(x)| \leq \frac{B}{|x|^n}, \quad x \neq 0$

*2.* $\int_{|x| \geq 2|y|} |K(x - y) - K(x)|\, dx \leq B$

*3.* $\int_{R_1 < |x| < R_2} K(x)\, dx = 0$ *for all* $0 < R_1 < R_2 < \infty$

*Consider* $T_\varepsilon f = \int_{|y| \geq \varepsilon} f(x - y) K(y)\, dy$, $\varepsilon > 0$, *avoiding the possible singularity at the origin. Then* $\|T_\varepsilon f\|_p \leq A \|f\|_p$ *for $A$ independent of $\varepsilon$, and the limit of $T_\varepsilon f$ in $L^p$ as $\varepsilon \to 0$ exists and*

$$\|Tf\|_p \leq A_p \|f\|_p$$

*for* $0 < p < \infty$.

**Remark 38.** A few examples of kernels satisfying the conditions above:

1. $K(x) = |x|^{-n} K(\theta)$, where $\int_{S^{n-1}} K(\theta)\, d\theta = 0$, $|K(\theta)| \leq B$.
2. $n = 1$, $K(x) = \frac{1}{\pi x}$.

(We will check the conditions later)

**Remark 39.** If $K(x)$ given is such that $|K(x)| \leq \frac{B}{|x|^n}$, and we let

$$K_\varepsilon(x) = \begin{cases} K(x) & |x| \geq \varepsilon \\ 0 & |x| < \varepsilon \end{cases}$$

Then $K_\varepsilon \in L^2$. (This removes the possible singularity near $x = 0$, and the decay rate is enough for $K$ to be square integrable near $\infty$).

**Lemma 40.** *For any* $\varepsilon > 0$ *we have*

$$\left|\hat{K}_\varepsilon(\xi)\right| \leq CB$$

*where $C$ depends only on dimension.*

**Proof.** First we prove the result for $\varepsilon = 1$ and then use dilations to prove the general case. If $\varepsilon = 1$, then

$$\begin{aligned}
\hat{K_1}(\xi) &= \lim_{R\to\infty} \int_{|x|\le R} e^{2\pi i x\cdot\xi} K_1(x)\,dx \\
&= \underbrace{\int_{|x|\le\frac{1}{|\xi|}} e^{2\pi i x\cdot\xi} K_1(x)\,dx}_{\mathbf{I}} + \underbrace{\lim_{R\to\infty}\int_{\frac{1}{|\varepsilon|}<|x|\le R} e^{2\pi i x\cdot\xi} K_1(x)\,dx}_{\mathbf{II}}
\end{aligned}$$

(The limit is necessary since $K_1$ is not necessarily in $L^1$). The first part we bound by

$$\begin{aligned}
|\mathbf{I}| &= \left|\int_{|x|\le\frac{1}{|\xi|}} \left(e^{2\pi i x\cdot\xi} - 1\right) K_1(x)\,dx\right| \qquad \text{(condition 3)} \\
&\le 2\pi \int_{|x|\le\frac{1}{|\xi|}} |x\cdot\xi||K_1(x)|\,dx \\
&\le 2\pi|B||\xi| \int_{|x|\le\frac{1}{|\xi|}} |x|^{-n+1}\,dx \\
&\le 2\pi|B||\xi| \int_0^{\frac{1}{|\xi|}} c\,dr \\
&\le C_0 B
\end{aligned}$$

For the second part, let $z = \frac{\xi}{2|\xi|^2}$, so $e^{2\pi i z\cdot\xi} = e^{i\pi} = -1$ and $|z| = \frac{1}{2|\xi|}$. Then by a change of variable,

$$\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} K_1(x)\,dx = \frac{1}{2}\int_{\mathbb{R}^n} [K_1(x) - K_1(x-z)]\, e^{2\pi i x\cdot\xi}\,dx$$

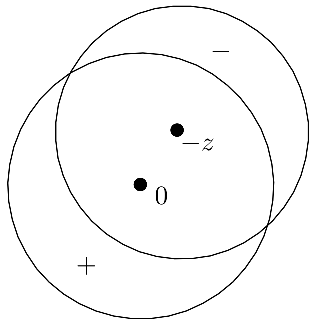
and similarly,

$$\begin{aligned}
\mathbf{II} &= \frac{1}{2}\lim_{R\to\infty}\left(\int_{\frac{1}{|\xi|}\le|x|\le R} [K_1(x) - K_1(x-z)]\, e^{2\pi i x\cdot\xi}\,dx\right) \\
&\quad -\frac{1}{2}\int_{\substack{|x+z|\ge 1/|\xi| \\ |x|\le 1/|\xi|}} K_1(x-z) e^{2\pi i x\cdot\xi}\,dx \\
&\quad +\frac{1}{2}\int_{\substack{|x|\ge 1/|\xi| \\ |x+z|\le 1/|\xi|}} K_1(x-z) e^{2\pi i x\cdot\xi}\,dx \\
&= \frac{1}{2}\lim_{R\to\infty}\left(\int_{\frac{1}{|\xi|}\le|x|\le R} [K_1(x) - K_1(x-z)]\, e^{2\pi i x\cdot\xi}\,dx\right) \\
&\quad +\frac{1}{2}\int_{\substack{|x|\ge 1/|\xi| \\ |x-z|\le 1/|\xi|}} K_1(x) e^{2\pi i x\cdot\xi}\,dx \\
&\quad -\frac{1}{2}\int_{\substack{|x-z|\ge 1/|\xi| \\ |x|\le 1/|\xi|}} K_1(x) e^{2\pi i x\cdot\xi}\,dx
\end{aligned}$$

noting that the two remainder terms comes from the change of variable

$$\int_{\frac{1}{|\xi|}\le|x|\le R} K_1(x) e^{2\pi i x\cdot\xi}\,dx = -\int_{\frac{1}{|\xi|}\le|x+z|\le R} K_1(x-z) e^{2\pi i x\cdot\xi}\,dx$$

and replacing the domain. See picture, where $-$ , $+$ indicate the regions that were added and subtracted in the formula above.




Summarizing, and taking absolute values and limits, we have

$$\begin{aligned}|\mathbf{II}| &\leq \frac{1}{2}\int_{\frac{1}{|\xi|}\leq|x|} |K_1(x)-K_1(x-z)|\,dx \\ &+\frac{1}{2}\int_{\frac{1}{2|\xi|}\leq|x|\leq\frac{3}{2|\xi|}} |K_1(x)|\,dx\end{aligned}$$

where we note that in each of the two exceptional regions, $|x-z|\geq\frac{1}{|\xi|}$ implies that $|x|\geq\frac{1}{2|\xi|}$ since $|x-z|\leq|x|+|z|=|x|+\frac{1}{2|\xi|}$ and $|x-z|\leq\frac{1}{|\xi|}$ implies that $|x|\leq\frac{3}{2|\xi|}$ since $|x-z|\geq|x|-|z|=|x|-\frac{1}{2|\xi|}$. The second term above is bounded by

$$\frac{1}{2}\int_{\frac{1}{2|\xi|}\leq|x|\leq\frac{3}{2|\xi|}} |K_1(x)|\,dx \leq \frac{1}{2}\int_{\frac{1}{2|\xi|}}^{\frac{3}{2|\xi|}} \frac{cBr^{n-1}}{r^n}\,dx \leq \frac{1}{2}\log(3)cB = C_2B$$

Noting that $2|z|=\frac{1}{|\xi|}$, the first term above is bounded by

$$\begin{aligned}\int_{|x|\geq\frac{1}{|\xi|}} [K_1(x)-K_1(x-z)]e^{2\pi i x\cdot\xi}\,dx &\leq \int_{|x|\geq 2|z|} |K_1(x)-K_1(x-z)|\,dx \\ &\leq B\end{aligned}$$

Adding all the estimates together gives the result that $|\hat{K}(\xi)|\leq CB$.

**General Case** $\varepsilon>0$. Replace the kernel $K_\varepsilon$ by $K=\varepsilon^n K_\varepsilon(\varepsilon x)$. Then $K(x)=0$ if $|x|\leq 1$, and

$$|K(x)|=\varepsilon^n|K_\varepsilon(\varepsilon x)|\leq\frac{B\varepsilon^n}{|\varepsilon x|^n}=\frac{B}{|x|^n}$$

Thus $K$ satisfies the the Lemma with $\varepsilon=1$ and from above we know that $|\hat{K}|\leq CB$ and thus $|\hat{K_\varepsilon}|\leq CB$ by the dilation property of the Fourier transform:

$$|\hat{K}(\xi)|=|\varepsilon^n(\varepsilon^{-n}\hat{K}_\varepsilon(\xi/\varepsilon))|$$

and thus $|\hat{K}_\varepsilon(\xi)|\leq|\hat{K}(\varepsilon\xi)|\leq CB$. This dilation trick is very useful and we will be using it again (probably...) □

Now we finish the proof of the theorem.

**Proof. (of Theorem 37)** As remarked in the main idea, we consider $T_\varepsilon f = K_\varepsilon * f$. By theorem 36

$$\|T_\varepsilon f\|_p \leq A_p \|f\|_p$$

for $1 < p < \infty$, and Lemma 40 shows that the bound is uniform in $\varepsilon$ (Marcinkiewicz interpolation bound depends on the bounds of the weak (1,1), weak (2,2) estimates. The weak (1,1) bound relies only on $p$ and $B$, the (2,2) estimate relies on the bound in Lemma 40). Now (following *Singular Integrals*, p37), let $g$ be a continuously differentiable function with compact support $g \in C_0^1$. Then

$$\begin{aligned}
T_\varepsilon g(x) &= \int_{|y| \geq \varepsilon} K(y) g(x-y)\, dy \\
&= \int_{|y| \geq 1} K(y) g(x-y)\, dy + \int_{\varepsilon \leq |y| \leq 1} K(y)[g(x-y) - g(x)]\, dy
\end{aligned}$$

where we have used the cancellation condition (condition 3). The first term is an $L^p$ function (convolution of $L^1$ and the $L^p$ function $K_1$). For the second term, by the continuous differentiability of $g$, $|g(x-y) - g(x)| \leq A|y|$, where $A$ is the uniform bound on the derivative of $g$ (note $g$ is continuously differentiable and compactly supported). Then

$$\begin{aligned}
\left| \left( \int_{|y| \leq 1} - \int_{\varepsilon \leq |y| \leq 1} \right) K(y)[g(x-y) - g(x)]\, dy \right| &\leq \left| \int_{|y| \leq \varepsilon} K(y)[g(x-y) - g(x)]\, dy \right| \\
&\leq \int_{|y| \leq \varepsilon} \frac{AB}{|y|^n} |y|\, dy \\
&= AB \int_0^\varepsilon \frac{c r^{n-1}}{r^{n-1}}\, dr \\
&= ABc\varepsilon \\
&\to 0
\end{aligned}$$

so that the second term converges uniformly as $\varepsilon \to 0$ to the integral on $|y| \leq 1$. As a function of $x$, this limit is in $L^p$, since it is compactly supported and the limit is uniform. Thus $T_\varepsilon g$ converges in $L^p$ to some $Tg$ and $\|Tg\|_p \leq A_p \|g\|_p$. Thus $T$ is a bounded operator on $C_0^1 \cap L^p$ with $L^p$ norm, and we extend to $L^p$ by continuity. For any $f \in L^p$, we can take a sequence $g_n \to f$ in $L^p$ with $g \in C_0^1$ and define $Tf = \lim_{n \to \infty} Tg_n$ ($Tg_n$ is Cauchy in $L^p$ since $g_n$ is Cauchy in $L^p$, and $L^p$ is a complete Banach space). In this case the inequality $\|Tf\|_p \leq A_p \|f\|_p$ is trivially satisfied. This completes the proof. □

## Hilbert Transform

The Hilbert Transform $Hf$ is defined by

$$Hf(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_{|y| \geq \varepsilon} \frac{f(x-y)}{y}\, dy$$

(working with 1 dimension), and the (convolution) kernel is $K(x) = \frac{1}{\pi x}$. Then via the usual contour integration methods (semicircular contour in the upper half plane with a small detour around $z = 0$) we have

$$\begin{aligned}
\hat{K}(\xi) &= \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \frac{e^{-2\pi i \xi x}}{\pi x}\, dx \\
&= -i \operatorname{sgn}(\xi)
\end{aligned}$$

noting that for negative $\xi < 0$,

$$\lim_{\varepsilon \searrow 0} \int_{|\mathrm{Re}\, z| \geq \varepsilon} \frac{e^{-2\pi i \xi z}}{\pi z} dz - \pi i \,\mathrm{Res}_{z=0} \frac{e^{-2\pi i \xi z}}{\pi z} = 0$$

(the little detour around the pole at 0 picks up half a residue, and for the upper half plane we need $-2\pi\xi > 0$ for Rouché's Theorem) and for negative $\xi < 0$, the sign reverses (change of variable in integration). Thus

$$\widehat{Hf}(\xi) = \hat{K}(\xi) = -i\,\mathrm{sgn}(\xi)\,\hat{f}(\xi)$$

This implies that

$$H^2 = -I$$

in $L^2$. A few properties involving dilations and translations:

1. $H\delta_h = \delta_h H,\ h > 0$
2. $H\delta_h = -\delta_h H,\ h < 0$
3. $H\tau_h = \tau_h H$ (already known for convolution operators)

These properties characterize the Hilbert transform.

**Proposition 41.** *Suppose $T$ bounded operator from $L^2 \to L^2$ satisfies*

1. *$T$ commutes with translations (i.e. convolution operator)*
2. *$T$ commutes with positive dilations.*
3. *$T$ anti-commutes with negative dilations.*

*Then $T = c \cdot H$.*

**Proof.** The first property shows that $\widehat{Tf}(\xi) = \hat{f}(\xi) m(\xi)$ for some $m(\xi) \in L^\infty$ (a multiplication operator from $L^2 \to L^2$). As for the second property, since $\mathcal{F}\delta_h = \frac{1}{|h|}\delta_{1/h}\mathcal{F}$, we have that $\delta_h \mathcal{F} = \frac{1}{|h|}\mathcal{F}\delta_{1/h}$, $\delta_{1/h}\mathcal{F}^{-1} = |h| \mathcal{F}^{-1}\delta_h$ and

$$\begin{aligned}
\delta_h(m \cdot \hat{f}) &= \delta_h(\mathcal{F} T \mathcal{F}^{-1} \hat{f}) \\
&= \frac{1}{|h|}\mathcal{F}\delta_{1/h} T \mathcal{F}^{-1}\hat{f} \\
&= \frac{1}{h}\mathcal{F} T \delta_{1/h}\mathcal{F}^{-1}\hat{f} && (\text{Properties } 2, 3) \\
&= \mathrm{sgn}(h)\,\mathcal{F} T \mathcal{F}^{-1}\delta_{1/h}\hat{f} \\
&= \mathrm{sgn}(h)\, m \cdot (\delta_h \hat{f})
\end{aligned}$$

Thus $m(h\xi)\hat{f}(h\xi) = \mathrm{sgn}(h) m(\xi)\ \hat{f}(h\xi)$ and $m(h\xi) = \mathrm{sgn}(h) m(\xi)$, and if $h \neq 0$ this proves that $m(\xi) = C\,\mathrm{sgn}(\xi)$, which proves the result since the Hilbert Transform in the Fourier domain corresponds to multiplication by $-i\,\mathrm{sgn}(\xi)$. □

# Week 7 (10/28/2009)

*Quick note: Stein proves general results about Calderon-Zygmund type operators and then shows that Riesz Transforms are a special case. We will treat Riesz Transforms first and then move to C-Z type operators next.*

## Riesz Transform

Note that for the Hilbert transform,

$$\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\,\hat{f}(\xi) = -\frac{i\xi}{|\xi|}\hat{f}(\xi)$$

Now we look into the higher dimensional analogue. For $j = 1, 2, \ldots, n$ in $\mathbb{R}^n$, we define $R_j f(x)$ by

$$\widehat{R_j f}(\xi) = -\frac{i\xi_j}{|\xi|}\hat{f}(\xi) \quad j = 1, 2, \ldots, n$$

Then $R_j: L^2 \to L^2$, $\|R_j\|_{2,2} \leq 1$, and

$$R_j f = K_j * f$$

for some $K_j$. We will also use the notation $R_{jk} = R_j R_k$ with corresponding kernel $K_{jk}$.

Note that $\hat{K}_j = -i\frac{\xi_j}{|\xi|}$ is homogeneous of degree 0, and by Proposition 17 (from week 4), this implies that $K_j$ is homogeneous of degree $-n$, so that $K_j(x) = \frac{\Omega_j(x)}{|x|^n}$ where $\Omega_j(x)$ is homogeneous of degree 0, i.e. $\Omega_j\left(\frac{x}{|x|}\right) = \Omega_j(x)$, constant along rays from the origin. The same holds for $K_{jk}$. We will show shortly that $K_j$, $K_{jk}$, etc. satisfy the conditions of Theorem 37 so that $R_j$, $R_{jk}$ map from $L^p$ to $L^p$ for $1 < p < \infty$.

**Relation to Laplace's Equation**

We relate this to the solution of Laplace's equation in $\mathbb{R}^n$

$$\begin{aligned} -\Delta u &= f \in L^p(\mathbb{R}^n) \\ u|_\infty &= 0 \end{aligned}$$

Taking the Fourier transform, we have that

$$\begin{aligned} \widehat{-\Delta u} &= \hat{f}(\xi) \\ 4\pi^2|\xi|^2\hat{u}(\xi) &= \hat{f}(\xi) \\ \hat{u}(\xi) &= \frac{\hat{f}(\xi)}{4\pi^2|\xi|^2} \end{aligned}$$

Note that

$$\begin{aligned} \widehat{u_{x_j x_k}} &= -4\pi^2\xi_j\xi_k\hat{u}(\xi) \\ &= \frac{-\xi_j\xi_k}{|\xi|^2}\hat{f}(\xi) \\ &= \widehat{R_{jk}f} \end{aligned}$$

so that $u_{x_j x_k} = R_{jk} f$, or $R_{jk} f = D_{x_j} D_{x_k}(-\Delta)^{-1} f$.

(As an aside, recall that the kernel of $(-\Delta)^{-1}$ is the fundamental solution, the Green's function:

$$(-\Delta)^{-1} f = c_n \int \frac{f(y)}{|x - y|^{n-2}}\,dy$$

for $n > 2$)

At any rate, the $L^p$ boundedness of $R_{jk}$ will imply that for $u \in C^2$ and $\Delta u \in L^p$,

$$\|u_{x_j x_k}\|_p = \|R_{jk} \Delta u\|_p \leq C \|\Delta u\|_p$$

**Riesz Kernel Formula**

Before continuing, we compute the kernel explicitly.

**Lemma 42.**

$$K_j(x) = \frac{\Omega_j(x)}{|x|^n} = \frac{c_n x_j}{|x|^{n+1}}$$

*i.e.* $\Omega_j(x) = \frac{c_n x_j}{|x|}$, *and the conditions of Theorem 37 are satisfied, so that* $R_j$ *is a bounded operator from* $L^p \to L^p$ *for* $0 < p < \infty$.

*Also,* $K_{jk}(x) = \tilde{c}_n \frac{x_j x_k}{|x|^{n+2}}$ *and has the same properties.*

**Proof.** As discussed above, since the Fourier transform is homogeneous of degree 0, the kernel is homogeneous of degree $-n$ and hence has the form above. We can perform an explicit computation to determine $\Omega_j$ (Follows *Fourier Analysis* by Javier D, Corollary to Proposition 4.3 and below Corollary 4.9).

First, we note by Proposition 17 that since $\frac{1}{|x|^{n-1}}$ (as a tempered distribution) is homogeneous of degree $1-n$, its Fourier transform $\hat{\varphi}(\xi)$ is homogeneous of degree $-1$. Furthermore, we note that the Fourier transform is rotationally invariant, since

$$\begin{aligned}
\hat{\varphi}(R\xi) &= \int \frac{1}{|x|^{n-1}} e^{-2\pi i x \cdot R\xi} dx \\
&= \int \frac{1}{|x|^{n-1}} e^{-2\pi i R^{-1} x \cdot \xi} dx \\
&= \int \frac{1}{|Rx|^{n-1}} e^{-2\pi i x \cdot \xi} dx \\
&= \int \frac{1}{|x|^{n-1}} e^{-2\pi i x \cdot \xi} dx \\
&= \hat{\varphi}(\xi)
\end{aligned}$$

and thus $\hat{\varphi}(\xi) = \frac{c}{|\xi|}$. Now we use the fact that

$$D_{x_j} \frac{1}{|x|^{n-1}} = (1-n) \frac{x_j}{|x|^{n+1}}$$

in the sense of distributions (since it is not integrable near $\infty$), and thus taking Fourier transforms of both sides yields

$$\hat{K_j}(\xi) = \frac{-i\xi_j}{|\xi|} = C_n \left( \frac{x_j}{|x|^{n+1}} \right)^{\wedge}$$

and thus $K_j = \frac{C_n x_j}{|x|^{n+1}}$ as desired. Taking an additional $D_{x_k}$ derivative proves the result for $K_{jk}$.

Now note that $\Omega_j(x)$ restricted to $\mathbb{S}^{n-1}$ is smooth, and in particular bounded, and that $\int_{\mathbb{S}^{n-1}} \Omega_j(x) dS^{n-1} = 0$, so that $K_j$ satisfies the first and third properties of Theorem 37.

We check the second property directly:

$$\begin{aligned}
\int_{|x|\geq 2|y|} |K_j(x-y) - K_j(x)|\,dx &= c_n \int_{|x|\geq 2|y|} \left| \frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right| dx \\
&= c_n \int_{|x|\geq 2|y|} \left| \frac{x_j(|x|^{n+1} - |x-y|^{n+1}) - y_j|x|^{n+1}}{|x-y|^{n+1}|x|^{n+1}} \right| dx \\
&\leq c_n \int_{|x|\geq 2|y|} \frac{C|y||x|^{n+1} + |y||x|^{n+1}}{|x-y|^{n+1}|x|^{n+1}}\,dx \\
&\leq B \int_{|x|\geq 2|y|} \frac{|y|}{|x|^{n+1}}\,dx \\
&\leq B' \int_{2|y|}^{\infty} \frac{|y|}{r^2}\,dr \\
&\leq B''
\end{aligned}$$

where we have used the mean value theorem on $||x|^{n+1} - |x-y|^{n+1}|$ along an arbitrary line segment from $x$ to $x-y$, so that it is bounded by $|y||x-y'|^n \leq C|y||x|^n$ ($|y| \leq 2|x|$). The proof for $K_{jk}$ is identical. $\square$

**A Short Application**

Consider the problem

$$\begin{aligned}
-\Delta u &= \operatorname{div} \mathbf{F} \\
u|_\infty &= 0
\end{aligned}$$

for $\mathbf{F}$ a vector field in $\mathbb{R}^k$. Taking the Fourier transform gives

$$\hat{u}(\xi) = \frac{\xi \cdot \hat{\mathbf{F}}(\xi)}{|\xi|^2}$$

where the Fourier transform of $\mathbf{F}$ is just a component-wise Fourier transform. Note $\hat{u}$ is homogeneous of degree $-1$, which allows us to "gain" a derivative in estimating $\nabla u$. Note

$$\widehat{D_{x_j} u} = \frac{\xi_j \xi \cdot \hat{\mathbf{F}}(\xi)}{|\xi|^2} = \sum_{k=1}^{n} \frac{\xi_j \xi_k \hat{\mathbf{F}}_k(\xi)}{|\xi|^2} = \sum_{k=1}^{n} \widehat{R_{jk}\mathbf{F}_k}(\xi)$$

and since $R_{jk}$ is a bounded operator from $L^p$ to $L^p$, this gives us the apriori estimate

$$\|\nabla u\|_p \leq C(u,p)\|\mathbf{F}\|_p$$

## Calderon-Zygmund Operators

Now we turn to general convolution operators with kernels that are homogeneous of degree $-n$, i.e. $K(x) = \frac{\Omega(x)}{|x|^n}$ with $\Omega(\omega) = \Omega\left(\frac{x}{|x|}\right)$ homogeneous of degree 0, and consider conditions on $\Omega(\omega)$ under which the conditions of Theorem 37 are satisfied. Note that Riesz Transforms are a special case.

To satisfy the first and third conditions, we should have that $|\Omega(\omega)| \leq B$ and that $\int_{\mathbb{S}^{n-1}} \Omega(\omega)\,d\omega = 0$. For the second condition, we perform a computation that leads us to a suitable condition on $\Omega$. First we define the **modulus of continuity** of $\Omega$ by

$$\eta_\Omega(\delta) = \sup_{\substack{|\omega|=|\omega'|=1 \\ |\omega-\omega'|<\delta}} |\Omega(\omega) - \Omega(\omega')|$$

Then note that

$$\begin{aligned} K(x-y)-K(x) &= \frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n} \\ &= \frac{\Omega(x-y)-\Omega(x)}{|x-y|^n}+\Omega(x)\left(\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right) \end{aligned}$$

and thus

$$\begin{aligned} \int_{|x|\geq 2|y|} |K(x-y)-K(x)|\,dx &\leq \int_{|x|\geq 2|y|} B\left|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right| dx \\ &\quad + \int_{|x|\geq 2|y|} \frac{|\Omega(x-y)-\Omega(x)|}{|x-y|^n}\,dx \end{aligned}$$

The first integral is bounded exactly like the computation in Lemma 42:

$$\begin{aligned} \int_{|x|\geq 2|y|} B\left|\frac{1}{|x-y|^n}-\frac{1}{|x|^n}\right| dx &\leq B\int_{|x|\geq 2|y|} \frac{||x|^n-|x-y|^n|}{|x|^n|x-y|^n}\,dx \\ &\leq B\int_{|x|\geq 2|y|} \frac{c|y|}{|x|^{n+1}}\,dx \\ &\leq B' \end{aligned}$$

For the second integral, we use the fact that for $|x|\geq 2|y|$,

$$\left|\frac{x-y}{|x-y|}-\frac{x}{|x|}\right|\leq C\frac{|y|}{|x|}$$

(this is the same trick as above). Then we have

$$\begin{aligned} \int_{|x|\geq 2|y|} \frac{|\Omega(x-y)-\Omega(x)|}{|x-y|^n}\,dx &\leq \int_{|x|\geq 2|y|} \frac{\eta_\Omega\Big(C\frac{|y|}{|x|}\Big)}{|x-y|^n}\,dx \\ &= \int_{|x|\geq 2|y|} \frac{\eta_\Omega\Big(C\frac{|y|}{|x|}\Big)}{\left|\frac{x}{|x|}-\frac{y}{|x|}\right|^n}\frac{dx}{|x|^n} \\ &\leq c_n\int_{2|y|}^{\infty} \eta_\Omega\Big(C\frac{|y|}{r}\Big)\frac{dr}{r} \\ &\leq c_n'\int_0^{C/2} \eta_\Omega(\delta)\frac{d\delta}{\delta} \end{aligned}$$

The third line follows from bounding the denominator below with triangle inequality and $|x|\geq 2|y|$ and then converting to polar. The last line follows from the change of variable $\delta = C\frac{|y|}{r}$. Thus, if this last quantity is bounded, we say that $\Omega$ is **Dini-continuous** on $\mathbb{S}^{n-1}$. In particular this is true if $\Omega$ is Hölder continuous, since in this case

$$\int_0^{C/2} \eta_\Omega(\delta)\frac{d\delta}{\delta}\leq \int_0^{C/2}\frac{d\delta}{\delta^{1-\alpha}}\leq BC^\alpha<\infty$$

Thus, with these conditions we have the following theorem.

**Theorem 43.** *Suppose $K(x)=\frac{\Omega(x)}{|x|^n}$ where $\Omega$ is homogeneous of degree $0$, satisfying*

1. $|\Omega(x)|\leq B$

2. $\int_{\mathbb{S}^{n-1}}\Omega(\omega)\,d\omega=0$

3. $\Omega(\omega)$ *is Dini-continuous, i.e.*

$$\int_0^1 \eta_\Omega(\delta)\frac{d\delta}{\delta} < \infty$$

*Then for* $1 < p < \infty$ *and* $f \in L^p$, *define*

$$T_\varepsilon(f) = \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\,dy$$

*as before. Then*

a) $\|T_\varepsilon f\|_p \leq A_p \|f\|_p$ *where* $A_p$ *does not depend on* $\varepsilon$.

b) $\lim_{\varepsilon\to 0} T_\varepsilon(f) = T(f)$ *exists in* $L^p$ *(this implies* $\|T\|_{p,p} \leq A_p$*)*

c) *If* $f \in L^2$, *then* $\widehat{Tf} = m(\xi)\ \hat{f}(\xi)$ *where* $m(\xi)$ *is homogeneous of degree* 0, *and can be computed by*

$$m(\xi) = \int_{\mathbb{S}^{n-1}} \left[\frac{\pi}{2} i\,\mathrm{sgn}(\xi \cdot y) + \log\left(\frac{1}{|\xi \cdot y|}\right)\right] \Omega(y)\,dS_y$$

*for* $|\xi| = 1$.

**Proof.** Since the conditions imply that $K(x)$ satisfy the hypotheses of Theorem 37, (a) and (b) are immediate. (c) is a long computation. See Stein, Chapter 2 Section 4.3. $\square$

**Remark 44.** For $K(x) = \frac{\Omega(x)}{|x|^n}$, $\Omega(x) = \Omega\left(\frac{x}{|x|}\right) \neq 0$,

1. If $f \geq 0$, and $f \in L^1$, then $Tf \notin L^1$.

   To verify this we check that $\widehat{Tf}(\xi) = m(\xi)\ \hat{f}(\xi)$ is not continuous. Since $m$ is homogeneous of degree 0, it cannot be continuous at 0 (unless $m$ is constant, which cannot happen unless $K$ is a dirac mass). Also, $\hat{f}(0) = \|f\|_{L^1} > 0$ since $f \geq 0$ (assuming $f$ is not identically zero), and thus we conclude that $\widehat{Tf}$ cannot be continuous at 0, and $Tf \notin L^1$.

2. There exists $f \in C_0(B)$ such that $Tf$ is unbounded near every point of $B$.

   This follows from Baire category theorem and the uniform boundedness principle somehow (?)

Now we pass from $L^p$ convergence of $T_\varepsilon$ to almost everywhere convergence by use of the maximal function.

**Theorem 45.** *Let* $\Omega$ *be as above,* $f \in L^p$, *for* $1 \leq p < \infty$ *and* $K(x) = \frac{\Omega(x)}{|x|^n}$

a) $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ *exists a.e.* $x$.

b) *Let* $T^* f = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|$, *then* $T^*$ *is weak (1,1)*

c) *If* $1 < p < \infty$, *then* $\|T^* f\|_p \leq A_p \|f\|_p$

For this Theorem it will be useful to use the following Lemma which we prove later.

**Lemma 46. (Cotlar)** *For* $x \in \mathbb{R}^n$,

$$T^* f(x) \leq M_{Tf}(x) + C M_f(x)$$

**Proof. (of Theorem 45)** Accepting Lemma 46 for now, we see that the proof of (c) is immediate:

$$\begin{aligned}\|T^*f\|_p &\leq \|M_{Tf}\|_p + C\|M_f\|_p\\ &\lesssim \|Tf\|_p + C\|f\|_p\\ &\lesssim \|f\|_p\end{aligned}$$

To prove (b), we again use the Calderon-Zygmund decomposition, and much of the details from the proofs of Theorems 35, 36 are the same. We again write $f = g + b$, and consider the enlarged cubes $\widehat{Q_j} = 3Q_j$ and the decomposition $\mathbb{R}^n = \hat{\Omega} \cup \hat{F}$, where we have the usual estimate on $\hat{\Omega}$,

$$|\hat{\Omega}| \leq \frac{3^n A}{\alpha}\|f\|_1$$

so that we only need to focus on $\hat{F}$. Following the same proof we estimate

$$|\{|T^*f| > \alpha\}| \leq |\{|T^*g| > \alpha/2\}| + |\{|T^*b| > \alpha/2\}|$$

and as in the proof of Theorem 35 and (c), the first is estimated by

$$|\{|T^*g| > \alpha/2\}| \leq \frac{4}{\alpha^2}\|T^*g\|_2^2 \leq \frac{C}{\alpha^2}\|g\|_2^2 \leq \frac{C'}{\alpha^2}\|g\|_1$$

using $g \in L^\infty$ and $g \in L^1$ to get the final estimate in $L^1$. For the bad part, we again focus on $\hat{F}$. We claim that for $x \in \hat{F}$,

$$T^*b(x) = \sup_{\varepsilon > 0} T_\varepsilon b(x) \leq \sum_{j=1}^{\infty} \int_{Q_j} |K(x-y) - K(x)||b(y)|dy + C M_b(x)$$

so that we can use the usual estimate for the first term and the maximal function estimate for the second term. To prove this claim, we fix $\varepsilon$ and separate each $Q_j$ into three cases:

- Case 1: For all $y \in Q_j$, $|x - y| > \varepsilon$
- Case 2: For all $y \in Q_j$, $|x - y| < \varepsilon$
- Case 3: There exists $y \in Q_j$ such that $|x - y| = \varepsilon$.

With the reminder that $T_\varepsilon b(x) = \sum_j \int_{Q_j} K_\varepsilon(x-y)b(y)dy$, we note that for the $Q_j$ in case 2, since $|x - y| < \varepsilon$, we have that $K_\varepsilon(x - y) = 0$ for $y \in Q_j$, and thus there is no contribution to $T_\varepsilon b(x)$ from $Q_j$ in case 2.

On the other hand, for the $Q_j$ in case 1, since $|x-y| > \varepsilon$, $K_\varepsilon(x-y) = K(x-y)$ and so the contribution of the integral over $Q_j$ is bounded above by $\int_{Q_j} |K(x-y)||b(y)|dy$.

For the $Q_j$ in case 3, there exists some $y \in Q_j$ such that $|x - y| = \varepsilon$. Then by construction since $x \in (3Q_j)^c$, $Q_j$ is contained in $\overline{B_{2\varepsilon}}$ . Then the contribution to $T_\varepsilon b$ for this $Q_j$ is bounded above by

$$\begin{aligned}\int_{Q_j} |K_\varepsilon(x-y)||b(y)|dy &\leq \int_{Q_j \cap B_{2\varepsilon}(x)} |K_\varepsilon(x-y)||b(y)|dy\\ &\leq \int_{Q_j \cap B_{2\varepsilon}(x)} \frac{B}{|x-y|^n}|b(y)|dy\\ &\leq \frac{B'}{|B_{2\varepsilon}(x)|}\int_{Q_j \cap B_{2\varepsilon}(x)} |b(y)|dy\\ &\leq B''|M_b(x)|\end{aligned}$$

Combining all three cases, and noting the bound is indepenent of $\varepsilon$ proves the claim. Now we use

$$\begin{aligned} |\{x\in \hat{F}: |T^*b(x)|>\alpha/2\}| &\le |\{x\in\hat{F}: CM_b(x)>\alpha/4\}| \\ &\quad + \left|\left\{x\in\hat{F}: \sum_{j=1}^{\infty}\int_{Q_j} |K(x-y)-K(x)|\,|b(y)|\,dy>\alpha/4\right\}\right| \end{aligned}$$

where the first term is bounded by $\frac{C}{\alpha}\|b\|_{L^1}\le\frac{C}{\alpha}\|f\|_{L^1}$ using the maximal operator bound, and the second term is bounded in the same way as in the proof of Theorem 36, by bounding the $L^1$ norm of the sum and using Markov. This proves (b).

Now (a) follows from (b),(c), similar in spirit to the proof of Corollary 4 (Week 1). From the proof of Theorem 37, we showed that $T_\varepsilon g$ converges uniformly if $g\in C^1$. Now decompose $f$ as $f=g+(f-g)$ so that $\|f-g\|_p\le\delta$ and $T_\varepsilon f=T_\varepsilon g+T_\varepsilon(f-g)$. Thus we only need to study the convergence of $T_\varepsilon(f-g)$, and we will only use the weak $(p,p)$ estimate on $T^*$ for $1\le p<\infty$. Consider (following Stein, Chapter 2, section 4.6.3)

$$\Lambda_{f-g}(x)=\left|\limsup_{\varepsilon\to 0} T_\varepsilon(f-g)(x)-\liminf_{\varepsilon\to 0} T_\varepsilon(f-g)(x)\right|$$

and note that $\Lambda_{f-g}(x)\le 2T^*(f-g)(x)$. Now using the weak estimate on $T^*$ (note that we have a weak $L^p$ estimate for $1\le p<\infty$), we have that for any $\alpha$,

$$|\{\Lambda_{f-g}(x)>\alpha\}|\le\frac{A_p}{\alpha}\|T^*(f-g)\|_p\le\frac{A_p'}{\alpha}\|f-g\|_p\le\frac{A_p'\delta}{\alpha}$$

and thus $\Lambda_{f-g}(x)=0$ a.e., and $T_\varepsilon(f-g)$ converges almost everywhere. □

We will prove Cotlar's Lemma (46) next time.

## Week 8 (11/4/2009)

**Proof. (of Lemma 46, Cotlar's Lemma)** By translation invariance, it suffices to show this result for $x=0$. More precisely, if $T^*f(0)\le CM_{Tf}(0)+C'\,M_f(0)$ for any $f$ (with a constant independent of $f$), we can apply this result to the translate $\tau_x f$ to show the same result for arbitrary $x$. So for $\varepsilon>0$ and $x=0$, we want to show that

$$|T_\varepsilon f(0)|\le C(M_f(0)+M_{Tf}(0))$$

with constant $C$ independent of $f$ and $\varepsilon$. Decompose $f=f_1+f_2$ where $f_1=\mathbf{1}_{|x|<\varepsilon}f$. In particular, $T_\varepsilon f(0)=Tf_2(0)$. Now we make the following claim.

**Claim:** For $|x|<\varepsilon/2$, $|Tf_2(x)-Tf_2(0)|\le CM_f(0)$

Accepting the claim for now, which implies that for $|x|<\varepsilon/2$,

$$|Tf_2(0)|-|Tf_2(x)|\le|Tf_2(x)-Tf_2(0)|\le CM_f(0)$$

and so

$$\begin{aligned} |T_\varepsilon f(0)| &= |Tf_2(0)| \\ &\le |Tf_2(x)|+CM_f(0) \\ &\le |Tf(x)|+|Tf_1(x)|+CM_f(0) \qquad (*) \end{aligned}$$

for $|x|<\varepsilon/2$. Now we consider two cases:

- $\frac{1}{3}|T_\varepsilon f(0)|\leq CM_f(0)$

  In this case, we already have the result, since

$$|T_\varepsilon f(0)|\leq 3CM_f(0)\leq 3C(M_f(0)+M_{Tf}(0))$$

- $\frac{1}{3}|T_\varepsilon f(0)|\geq CM_f(0)$

  In this case, let $\lambda=|T_\varepsilon f(0)|$, and $B=B_{\varepsilon/2}(0)=\{x\colon |x|<\varepsilon/2\}$. Note for any $x\in B$, from the previous calculation we have that by $(*)$ above and the assumption,

$$\lambda\leq|Tf(x)|+|Tf_1(x)|+\lambda/3$$

  and so either $|Tf(x)|\geq\lambda/3$ or $|Tf_1(x)|\geq\lambda/3$ must be true or else $\lambda<\lambda$, a contradiction. Therefore,

$$|B| \;\leq\; |\{x\in B\colon |Tf(x)|\geq\lambda/3\}|+|\{x\in B\colon |Tf_1(x)|\geq\lambda/3\}|$$

  For the first term we use Chebyshev's inequality and the definition of the maximal function:

$$|\{x\in B\colon |Tf(x)|\geq\lambda/3\}|\leq\frac{3\int_B |Tf(x)|\,dx}{\lambda}\leq\frac{3|B|}{\lambda}M_{Tf}(0)$$

  For the second term, we use the weak (1,1) estimate for $T$ and the fact that $f_1$ is supported on and equal to $f$ on $\{|x|<\varepsilon\}=2B$ to get

$$\begin{aligned}|\{x\in B\colon |Tf_1(x)|\geq\lambda/3\}| &\leq \frac{C'}{\lambda}\|f_1\|_{L^1(2B)}\\ &\leq \frac{2C'|B|}{\lambda}M_f(0)\end{aligned}$$

  Combining the results, we have that

$$|B|\leq\frac{3|B|}{\lambda}M_{Tf}(0)+\frac{2C'|B|}{\lambda}M_f(0)$$

  and so

$$|T_\varepsilon f(0)|=\lambda\leq 3M_{Tf}(0)+2C'M_f(0)$$

Now combining the two cases together, letting $\tilde{C}=\max(3C,3,2C')$ we have that

$$|T_\varepsilon f(0)|\leq\tilde{C}(M_{Tf}(0)+M_f(0))$$

as desired. What remains is the proof of the claim:

**Proof of Claim:** We want to show that for $|x|<\varepsilon/2$, $|Tf_2(x)-Tf_2(0)|\leq CM_f(0)$.

$$\begin{aligned}|Tf_2(x)-Tf_2(0)| &= \left|\int_{|y|\geq\varepsilon}(K(x-y)-K(-y))f(y)\,dy\right|\\ &\leq \int_{|y|\geq\varepsilon}|K(x-y)-K(-y)||f(y)|\,dy\\ &\leq C\varepsilon\int_{|y|\geq\varepsilon}\frac{1}{|y|^{n+1}}|f(y)|\,dy\\ &\leq CM_f(0)\end{aligned}$$

The third line uses $|K(x-y)-K(-y)|\leq\frac{C|x|}{|y|^{n+1}}$ with the gradient condition. Using Dini continuity needs a bit more justification and is quite technical. The fourth line uses integration by parts in the following way. First we convert to polar:

$$C\varepsilon\int_{|y|\geq\varepsilon}\frac{1}{|y|^{n+1}}|f(y)|dy \;=\; C\varepsilon\int_{\varepsilon}^{\infty}\frac{1}{r^{n+1}}\left(\int_{\mathbb{S}^{n-1}}|f(r,\omega)|dS^{n-1}\right)r^{n-1}dr$$

Note that

$$\int_{B_R}|f(x)|dx=\int_0^R\left(\int_{\mathbb{S}^{n-1}}|f(r,\omega)|dS^{n-1}\right)r^{n-1}dr$$

Therefore we will apply integration by parts with $u(r)=\frac{1}{r^{n+1}}$ and $v(r)=\int_{B_r}|f(x)|dx$. We will also use the inequality $v(r)\leq c_n r^n M_f(0)$ (definition of maximal function). Then continuing above, we have that

$$\begin{aligned}
&= C\varepsilon\left[\frac{c_n r^n M_f(0)}{r^{n+1}}\right]_{\varepsilon}^{\infty}-C\varepsilon\int_{\varepsilon}^{\infty}\frac{-(n+1)}{r^{n+2}}r^n M_f(0)dr\\
&= -Cc_n M_f(0)+C(n+1)M_f(0)\\
&\leq C_n M_f(0)
\end{aligned}$$

as desired. □

(Other proofs in Stein, Javier D....)

**Fourier Series Convergence**

We now know enough to understand the convergence of Fourier series, as far as singular operators are concerned. There are a few more technical points. Recall that we want to study the convergence of

$$\int_{-\pi}^{\pi}\frac{\sin\lambda y}{y}f(x-y)dy$$

as $\lambda\to\infty$. Note that the Dirichlet kernel for the Fourier series has $\lambda=N+\frac{1}{2}$ and the denominator is $\sin\pi y$ instead of $y$, which is a small difference. Can get $L^p\to L^p$ uniformly in $\lambda$ using a slick trick by representing the operator by something like

$$M_{-N}(I+H)M_N-M_{N+1}(I+H)M_{N+1}$$

where $M_N$ is multiplication by $e^{2\pi iNx}$ (modulation). This gives boundedness in $L^p$ (uniformly in $\lambda$, using uniform boundedness principle).

Then to get a.e. convergence, we need the weak (1,1) estimate on $\sup_\lambda T_\lambda$, and this is much harder. (Need more references now)

## $H^1$ and BMO Spaces

We note that in singular integral estimates, we have boundedness of operators in $L^p$ for $1<p<\infty$, and which fail to be bounded for $p=1$ or $p=\infty$. This motivates a search for perhaps a subspace of $L^1$ for which we do have boundedness, whose dual space then corresponds to a slightly larger space than $L^\infty$. The Hardy space will serve as the replacement, and the dual is BMO, the space of functions with bounded mean oscillations.

We denote the Hardy spaces by $\mathcal{H}^p$, for $0 < p \leq \infty$, which we will define in various ways and show that all the definitions are equivalent. Also we will see that $\mathcal{H}^p = L^p$ for $p > 1$.

The following is a survey of results, omitting all proofs. We will return in detail later.

First we start with the following known result. Let $P_y$ be the Poisson kernel in the upper half plane.

**Theorem 47.** *If* $1 < p \leq \infty$, *and if* $\sup_{y>0} \|u(\,\cdot\,, y)\|_{L^p(\mathbb{R}^n)} < \infty$ *(i.e. each slice is bounded) and* $\Delta u = 0$ *in* $\mathbb{R}^{n+1}_+$ *(harmonic in the upper half plane), then*

$$u(x, y) = P_y * f$$

*for some* $f$*. For* $p = 1$*, we have the same result except we have a Radon measure* $\mu$ *in place of* $f$*.*

In general, we can consider the supremum over a cone-like region (simplifies discussion of limits among other issues). Let $\Gamma_{x_0} = \{(x, y)\colon \|x - x_0\| \leq y, y > 0\}$ (cone with origin $x_0$ in the upper half plane), and consider

$$u^*(x) = \sup_{(z,y) \in \Gamma_x} |u(z, y)|$$

Then we can show that $u^* \in L^p \Longleftrightarrow u = P_y * f$ for $f \in L^p$, $1 < p \leq \infty$. For $p = 1$, the same result holds with $f$ replaced by a Radon measure $\mu$.

**Possible First Definition of $\mathcal{H}^p(\mathbb{R}^n)$**

If $f \in L^p(\mathbb{R}^n)$, then $f \in \mathcal{H}^p(\mathbb{R}^n)$ iff $P_y * f \in L^p(\mathbb{R}^n)$. We can then consider this definition for when $0 < p \leq \infty$. By the above result, we have that $\mathcal{H}^p = L^p$ for $p > 1$ for this definition. Also, by the definition of $u^*$, $f \leq u^*$ (since the cone includes the $y = 0$ hyperplane), and thus $\mathcal{H}^p \subset L^p$ for all $p$. The interesting case is when $0 < p \leq 1$. For instance, for $f \in L^1$, it is not necessarily the case that $f \in \mathcal{H}^1$. This definition works nicely, but it is highly dependent on dimension and hard to characterize in the range of $p$ in $(0, 1)$.

**Theory of M. Riesz**

For $f$ defined on $\mathbb{R}$, define $F(z)$ to be the Cauchy integral of $f$,

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(\xi)}{\xi - z} \, d\xi$$

We are then interested in the value of $F$ towards the boundary. For what follows, there is an analogous theory for the Cauchy integral over the unit disk.

Then define $f \in \mathcal{H}^p$ if and only if

$$\sup_{y>0} \int_{\mathbb{R}} |F(x + iy)|^p \, dx < \infty$$

and this definition makes sense for $p > 0$. This says that $F$ restricted to lines $z = iy$ is in $L^p$ with uniformly bounded norm. Under this definition, it turns out that

$$\mathcal{H}^p(\mathbb{R}) = \{f \in L^p\colon Hf \in L^p\}$$

where $H$ is the Hilbert transform. In the smooth case, if we write $F(z) = u + iv$, fix $y_0 > 0$, and consider $f = u|_{y=y_0}$, then

$$v|_{y=y_0} = Hf$$

and so

$$F|_{y=y_0} = f + iHf$$

and $F|_{y=y_0} \in L^p$ if both $f$ and $Hf$ are in $L^p$. Then for $p > 1$, by the boundedness of the Hilbert transform in $L^p$, we have that $Hf \in L^p$. Furthermore, we have a uniform bound for all $y$ since for harmonic functions, the $p$-norm of each strip $z = x + iy$ is bounded by the $p$-norm at the boundary $y = 0$ (by the Fatou Theorems, which we prove later). Thus $\mathcal{H}^p = L^p$ for $p > 1$.

For higher dimensions, we use Riesz Transforms to define $\mathcal{H}^p(\mathbb{R}^n)$.

Given $f \in L^p(\mathbb{R}^n)$, the Poisson integral $u_0 = P_y * f$ gives a harmonic function in the upper half plane with boundary value $f$, i.e.

$$\begin{aligned} \Delta u_0 &= 0 \text{ on } \mathbb{R}^{n+1}_+ \\ u_0\big|_{y=0} &= f \end{aligned}$$

Note that $\widehat{P_y}(\xi) = e^{-2\pi|\xi|y}$. Then define $Q_j^y$ such that $\widehat{Q_j^y}(\xi) = i\frac{\xi_j}{|\xi|}e^{-2\pi|\xi|y}$. It turns out that inverting gives

$$Q_j^y(x) = y^{-n}\frac{c_n x_j/y_j}{(1 + |x/y|^2)^{(n+1)/2}}$$

Then as $y \to 0$, we have that $P_y * f \to f$ and $Q_y^j * f \to R_j f$. Then the study of the conjugate harmonic functions $R_j f$ leads to the second possible definition of $\mathcal{H}^p$:

**Second Possible Definition of $\mathcal{H}^p(\mathbb{R}^n)$**

$$\mathcal{H}^p(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) \colon R_j f \in L^p(\mathbb{R}^n), j = 1, \ldots, n\}$$

for $p > 1 - \frac{1}{n}$.

We will be focusing on $\mathcal{H}^1$ and its dual BMO. As an aside, it turns out that for $0 < p < 1$, the dual of $\mathcal{H}^p$ is $\mathrm{Lip}^{\alpha_p}$ for some $\alpha_p$.

The definitions above are all equivalent, and we have the following equivalent characterizations of $\mathcal{H}^1$.

1. $f \in \mathcal{H}^1$ if and only if $u^* \in L^1$, and $\|f\|_{\mathcal{H}^1} \approx \|u^*\|_{L^1}$
2. $f \in \mathcal{H}^1$ if and only if $M_\Phi f \in L^1$, and $\|f\|_{\mathcal{H}^1} \approx \|M_\Phi f\|_{L^1}$

   where $\Phi_t(x) = t^{-n}\Phi(x/t)$ and $\Phi$ is an approximate identity, and

   $$M_\Phi f(x) = \sup_{t>0} |\Phi_t * f|(x)$$

   (the Grand Maximal Function) which is not quite the Hardy Littlewood maximal function, since we use $f$ instead of $|f|$, so that it measures cancellations as well. This definition is motivated from noting that we should not have to restrict ourselves to the Poisson kernel, and it turns out that the resulting definition is equivalent.
3. $\mathcal{H}^1 = \{f \in L^1(\mathbb{R}^n) \colon R_j f \in L^1(\mathbb{R}^n), j = 1, \ldots, n\}$
4. $(\mathcal{H}^1)^* = \mathrm{BMO}$

5. $f \in \mathcal{H}^1$ if and only if $f \in L^1$ and $f = \sum_{k=1}^{\infty} \lambda_k a_k$ where $\lambda_k \in \mathbb{R}$ s.t. $\sum_{k=1}^{\infty} |\lambda_k| < \infty$ and the $a_k$ satisfy

   a. $\mathrm{supp}(a_k) \subset B$

   b. $|a_k|(x) \leq \frac{1}{|B|}$

   c. $\int_B a_k = 0$

   In this case $\|f\|_{\mathcal{H}^1} \approx \sum_{k=1}^{\infty} |\lambda_k|$. The $a_k$ are called the $\mathcal{H}^1$ atoms of $f$ and this decomposition is called the atomic decomposition of $f$.

## Week 9 (11/11/2009)

(Again) For a typical singular operator $T$, we see that $\|Tf\|_p \leq A_p \|f\|_p$ for $1 < p < \infty$, and also we have weak (1,1) estimate. For $p = 1$, we want $\|Tf\|_1 \leq A_1 \|f\|_1$, but this is not true in general. But perhaps we will have $\|Tf\|_{L^1} \leq A_1 \|f\|_{\mathcal{H}^1}$. In fact, maybe we can define $\mathcal{H}^1$ as the $f \in L^1$ such that $Tf \in L^1$ for "typical" singular operators $T$. Such definitions do not give much information about $\mathcal{H}^1$.

For example,

$$\begin{cases} -\Delta u = f \text{ in } \mathbb{R}^n \\ u|_{\infty} = 0 \end{cases}$$

If $f \in L^p$, then $u \in W^{2,p}$ for $1 < p < \infty$. If $f \in L^1$, then it is not true that $u \in W^{2,1}$. By Sobolev embedding, we note that $W^{2,1} \subset C^0$, but it is not hard to find an $f \in L^1$ where the corresponding $u$ is not continuous. In 1 dimension, for instance, take $u(x) = \log|x|$, then $u_{xx}(x) = -\frac{1}{x^2}$, which is integrable (too simple...). As for $p = \infty$, even if $f$ is continuous (uniformly even), $u$ may not be $C^2$. Then we can ask what space gives us $\|Tf\|_? \leq A_\infty \|f\|_{L^\infty}$. This will turn out to be the dual of $\mathcal{H}^1$, BMO.

Now we will work towards showing the equivalences of all the definitions.

### Fatou Theorems

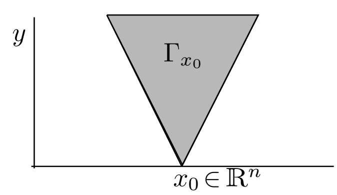
**Theorem 48.** *Suppose $f \in L^1(\mathbb{R}^n)$, $1 \leq p \leq \infty$, and let $u(x, y) = P_y * f(x)$. Let $\alpha > 0$ fixed. Then*

1. $\sup_{(x,y) \in \Gamma_\alpha(x_0)} |u(x, y)| \leq A_\alpha M_f(x_0)$

2. $\lim_{\substack{(x,y) \to (x,0) \\ (x,y) \in \Gamma_\alpha}} u(x, y) = f(x_0)$ *for a.e.* $x_0 \in \mathbb{R}^n$.

*where* $\Gamma_\alpha(x_0) = \{(x, y) \in \mathbb{R}^{n+1}_+ : |x - x_0| \leq \alpha y\}$.

Pictorally, $\Gamma_\alpha(x_0)$ looks like a cone:



**Proof. (of (a))** First we note that $P_y(x) = \frac{c_n y}{(|x|^2 + y^2)^{(n+1)/2}}$ and so $P_y(x-t) \leq A_\alpha P_y(x)$ for $|t| \leq \alpha y$, since

$$\begin{aligned}
P_y(x-t) &= \frac{c_n y}{(|x-t|^2 + y^2)^{(n+1)/2}} \\
&\leq \frac{c_n y}{(|x|^2 - 2|t||x| + |t|^2 + y^2)^{(n+1)/2}} \\
&= \frac{c_n y}{(|x|^2 + (y - |t|)^2)^{(n+1)/2}} \\
&\leq \frac{c_n y}{((1-\alpha)^2 |x|^2 + (1-\alpha)^2 y^2)^{(n+1)/2}} \\
&= A_\alpha P_y(x)
\end{aligned}$$

with $A_\alpha = \frac{1}{(1-\alpha)^{n+1}}$ when $|t| \leq \alpha y$. By definition we have

$$u(x_0 - t, y) = \int_{\mathbb{R}^n} P_y(x_0 - t - \xi) f(\xi) d\xi$$

and thus we have that

$$\begin{aligned}
\sup_{(x,y) \in \Gamma_\alpha(x_0)} |u(x,y)| &= \sup_{|t| \leq \alpha y} |u(x_0 - t, y)| \\
&\leq A_\alpha \sup_{y>0} \int_{\mathbb{R}^n} P_y(x_0 - \xi) |f(\xi)| d\xi
\end{aligned}$$

we want to show that this is less than $A_\alpha M_f(x_0)$. Without loss of generality, consider $x_0 = 0$. (Otherwise just translate to origin). We will do a dyadic decomposition:

$$\begin{aligned}
&= \sup_{y>0} \int_{\mathbb{R}^n} \frac{c_n y}{(|\xi|^2 + y^2)^{\frac{n+1}{2}}} |f(\xi)| d\xi \\
&\leq \sup_{y>0} \left( \int_{|\xi| \leq y} + \int_{y \leq |\xi| \leq 2y} + \ldots \right) \frac{c_n y}{(|\xi|^2 + y^2)^{\frac{n+1}{2}}} |f(\xi)| d\xi \\
&\leq \sup_{y>0} \frac{C_n}{y^n} \int_{|\xi| \leq y} |f(\xi)| d\xi + \sum_{k=0}^{\infty} \frac{C_n 2^{n-k}}{(2^{k+1} y)^n} \int_{2^k y \leq |\xi| \leq 2^{k+1} y} |f(\xi)| d\xi \\
&\leq C_n' M_f(0) + C_n'' \sum_{k=0}^{\infty} \frac{1}{2^k} M_f(0) \\
&\leq C M_f(0)
\end{aligned}$$

The other way to show this estimate is to convert to polar and integrate by parts, as in the proof of Cotlar's Lemma 46. So if $f \in L^p$ for $1 < p \leq \infty$, we have that $u^*(x) \in L^p$. $\square$

**Proof. (of (b))** Consider $u(x_0 - t, y) - f(x_0)$ for $|t| \leq \alpha y$, which is

$$\int_{\mathbb{R}^n} P_y(x_0 - t - \xi)(f(\xi) - f(x_0)) d\xi$$

since $\int P_y = 1$. Then for a fixed $y$, and using the same observation as in the proof of (a), that $P_y(x-t) \leq A_\alpha P_y(x)$ for $|t| \leq \alpha y$, we have

$$\begin{aligned}
\sup_{|t| \leq \alpha y} |u(x_0 - t, y) - f(x_0)| &\leq A_\alpha \int_{\mathbb{R}^n} P_y(x_0 - \xi) |f(\xi) - f(x_0)| d\xi \\
&\leq A_\alpha \left( \int_{|\xi| \leq \delta} + \int_{|\xi| \geq \delta} \right) P_y(\xi) |f(x_0 - \xi) - f(x_0)|
\end{aligned}$$

In the second term, the integrand tends to 0 uniformly, and the first term tends to 0 for continuous $f$. Approximating gives the result in general (This is just the usual approximation of identity estimate), which gives the result. $\square$

Combining the two statements, we have that $|f(x)| \leq u^*(x)$ a.e. $x$ since $u(x, y) \to f(x)$ as $y \to 0$ for a.e. $x$.

**Theorem 49. (Fatou)** *Suppose $u$ is given on $\mathbb{R}^{n+1}_+$. Then $u$ is the Poisson integral of some $f \in L^\infty(\mathbb{R}^n)$ if and only if $u$ is a bounded harmonic function on $\mathbb{R}^{n+1}_+$*

**Proof.** We already proved the forward direction in the previous theorem. Conversely, suppose that $u$ is a bounded harmonic function on $\mathbb{R}^{n+1}_+$. Let $f_k(x) = u\left(x, \frac{1}{k}\right) \in L^\infty(\mathbb{R}^n) \cap C^\infty$ (smoothness follows from $u$ being harmonic). Then let $u_k(x, y) = P_y * f_k$, which is harmonic in $\mathbb{R}^{n+1}_+$, with $u_k\Big|_{y=0} = f_k$. Study $u(x, y + 1/k), u_k(x, y)$, which are both bounded harmonic and agree at $y = 0$. This implies by the maximum principle that $u(x, y + 1/k) = u_k(x, y)$.

(Aside: To handle the unbounded domain, can use an extension and Liouville type argument, or use a scaling argument, pulling points near infinity back towards the origin and establish a contradiction)

Now fix $y > 0$. Then as $k \to \infty$ we have that $u_k(x, y) = u(x, y + 1/k) \to u(x, y)$ pointwise. On the other hand, we have that since $f_k$ is uniformly bounded, we can take a subsequence if needed so that $f_k \to f$ weakly (in weak* topology). In this case then $P_y * f \to P_y * f$, by weak* convergence $\langle f_k, P_y(x - \xi)\rangle \to \langle f, P_y(x - \xi)\rangle$. $\square$

By the previous theorem, we also have that

$$u \,\text{bdd harmonic} \Longleftrightarrow u \,\text{is Poisson integral} \Longrightarrow u \,\text{has nontangential limit as}\, y \to 0 \,a.e.$$

This generalizes to $L^p$ as well:

**Theorem 50. (Generalization of Fatou)** *Let $u$ be harmonic in $\mathbb{R}^{n+1}_+$, $1 \leq p \leq \infty$. If $\sup_{y>0 \|u(\cdot, y)\|_{L^p} < \infty}$ then $u(x, y) = P_y * f$ for some $f \in L^p$ for $p > 1$, and $u(x, y) = P_y * \mu$ for $p = 1$, where $\mu$ is a Radon measure.*

**Proof.** This is accomplished in two steps:

1. First we show $|u(x, y)| \leq C y^{-n/p}$ for $y > 0$. Taking a ball of radius $y/2$ around $(x, y)$, we have that

$$\int_{B_{y/2}(x, y)} |u|^p \leq C y$$

where the bound follows from integrating along strips where $y$ is constant. This implies that

$$\frac{1}{|B_{y/2}(x, y)|} \int_{B_{y/2}(x, y)} |u|^p \leq C y^{-n}$$

noting that $|B_{y/2}(x, y)| = c_{n+1} y^{n+1}$. Now using the mean value property for harmonic functions, we have that

$$|u(x, y)| \leq \left(\frac{1}{|B_{y/2}(x, y)|} \int_{B_{y/2}(x, y)} |u|^p\right)^{1/p} \leq C y^{-n/p}$$

2. Now we use the same technique as before, and look at the same $f_k$, $u_k$ defined above. Part 1 shows that $u(x, y + 1/k)$ and $P_y * f_k$ are bounded for $y > 0$, and thus $u(x, y + 1/k) = P_y * f_k$, since both are bounded harmonic and have the same boundary values. Now we have that $f_k \to f$ weakly (weak* topology) in $L^p$ (uniform boundedness follows from the assumption), which gives the result, where the limit is an $L^p$ function if $p > 1$ and a Radon measure if $p = 1$ (the closure of $L^1$ in weak* topology is the space of Radon measures).

□

Summarizing:

- For $1 < p \leq \infty$, $u^* \in L^p(\mathbb{R}^n)$. if $f \in L^p(\mathbb{R}^n)$. Thus if we define $\mathcal{H}^p(\mathbb{R}^n) = \{f \in L^p : u^* \in L^p\}$, then $\mathcal{H}^p = L^p$.

- For $p = 1$, $\mathcal{H}^1(\mathbb{R}^n) = \{f \in L^1 : u^* \in L^1\} \neq L^1$. Recall that $M_f \notin L^1$ for some $f \in L^1$. If $M_f \in L^1$, then $f \in \mathcal{H}^1(\mathbb{R}^n)$ by Theorem 48. In particular, we saw in the first week that if $f \geq 0$, then $f \log^+ f \in L^1 \Longleftrightarrow f \in \mathcal{H}^1$. In particular, $L^p$ $(p > 1)$ functions satisfy this property.

Now we turn towards the Grand Maximal function $M_\Phi f$, where $\Phi_t(x) = t^{-n}\Phi(x/t)$ and $\Phi$ is an approximate identity, and

$$M_\Phi f(x) = \sup_{t>0} |\Phi_t * f|(x)$$

Recall that if $f$ is a distribution, then for $\Phi \in \mathcal{S}$, $f * \Phi \in C^\infty$. The same holds for $f * \Phi_t$. Note that the Poisson kernel is not in $\mathcal{S}$, but we still want to make sense of $P_t * f$ for $f$ in some class of distributions.

**Definition 51.** *We say that $f$ is a **bounded distribution** if $f * \Phi \in L^\infty$ for all $\Phi \in \mathcal{S}$.*

There are easy examples for which $f * \Phi$ is not bounded at $\infty$ for some $\Phi$, for instance polynomials will work (a polynomial convolved with a bump will still be unbounded near $\infty$).

**Fact:** If $f$ is a bounded distribution and $h \in L^1(\mathbb{R}^n)$, then $f * h$ is also a bounded distribution. This is straightforward to check: For all $\Phi \in \mathcal{S}$,

$$\langle f * h, \Phi \rangle = \left\langle f * \tilde{\Phi}, \tilde{h} \right\rangle$$

the $\tilde{}$ denotes the reflection $\tilde{f}(x) = f(-x)$. Since $f * \tilde{\Phi} \in L^\infty$ and $\tilde{h} \in L^1$, thus $f * h$ is well defined. Also, for any $\Psi \in \mathcal{S}$, we have that

$$f * h * \Psi = f * (h * \Psi) \in L^\infty$$

since $h * \Psi \in \mathcal{S}$.

Now if $f$ is a bounded distribution, then $P_t * f$ is well defined as a bounded distribution.

# Week 10 (11/18/2009)

**Theorem 52.** *Let $f$ be a bounded distribution, $0 < p \leq \infty$. Then the following are equivalent:*

1. *There exists $\Phi \in \mathcal{S}$ such that $\int_{\mathbb{R}^n} \Phi \, dx = 1$ and $M_\Phi f \in L^p(\mathbb{R}^n)$.*

2. *For any* $\Phi \in \mathcal{S}$ *such that* $\int_{\mathbb{R}^n} \Phi \, dx = 1$*, we have that* $M_\Phi f \in L^p(\mathbb{R}^n)$*.*

3. $u^* \in L^p(\mathbb{R}^n)$*, where* $u^*(x) = \sup_{(z,y)\in\Gamma(x_0)} |P_y * f(x)|$*.*

**Proof. (Idea)** This is a very technical proof, so we sketch the necessary tools here.

- First, to show (1) ⇒ (2), we use a Littlewood-Paley decomposition. For any two functions $\Phi, \Psi \in \mathcal{S}$, we can find a decomposition of the form

$$\Psi = \sum_{k=1}^{\infty} \eta^{(k)} * \Phi_{2^{-k}}(x)$$

  more easily described in the Fourier domain:

$$\hat{\Psi} = \sum_{k=1}^{\infty} \hat{\eta}^{(k)} \hat{\Phi}$$

  where $\hat{\eta}^{(k)}$ is a smooth cutoff function which is 1 at dyadic intervals $[2^k, 2^{k+1}]$. We can also choose $\eta^{(k)}$ so that $\|\eta^{(k)}\|_{m,\alpha} \leq M(m,\alpha)$ where $\|\cdot\|_{m,\alpha}$ are the family of seminorms that come from the Frechet space $\mathcal{S}$.

- To show (2) ⇒ (3), the idea is to decompose the Poisson kernel in terms of functions in $\mathcal{S}$. We can write

$$P_1(x) = \sum_{k=0}^{\infty} 2^{-k} \Phi_{2^k}^{(k)}(x)$$

  taking $P_y$ for $y = 1$, and $\Phi^{(k)} \in \mathcal{S}$ and satisfies $\|\Phi^{(k)}\|_{m,\alpha} \leq M(m,\alpha)$ where $\|\cdot\|_{m,\alpha}$ are the family of seminorms that come from the Frechet space $\mathcal{S}$.

- To show (3) ⇒ (1), we can get a Schwarz function in terms of $P_y$ by use of a smooth function with rapid decay. First we construct $\eta \in C^\infty(1,\infty)$ such that $\int_1^\infty \eta(s)\,ds = 1$ and $\int_1^\infty s^k \eta(s)\,ds = 0$ for $k = 1, 2, \ldots$ Given such a function, we then examine

$$\Phi(x) = \int_1^\infty \eta(s) P_s(x)\,ds$$

  and then take the Taylor expansion (to show smoothness). The $\eta$ that satisfies the desired properties exists and is of the form

$$\eta(s) = \frac{1}{\pi s} \operatorname{Im} \exp(1 - \alpha(1-s)^{-1/4}),\ \alpha = e^{-i\pi/4}$$

□

This gives the equivalence of the Hardy space definition using the Grand maximal function. Now we turn to $\mathcal{H}^p$ atomic decomposition.

(Fefferman's paper) $\mathcal{H}^p$ atoms $0 < p \leq 1$. We say $a$ is an $\mathcal{H}^p$ **atom** if the following three conditions hold:

1. $a(x)$ is supported in a ball $B$

2. $|a|(x) \leq \frac{1}{|B|^{1/p}}$

3. (Moment condition) $\int_B x^\beta a(x)\,dx = 0$ for all $\beta$ such that $|\beta| \leq n(p^{-1} - 1)$

Note that under these conditions, we have that $\int_{\mathbb{R}^n} |M_\Phi a|^p\,dx \leq C_n$:

**Proof.** Using the second condition,

$$M_\Phi a(x) = \sup_{t>0} |\Phi_t * a(x)| \leq \frac{C}{|B|^{1/p}}$$

Now for $x \in 2B$, we have an easy estimate:

$$\int_{2B} |M_\Phi\, a(x)|^p\,dx \leq C_n$$

For $x \notin 2B$, we have that

$$\begin{aligned} a * \Phi_t(x) &= \int_B a(y)\,\Phi_t(x-y)\,dy \\ &= \int_B a(y)[\Phi_t(x-y) - q_t(x-y)]\,dy \end{aligned}$$

where $q_t$ is the Taylor polynomial of $\Phi_t$ with center 0 and degree $d = \lfloor n(p^{-1} - 1) \rfloor$, using the third condition to insert into the integral. Now we estimate

$$|\Phi_t(x-y) - q_t(x-y)| \leq C_d \frac{|y-x|^{d+1}}{t^{d+1+n}}$$

noting that $\Phi_t(\xi) = t^{-n}\Phi(\xi/t)$. If $y \in B$ and $x \notin 2B$, this becomes like a singular operator estimate:

$$\begin{aligned} |a * \Phi_t(x)| &\leq \int_B |a(y)|\,|\Phi_t(x-y) - q_t(x-y)|\,dy \\ &\leq \int_B \frac{C'}{|B|^{1/p}} \frac{|y-x|^{d+1}}{t^{d+1+n}} \\ &\leq \frac{C'}{|B|^{1/p} t^{d+1+n}} \int_0^R r^{d+n}\,dr \\ &\leq \frac{C'}{|B|^{1/p}} \left(\frac{R}{t}\right)^{d+1+n} \end{aligned}$$

Above $R$ is the radius of $B$. Let $y_0$ be the center of $B$ for what follows. Now we choose a particular $\Phi$ to be supported in $B_1$, and with this choice we note that for $t < |x - y_0|$, $\Phi_t(x - y_0) = q_t(x - y_0) = 0$, since $\frac{x - y_0}{t} > 1$. Thus, we consider the case $t \geq |x - y_0|$, so that

$$M_\Phi a(x) \leq \frac{C'}{|B|^{1/p}} \left(\frac{R}{|x - y_0|}\right)^{d+1+n}$$

Now combining the two estimates, we have that

$$\begin{aligned} \int_{\mathbb{R}^n} |M_\Phi\, a|^p &\leq \int_{2B} |M_\Phi a|^p + \int_{(2B)^c} |M_\Phi a|^p \\ &\leq C_n + \frac{C''}{|B|} R^{d+1+n+p} \int_{2R}^\infty \frac{r^{n-1}}{r^{d+1+n+p}} dr \\ &\leq C_n'' \end{aligned}$$

□

This shows that if $a$ is an $\mathcal{H}^p$-atom, then $a\in\mathcal{H}^p$ using the definition with the Grand maximal function. As a direct Corollary, we have that if $f(x)=\sum_{k=1}^{\infty}\lambda_k a_k$ where $a_k$ are $\mathcal{H}^p$-atoms and $\sum_{k=1}^{\infty}|\lambda_k|<\infty$. Then $f\in\mathcal{H}^p$. This is because using $\Phi$ as above, we have

$$M_\Phi f(x)\leq\sum_{k=1}^{\infty}|\lambda_k|M_\Phi a_k(x)$$

so that

$$\int_{\mathbb{R}^n}|M_\Phi f(x)|^p\,dx\leq C'\sum_{k=1}^{\infty}|\lambda_k|C_n<\infty$$

using Jensen's to show that $\left(\sum_{k=1}^{\infty}\frac{|\lambda_k|}{C}M_\Phi a_k\right)^p\leq\sum_{k=1}^{\infty}\frac{|\lambda_k|}{C}|M_\Phi f(x)|^p$.

Thus we have shown that functions of the form $\sum_{k=1}^{\infty}\lambda_k a_k$ are in $\mathcal{H}^p$. We have a result that shows that every function in $\mathcal{H}^p$ has such a decomposition. Here we focus on the case $p=1$.

**Theorem 53.** *$\mathcal{H}^1$ decomposition. For any $f\in\mathcal{H}^1$, we can find $a_k$ $\mathcal{H}^1$-atoms and $\lambda_k$ for which $\sum_{k=1}^{\infty}|\lambda_k|\leq C_0\|M_\Phi f\|_{L^1}$ so that*

$$f(x)=\sum_{k=1}^{\infty}\lambda_k\,a_k(x)$$

*Furthermore, we can define $\|f\|_{\mathcal{H}^1}$ to be $\sum_{k=1}^{\infty}|\lambda_k|$. Then $\|M_\Phi f\|_{L^1}$ is comparable to $\|f\|_{\mathcal{H}^1}$.*

The proof is long and technical, involving the Calderón-Zygmund type decomposition on a dyadic choice of $\alpha_k$ (in fact, we examine the sets $\{M_\Phi f>\alpha\}$ and $\{M_\Phi f\leq\alpha\}$), and patching everything together.

**Remark 54.** The third moment property for $\mathcal{H}^p$ atoms is necessary. For $p=1$, this condition is just $\int a(x)\,dx=0$. If $a=\mathbf{1}_B$ for instance, we can show that $\Phi_t a(x)\approx\frac{C}{|x|^n}$, which is not integrable. The cancellation gives us an additional $|x|^{-1}$.

This gives a rough sketch of all the equivalent definitions of $\mathcal{H}^1$. Now we turn back to estimates on singular operators in the case $p=1$, filling in the gap.

**Theorem 55.** *If $T$ is a singular integral operator of Calderon-Zygmund type, then $T$ is bounded from $\mathcal{H}^1\to L^1$.*

**Remark 56.** Another quick remark for why the cancellation condition is good. Earlier we discussed that in order for $Tf\in L^1$ it must be the case that $\hat{Tf}$ is continuous. Since $\hat{T}$ is homogeneous of degree 0, we have a problem near the origin unless $\hat{f}(0)=\int f=0$. This gives a necessary condition for when $Tf\in L^1$.

We now have all the equivalent definitions for the Hardy space, and we will make use of the atomic decomposition.

**Proof.** For all $f\in\mathcal{H}^1$ we can write $f=\sum_{k=1}^{\infty}\lambda_k a_k$ as in the atomic decomposition. It then suffices to check this estimate for each atom $a_k$, and the result follows by subadditivity. Thus we show that $\|Ta\|_{L^1}$ for an $\mathcal{H}^1$ atom $a$. Recall $a$ is supported in some ball $B$, bounded by $1/|B|$, and $\int_B a=0$. As before we estimate $|Ta|$ on $2B$ and $(2B)^c$:

On $2B$ we can use just the $L^2$ bound:

$$\int_{2B} |Ta|^2\,dx \leq A^2\|a\|_{L^2}^2 \leq A^2\int_B |a|^2 = A^2\frac{|B|}{|B|^2} = \frac{A^2}{|B|}$$

Then

$$\int_{2B} |Ta|\,dx \leq |2B|^{1/2}\|Ta\|_{L^2}^{1/2} = |2B|^{1/2}\frac{A}{|B|^{1/2}} = C_0 A$$

Now for $(2B)^c$, we use the usual singular operator trick:

$$\begin{aligned} Ta(x) &= \int_B K(x-y)a(y)\,dy \\ &= \int_B [K(x-y) - K(x-y_0)]a(y)\,dy \end{aligned}$$

Then

$$\begin{aligned} \int_{(2B)^c} |Ta(x)|\,dx &\leq \int_B |a(y)|\,dy \int_{(2B)^c} |K(x-y) - K(x-y_0)|\,dx \\ &\leq C_0 A \int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,dx \\ &\leq C_0 A' \end{aligned}$$

Combining the two estimates gives the result. □

# Week 11 (11/25/2009)

## The Space BMO

First we define the space BMO.

**Definition:** A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in BMO($\mathbb{R}^n$) if

$$\sup_Q \frac{1}{|Q|}\int_Q |f - f_Q|\,dx \leq A < \infty$$

Here $Q$ are cubes in $\mathbb{R}^n$, and $f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx$. The optimal upper bound is taken to be the norm $\|f\|_{\text{BMO}}$. If desired we can also use balls instead of cubes. Note that constants have 0 norm, and thus BMO functions are defined up to an additive constant (BMO functions that differ by an additive constant are equivalent). This means that we may replace $f_Q$ by arbitrary constants $c_Q$ in the definition.

**Remark 57.** We have the following facts about BMO:

1. $L^\infty \subset \text{BMO}$, $\|f\|_{\text{BMO}} \leq 2\|f\|_\infty$

2. $W^{1,n}(\mathbb{R}^n) \subset \text{BMO}$, by Poincaré inequality.

3. $\log|x| \in \text{BMO}$, $\log|x| \notin L^\infty$:

$$\frac{1}{R^n}\int_{B_R(0)} |\ln x - \ln R|\,dx = \int_{B_1(0)} |\ln x|\,dx$$

Now we return to the Calderón-Zygmund singular operators to fill in the gap concerning $L^{\infty}$:

**Theorem 58.** *If $T$ is a Calderón-Zygmund type operator, then $T$ maps $L^{\infty}$ to* BMO.

**Proof.** Let $f \in L^{\infty}(\mathbb{R}^n)$, and let $g = Tf = K * f$ with $K = \frac{\Omega(y)}{|y|^n}$. We want to show that $g$ is in BMO. Note that the BMO norm is scaling and translation invariant by definition, so if $g_{a,\lambda} := g(a + \lambda x)$, then

$$\|g_{a,\lambda}\|_{\mathrm{BMO}} = \|g\|_{\mathrm{BMO}}$$

The same is true for $L^{\infty}$. Therefore it suffices to check

$$\int_Q |g - g_Q|\, dx \leq C_A \|f\|_{\infty}$$

for the unit cube $Q$ centered at the origin (note that $|Q| = 1$). Now write

$$f = f\,\mathbf{1}_{2Q} + f\,\mathbf{1}_{(2Q)^c} = f_1 + f_2$$

Then $g = g_1 + g_2$, with $g_i = K * f_i$, and

$$\|g_1\|_{L^1(Q)} \leq \|g_1\|_{L^2(Q)} \leq \|\hat{K}\|_{\infty} \|f_1\|_{L^2} \leq C_n \|f\|_{\infty}$$

with $C_n = \|\hat{K}\|_{\infty} 2^{n/2}$ (Hölder).

Thus $\int_Q |g_1 - g_{1,Q}| \leq C_n \|f\|_{\infty}$. As for $g_2$, we compute

$$\int_Q |g_2 - C_Q|\, dx$$

where $C_Q = \int_{\mathbb{R}^n} K(-y)\, f_2(y)\, dy = g_2(0)$ (recall that we can replace $g_{2,Q}$ with any constant $C_Q$). Note $g_2(x) = \int_{\mathbb{R}^n} K(x - y)\, f_2(y)\, dy$

Now

$$\begin{aligned}
\int_Q |g_2(x) - C_Q|\, dx &\leq \int_Q \int_{\mathbb{R}^n} |K(x - y) - K(-y)|\, |f_2(y)|\, dy\, dx \\
&\leq \|f\|_{\infty} \int_Q \int_{(2Q)^c} |K(x - y) - K(-y)|\, dy\, dx \\
&\leq B \|f\|_{\infty}
\end{aligned}$$

recalling that $\int_{(2Q)^c} |K(x - y) - K(-y)|\, dy \leq \int_{|y| \geq |2x|} |K(x - y) - K(y)|\, dy \leq B$, since $K$ is a Calderón-Zygmund type kernel.

Thus $g \in \mathrm{BMO}$. $\square$

In particular, the Riesz Transforms map $L^{\infty}$ to BMO.

Now note that $\langle R_i f,\, g\rangle = \langle f,\, R_i^* g\rangle = -\langle f,\, R_i g\rangle$ (can check that $R_i^* = -R_i$ on a common domain). Then if $f \in \mathcal{H}^1$, then $R_i f \in L^1$ and suppose that $g \in L^{\infty}$. Then $R_i^* g \in \mathrm{BMO}$ by what we just proved. In particular, $R_i^* g$ defines a linear functional on $\mathcal{H}^1$.

We want to show that $\mathcal{H}^1 = \{f \in L^1 \colon R_i f \in L^1, i = 1, \ldots, n\}$. By our previous results, we know that $\mathcal{H}^1 \subset \{f \in L^1 \colon R_i f \in L^1, i = 1, \ldots, n\}$. We want to show that it is enough to check just the Riesz transforms. (What we know is that if $M_\Phi f \in L^1$ for some $\Phi \in \mathcal{S}$, then $f \in \mathcal{H}^1$).

We also know that $\mathrm{BMO} \supset \{R_i g \colon i = 1, \ldots, n, g \in L^\infty\} = \bigcup_{i=1}^n R_i(L^\infty)$.

**Remark 59.** Establishing that $(\mathcal{H}^1)^* = \mathrm{BMO}$ implies equality of both definitions. ??

**Theorem 60.** $(\mathcal{H}^1)^* \subset \mathrm{BMO}$

**Proof.** Let $f \in \mathcal{H}^1$. Consider $G(f) = (f, R_1 f, \ldots, R_n f) \in (L^1)^{n+1}$. Note that $g(\mathcal{H}^1)$ is a closed subspace. Now let $l \in (\mathcal{H}^1)^*$. Then define $\tilde{l}(G(f)) = l(f)$, a functional on $g(\mathcal{H}^1)$. By Hahn-Banach, we can extend to $(L^1)^{n+1}$, so that $\tilde{l} \in ((L^1)^{n+1})^* = (L^\infty)^{n+1}$. So there exists $g_0, \ldots, g_n \in L^\infty$ such that

$$\tilde{l}(f_0, \ldots, f_n) = \int \sum_{i=0}^n f_i g_i$$

(reminder, make use Riesz Representation on $\tilde{l} \circ \pi_i$ where $\pi_i$ is projection to $i$-th component) Thus

$$l(f) = \int \left( f g_0 + \sum_{i=1}^n R_i f g_i \right) = \int_{\mathbb{R}^n} f \left( g_0 - \sum_{i=1}^n R_i g_i \right)$$

from integration by parts. Then $g_0 - \sum_{i=1}^n R_i g_i \in \mathrm{BMO}$. This shows that $(\mathcal{H}^1)^* \subset \mathrm{BMO}$. □

It remains to show that BMO functions define a continuous linear functional on $\mathcal{H}^1$. Towards this result we have the following Lemma:

**Lemma 61.** *If $f \in \mathrm{BMO}(\mathbb{R}^n)$ then $\frac{f(x)}{(1+|x|)^{n+1}} \in L^1(\mathbb{R}^n)$.*

**Proof.** We want to show that

$$\int_{\mathbb{R}^n} \frac{|f - f_{B_1}|}{(1+|x|)^{n+1}} dx \leq C_n \|f\|_{\mathrm{BMO}}$$

which implies the result. Now note that

$$\frac{1}{2^{nk}} \int_{B_{2^k}} |f - f_{B_{2^k}}| \leq C_n \|f\|_{\mathrm{BMO}}$$

by definition of BMO. This shows that

$$\begin{aligned} |f_{B_{2^{k+1}}} - f_{B_{2^k}}| &\leq \frac{c_n}{2^{nk}} \int_{B_{2^k}} |f - f_{B_{2^{k+1}}}| \\ &\leq \frac{c_n}{2^{nk}} \int_{B_{2^{k+1}}} |f - f_{B_{2^{k+1}}}| \\ &\leq C_n \|f\|_{\mathrm{BMO}} \end{aligned}$$

Now we have that $|f_{B_{2^k}} - f_{B_1}| \leq c_n k \|f\|_{\mathrm{BMO}}$ by triangle inequality, for $k = 1, 2, \ldots$ and so

$$\int_{B_{2^k}} |f - f_{B_1}| dx \leq C_n k 2^{kn} \|f\|_{\mathrm{BMO}}$$

so that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|f - f_{B_1}|}{(1+|x|)^{n+1}}\,dx &\leq \sum_k \int_{B_{2^{k+1}}-B_{2^k}} \frac{|f - f_{B_1}|}{(1+|x|)^{n+1}}\,dx\\ &\leq \sum_k \frac{C_n\,k\,2^{kn}\|f\|_{\mathrm{BMO}}}{(2^k)^{n+1}}\,dx\\ &\leq C_n'\,\|f\|_{\mathrm{BMO}}\end{aligned}$$

□

(Notes say that this result implies that $P_y * f \in L^1$ also...??)

**Remark 62.** Note that $\ln|P(x)| \in \mathrm{BMO}$ and $\ln|x| \in \mathrm{BMO}$. Then $f = \ln u \in \mathrm{BMO}$ where $u$ satisfies $u \geq 0$, $\sum_{ij} \partial_{x_i}(a^{ij}(x)\,u_{x_j}) = Lu = 0 \in B$ (Moser estimate)

**Proof.** We will show that for $f \in \mathrm{BMO}$, $e^{\beta|f|} \in L^1$. Combining these gives

$$\int_B e^{-\beta \ln u}\,dx \int_B e^{\beta \ln u}\,dx \leq C$$

Thus $Lu = 0$, $u \geq 0$ in $B_1$ implies

$$\int_{B_{1/2}} u^{-\beta}\,dx \int_{B_{1/2}} u^{\beta}\,dx \leq C_0$$

(Moser's inequality), (essentially Harnack?) ?? □

Note $f \in \mathrm{BMO}$ implies $e^{\beta|f|} \in L^1$. This shows that $f \log f \in L^1$ for $f \geq 0$ implies that $f \in \mathcal{H}^1$, so we should expect such an estimate. ??

(What's missing... Stein's Harmonic Analysis, the large one has most of these)

- **John-Nirenberg inequality**
- Reverse (?) Holder inequality (F. Gehrig), if $\left(\frac{1}{|Q|}\int_Q |f|^p\right)^{1/p} \leq m\frac{1}{|Q|}\int_Q |f|$, $p > 1$, then $f \in L^{p+\delta}$, $\delta > 0$, depending on $p, n$. Useful for Q.C. (?) Mapping
- General rule: If you have translation/dilation, then use C-Z.
- Bourgain theorem on harmonic measure...
- Finishing Duality, $\mathrm{BMO} \subset (\mathcal{H}^1)^*$, using atomic decomposition.

# Week 12 (12/2/2009)

### Summary

If $\mathcal{H}^1 = \overline{\{\sum_{k=1}^{\infty} \lambda_k f_k,\, f_k \in L^2_{B_k}\}}^{L^1\text{closure}}$, where $\sum |\lambda_k|\,|B_k|^{1/2}\,\|f_k\|_{L^2} < \infty$, then

1. $f \in \mathcal{H}^1 \Longrightarrow M_\Phi f \in L^1 \Longrightarrow u^* \in L^1$
2. $f \in \mathcal{H}^1 \Longrightarrow R_j f \in L^1, j = 1, , ..., n,$ and $Tf \in L^1$, $T$ of C-Z type.
3. $f \in L^\infty \Longrightarrow R_j f \in \mathrm{BMO}, j = 1, ..., n$ and $Tf \in \mathrm{BMO}$ (John Nirenberg, $L\log L \leftrightarrow \exp(\,\cdot\,)$)

4. $l\in(\mathcal{H}^1)^* \Longrightarrow \exists g\in \text{BMO}\ s.t.\ l(f)=\langle f,\, g\rangle$ for all $f\in\mathcal{H}^1$ (Prove on $L^2(B)$ first with Riesz representation and patch up by continuity)

5. If $g\in\text{BMO}$, then $g\in(\mathcal{H}^1)^*\Longrightarrow(\mathcal{H}^1)^*=\text{BMO}$ (Here we noted that

$$\langle R_i f, h\rangle = -\langle f, R_i h\rangle$$

with $f\in\mathcal{H}^1, h\in L^\infty$, and the closed graph theorem shows $\bigcup_{i=0}^n R_i L^\infty = \text{BMO}$.

We are only missing the full proof of the $\mathcal{H}^1$-atomic decomposition, which gives the reverse implication in (1).

## Square Functions (Lusin Area-Integrals)

Now we continue investigating properties of $H^1$ and BMO.

**Lemma 63.** *If $g\in\text{BMO}$, then $u(x,t)=P_t * g(x)$ satisfies*

$$\sup_x \int_{\substack{|y-x|<h\\ 0<t<h}} t\,|\nabla u|^2(y,t)\,dy\,dt \le C_n\,\|g\|_{\text{BMO}}^2 h^n$$

*for all $h>0$.*

The integration is over the truncated cone of height $h$, with origin $x$ and slope 1. Denoting $K(t,x)=P_t(x)$ (easier notation), recall that $K(t,x)=\frac{c_n\,t}{(t^2+|x|^2)^{(n+1)/2}}$, and if we dilate the integral $y$, $x,t$ by $h$, we may assume that $h=1$:

$$u(hy',ht')=h^n\int K(ht',hy'-hz')\,g(hz')\,dz'=\int K(t',y'-z')\,g(hz')\,dz'$$

$$\nabla u(hy',ht')=h^n\int\ \nabla K(ht',hy'-hz')\,g(hz')\,dz'=\frac{1}{h}\int\ \nabla K(t',y'-z')g(hz')\,dz'$$

where we note that $\delta_h K=\frac{1}{h^n}K$ and $\delta_h\nabla K=\frac{1}{h^{n+1}}\nabla K$ using direct computation:

$K(ht,hx)=\frac{1}{h^n}K(t,x)$, $\partial_{x_i}K(t,x)=\frac{c_n'\,t\,(2x_i)}{(t^2+|x|^2)^{(n+1)/2+1}}$, and

$$\begin{aligned}\partial_t K(t,x) &= \frac{c_n(t^2+|x|^2)^{(n+1)/2}-c_n'\,t\,(t^2+|x|^2)^{(n+1)/2-1}(2t)}{(t^2+|x|^2)^{n+1}}\\ &= \frac{c_n-c_n'\,t^2(t^2+|x|^2)^{-1}}{(t^2+|x|^2)^{\frac{n+1}{2}}}\end{aligned}$$

so that $(\partial_t K)(ht,hx)=\frac{1}{h^{n+1}}\partial_t K(t,x)$. Thus $(\nabla K)(ht,hx)=\frac{1}{h^{n+1}}(\nabla K)(t,x)$,

(in notes: "counting dimensions (dilation factor), each $y_i$ gives 1, $t$ gives 1, $\frac{\partial}{\partial y_i}$ gives $-1$, $\frac{\partial}{\partial y}$ gives $-1$, $u\to 0$ and $\nabla u\to -1$, so $1-2+(n+1)=n$")

Therefore, we have that

$$\int_{\substack{|y-x|<h\\ 0<t<h}} t\,|\nabla u|^2(y,t)\,dy\,dt = h^n\int_{\substack{|y-x|<1\\ 0<t<1}} t\,|\nabla K*\delta_h g|^2(y,t)\,dy\,dt$$

and since $\|\delta_h g\|_{\text{BMO}}=\|g\|_{\text{BMO}}$, we have reduced the problem to the case where $h=1$.

**Proof.** From discussion above, we may assume $h=1$. Now decompose $g=g_1+g_2$, $g_1=g\mathbf{1}_{2Q}$. We may assume without loss of generality that $g_{1,2Q}=\frac{1}{|2Q|}\int_{2Q} g_1\,dx=0$, since the area integral and the BMO norm are both invariant up to constants. Now let $u=u_1+u_2$ where $u_i=P_t * g_i$. Have

$$\int_{\substack{|y|<1\\0<t<1}} t\,|\nabla u|^2(y,t)\,dy dt$$

Note that $u_2$ is the easy part, since it solves the boundary problem on $2Q$ with boundary value 0. This gives (PDE estimate)

$$|\nabla u_2|_{L^\infty(Q_1)}\leq C\sup_{Q_{3/2}}|u_2(y,t)|\leq C\int_{(2Q)^c}\frac{|g(x)|}{(1+|x|^2)^{\frac{n+1}{2}}}\,dx\leq C\,\|g\|_{\text{BMO}}$$

Then

$$\int_{\substack{|y|<1\\0<t<1}} t\,|\nabla u_2|^2(y,t)\,dy dt\leq C\,\|g\|^2_{\text{BMO}}$$

As for $u_1$, we have

$$\int_{\substack{|y|<1\\0<t<1}} t\,|\nabla u_1|^2(y,t)\,dy dt\ \leq\ \int_0^\infty dt\int_{\mathbb{R}^n} t\,|\nabla u_1(y,t)|^2\,dy$$

(In notes: Since $g_1$ is supported in a ball, $\nabla u_1$ decays fast ??)

Using Plancherel, we have

$$\begin{aligned}
&= \int_0^\infty\int_{\mathbb{R}^n} t\,|\xi|^2\,e^{-2t|\xi|}|\hat{g}_1(\xi)|^2\,d\xi dt\\
&= \int_{\mathbb{R}^n}|\hat{g}_1(\xi)|^2\frac{|\xi|}{2}\int_0^\infty 2t|\xi|e^{-2t|\xi|}dt d\xi\\
&= c\int_{\mathbb{R}^n}|\hat{g}_1(\xi)|^2\,d\xi\\
&= c\,\|g_1\|^2_{L^2(2Q)}
\end{aligned}$$

noting $\int_0^\infty 2t\,|\xi|e^{-2t|\xi|}=\frac{1}{2|\xi|}$ (expectation of exponential distribution). This apparently proves the desired result (not sure how to bound this last term in terms of $\|g_1\|^2_{\text{BMO}}$ though...) $\square$

**Remark 64.** Let $G$ be the Green's function on $B_R$ with pole at $q=(\mathbf{0},3/2)$. Then

$$\iint_{\tilde{Q}_1} t\,|\nabla u_1(y,t)|^2\,dt dy\leq c_0\iint_{B_R} G(y,t)\,|\nabla u_1(y,t)|^2\,dy dt$$

(not sure why...)

Using Green's identity, we then have

$$\begin{aligned}
c_0\iint_{B_R} G\,|\nabla u_1|^2+u_1^2(0) &= \iint_{B_R} G\,\Delta(u_1^2)-u_1^2\Delta G\\
&= \int_{\partial B_R} G\,\frac{\partial u_1^2}{\partial\nu}-\int_{\partial B_R}\frac{\partial G}{\partial\nu}\,u_1^2
\end{aligned}$$

note that $-\Delta G=\delta$ and $\Delta(u_1^2)=2u_1\Delta u_1+|\nabla u_1|^2=|\nabla u_1|^2$ ($u_1$ is harmonic). Thus

$$\begin{aligned}
\iint_{\tilde{Q}_1} t\,|\nabla u_1(y,t)|^2\,dt dy &\leq C_0\|g_1\|^2_{L^2(2Q_1)}-u_1^2(0)+C\int_{\partial B_R} u_1^2\\
&\leq C\,\|g_1\|^2_{L^2(2Q_1)}
\end{aligned}$$

much like the $L^2$ boundary $\to H^{1/2}$ estimate in PDE.

**Lemma 65.** *Any function $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ where the Poisson integral $P_t * g$ satisfies Lusin's Area integral estimate*

$$\iint_{\tilde{Q}_h} t\,|\nabla(P_t * g)|^2\,dydt \le B^2 h^n$$

*is an element in $(\mathcal{H}^1)^* = \mathrm{BMO}$.*

**Proof. (Sketch)** Let $U(y,t) = P_t * g(y)$ and $V(y,t) = P_t * f(y)$ and consider

$$\begin{aligned}
2\int_0^\infty \int \langle t\nabla u, \nabla v\rangle\,d\xi dt &= 4\int_0^\infty \int t\,|\xi|^2 e^{-2t|\xi|}\hat{f}(\xi)\hat{g}(\xi)\,d\xi dt \\
&= \left\langle \hat{f}(\xi), \hat{g}(\xi)\right\rangle \\
&= \langle f(x), g(x)\rangle
\end{aligned}$$

Take $f \in \mathcal{H}^1$. Using a decay estimate for $t\nabla v$, we show that $\int\int \langle t\nabla u, \nabla v\rangle < \infty$, from which it follows by the above computation that $\int f^* g < \infty$, and since this holds for all $f \in \mathcal{H}^1$, we have that $g \in \mathrm{BMO}$. The assumption on $g$ allows us to use the maximal function.

Again, we will show that $l(f) = 2\int_0^\infty \int \langle t\nabla u, \nabla v\rangle\,d\xi dt$ for $f \in \mathcal{H}^1$ is well defined and bounded, then $l$ is a linear functional implies that $g \in \mathrm{BMO}$ by the above computation. The proof makes use of the characterization of $\mathcal{H}^1$ as the $L^1$ closure of $\{\sum \lambda_k g_k, g_k \in L^2(B_k)\}$. $\square$

Scattered notes... to decipher later...

- Let $\Phi \in C_0^\infty(B_1)$ with $\int \Phi = 0$, and define

$$\begin{aligned}
S_\Phi f(x) &:= \left(\int_0^\infty |f * \Phi_t|^2 \frac{dt}{t}\right)^{1/2} \\
\Sigma_\Phi f(x) &:= \left(\iint_{\Gamma_x} |f * \Phi_t|^2 \frac{dtdy}{t^{n+1}}\right)^{1/2}
\end{aligned}$$

  with $\Gamma_x = \{(y,t) : |y - x| < t\}$. By a tedious computation involving Fubini, we have

$$\|\Sigma_\Phi f\|_{L^2}^2 = c_0 \|S_\Phi f\|_{L^2}^2 \le A\,\|f\|_{L^2}^2$$

  where the last inequality is a computation involving Fourier transform:

$$\begin{aligned}
\|S_\Phi f\|_{L^2}^2 &= \int_{\mathbb{R}^n} \int_0^\infty |\hat{f}(\xi)|^2\,|\hat{\Phi}(t,\xi)|^2 \frac{dt}{t}\,d\xi \\
&\le \left(\sup_\xi \int_0^\infty |\hat{\Phi}(t,\xi)|^2\,\frac{dt}{t}\right)\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi \\
&\le C\,\|f\|_{L^2}^2
\end{aligned}$$

  Note that $\int \Phi = 0$ holds if and only if $|\hat{\Phi}(u)| \le C\,|u|$ and for $u$ large, $|\hat{\Phi}(u)| \le \frac{C}{|u|}$.

- (Hardy Sobolev estimate, side remark)

$$\begin{aligned}
\int_\Omega \frac{u^2}{d^2(x,\partial\Omega)} &\le C_\Omega \int_\Omega |\nabla u|^2 \\
\int_\Omega \frac{u^2}{d(x,\partial\Omega)} &\le C_d \int_\Omega d(x,\partial\Omega)\,|\nabla u|^2
\end{aligned}$$

## Carleson measures

Carleson addresses the following question: Which positive measures $\mu$ on $\mathbb{R}^{n+1}$ have the property that

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^2(x)\, d\mu(x,t) < C_\mu \|f\|_{L^2}^2$$

If $f = \mathbf{1}_Q$, then $P_t * f(x) \geq c_0$ for all $(x,t) \in \tilde{Q}_{1/2}$, where $\tilde{Q}_{1/2} = \{|x_i| \leq 1, i = 1, \ldots, n+1, x_{n+1} \geq 0\}$ (cube in the upper halfspace). This implies that

$$\begin{aligned} |Q_{1/2}| \simeq C_\mu \|f\|_{L^2}^2 \geq \int_{\mathbb{R}_+^{n+1}} |P_t * f|^2 d\mu(x,t) &\geq C_0^2 \int_{\tilde{Q}_{1/2}} d\mu(x,t) \\ &= C_0^2 \mu(\tilde{Q}_{1/2}) \end{aligned}$$

Thus, $\mu(\tilde{Q}) \leq C_0|Q|$, called the Carleson condition. We then define a Carleson measure to be a measure on $\mathbb{R}^{n+1}$ such that $\mu(\tilde{Q}) \leq C_0|Q|$. The computation above shows that any positive measure satisfying the property

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^2(x)\, d\mu(x,t) < C_\mu \|f\|_{L^2}^2$$

is a Carleson measure.

It turns out that the converse is also true:

**Lemma 66. (Carleson)** *If $\mu$ is a Carleson measure on $\mathbb{R}_+^{n+1}$, then*

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^p(x)\, d\mu(x,t) < C_\mu \|f\|_{L^p}^p$$

*for $1 < p \leq \infty$. Also,*

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^2(x)\, d\mu(x,t) \leq C \int_{\mathbb{R}^n} |Mf|^p\, d\mu^*$$

*for $1 \leq p \leq \infty$. ($\mu^*$ is probably the restriction of $\mu$ on $\mathbb{R}^n$, or something...)*

**Observation.** Let $\varphi, \psi \in \mathcal{S}$, with $\int \varphi = 1$, $\int \psi = 0$. Define

$$\begin{aligned} (\widehat{P_t f})(\xi) &= \hat{\varphi}(t\xi)\, \hat{f}(\xi) \\ (\widehat{Q_t f})(\xi) &= \hat{\psi}(t\xi)\, \hat{f}(\xi) \end{aligned}$$

(similar to square functions). If $f \in L^2(\mathbb{R}^n)$, then

$$\int_0^\infty \|Q_t f\|_{L^2(\xi)}^2 \frac{dt}{t} \leq C_\psi \|f\|_{L^2}^2$$

Since $\|Q_t f\|_{L^2}^2 = \|\widehat{Q_t f}\|_{L^2}^2 = \|\hat{\psi}(t\xi)\, \hat{f}(\xi)\|_{L^2}^2$, $C_\psi = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{dt}{t}$.

**Theorem 67.** *If $f \in \mathrm{BMO}$, then*

$$d\mu(x,t) := |Q_t f|^2 \frac{dx\, dt}{t}$$

*is a Carleson measure.*

**Proof.**

1. Show that $Q_t f$ is well defined for $x \in \mathbb{R}^n$:

   - $Q_t 1 = 0$ (can take out constant for BMO)

   - $\int_{\mathbb{R}^n} \frac{|f - f_{Q_1}|}{(1+|x|)^{n+1}} dx < \infty$ since $f \in \text{BMO}$ (thus product with Schwarz function also $< \infty$)

   These two imply that $Q_t f$ is well defined. (in notes: take off average, gain $\frac{1}{t}$...)

2. It suffices to verify the Carleson condition for $\tilde{Q} = \tilde{Q}_1$, noting that BMO is invariant under translation and dilation, and the same is true for $\frac{dt}{t}$. Furthermore, we may assume without loss of generality that $f_{2Q_1} = 0$ since $Q_t 1 = 0$ (doesn't see constants)

Finally, check that

$$\iint_{\tilde{Q}_1} |Q_t f|^2(x) \frac{dx\,dt}{t} \leq c_\psi \|f\|_{\text{BMO}} |Q|$$

Let $f = f_1 + f_2$ with $f_1 = f\,\mathbf{1}_{2Q_1}$. Then $Q_t f = Q_t f_1 + Q_t f_2$. As before, $Q_t f_2$ is not a problem. Otherwise,

$$\begin{aligned}
\iint_{\widetilde{Q_1}} |Q_t f_1|^2 \frac{dx\,dt}{t} &\leq \iint_{\mathbb{R}_+^{n+1}} |Q_t f_1|^2 \frac{dx\,dt}{t} \\
&= \left(\sup_\xi \int_0^\infty \frac{|\hat{\psi}(t\xi)|^2}{t} dt\right) \int |\hat{f}_1(\xi)|^2 \\
&= C_\psi \|f\|_{L^2}^2
\end{aligned}$$

the same estimate as before. Now for $(x, t) \in \tilde{Q}_1$, and letting $K$ be the kernel for the operator $Q_t$,

$$\begin{aligned}
|Q_t f_2(x)| &\leq \int_{(2Q_1)^c} \frac{1}{t^n} \left|\frac{K(x-z)}{t}\right| |f(z)| dz \\
&\leq C_\psi \int_{\mathbb{R}^n} \frac{t\,|f(z)|\,dz}{(t^2 + (x-\xi)^2)^{\frac{n+1}{2}}} \\
&\leq C_\psi\, t\, \|f\|_{\text{BMO}}
\end{aligned}$$

so that

$$\iint_{\widetilde{Q_1}} |Q_t f_2|^2(x) \frac{dx\,dt}{t} \leq C_\psi \|f\|_{\text{BMO}}^2 |Q|$$

□

This theorem shows that the space BMO can be embedded into the space of Carleson measures.

A similar theorem for singular operators $T$ of C-Z type. $T$ bounded in $L^2$ $\leftrightarrow T1 \in \text{BMO}$ (? Levy Theorem ?)

## Week 13 (12/9/2009)

We return to the Carleson lemma.

**Theorem 68.** *Suppose $d\mu(x,t)$ is a Carleson measure on $(x,t)\in\mathbb{R}_+^{n+1}$. Then*

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^2\, d\mu(x,t) \le C_\mu \|f\|_{L^2}^2$$

**Ideas.** (BMO $\longleftrightarrow$ Square function $\longleftrightarrow$ Carleson measures)

Suppose $Q_t$ is an operator such that $\widehat{Q_t f}(\xi) = \hat{\psi}(t\xi)\hat{f}(\xi)$, where $\psi \in \mathcal{S}$ or has sufficient decay ($\hat{\psi}(0) = 0$ works).

- If $g \in$ BMO, then $\frac{|Q_t g|^2}{t} dx\,dt$ is a Carleson measure (proved last time)
- If $g \in$ BMO, then

  $$\int_{\tilde{Q}_h} t\,|\nabla Q_t g|^2(x,t)\,dx\,dt \le B^2 h^n$$

  This is the square function (which in 2D is the area of the image $u(Q_h)$)
- Also, if $g \in L^1$, and

  $$\int_{\tilde{Q}_h} t\,|\nabla Q_t g|^2 dx\,dt \le B^2 h^n$$

  then $g \in$ BMO. (converse type result)

**Proof. (Sketch)** Consider

$$\int\!\!\int \langle t\nabla P_t * g, \nabla P_t * f\rangle\, d\xi dt$$

with $f \in \mathcal{H}^1$. Taking the Fourier transform ("polarization or whatever") we get

$$\int_0^\infty \int t e^{-2t|\xi|}|\xi|^2 \hat{g}(\xi)\hat{f}(\xi)\,dt\,d\xi = \left\langle \hat{g}, \hat{f} \right\rangle = \langle g, f\rangle$$

where we integrate in $t$ first.

Need $\langle t\nabla P_t * g, \nabla P_t * f\rangle$ to make sense (i.e. each part should be in $L^2$).

If we consider $\widehat{Q_t g} = t\xi e^{-t|\xi|}\hat{g}$, then we can use the previous lemmas to show that $t\nabla P_t * g \in L^2$ also. We will show the second term is also $L^2$ in what follows. $\square$

Consider $F(x,t) \in L^1(\mathbb{R}^n + \mathbb{R}_+)$, and $F^*(x) = \sup_{(y,t)\in\Gamma_x} |F(y,t)|$ where $\Gamma_x$ is the usual cone for the nontangential maximal function. Note that if $F(x,t) = P_t * f$, then we already have the estimate $F^*(x) \le CMf(x)$.

Define the space

$$\mathcal{N} = \left\{ F \in L^1(\mathbb{R}_+^{n+1}) : F^* \in L^1 \right\}$$

noting that if $F = P_t * f$, then we have the Hardy space, i.e. $\mathcal{H}^1 \subset \mathcal{N}$. Define $\|F\|_{\mathcal{N}} := \|F^*\|_{L^1}$.

Given a ball $B_r(x_0) = B$, we examine tents $T(B) = \{(x,t) : |x - x_0| \le r - t\}$. Having defined $T(B)$ for balls, we can extend the definition to open sets $O \subset \mathbb{R}^n$ as well:

$$T(O) = \bigcup_{x\in O} T(B_{d_x}(x))$$

with $d_x = d(x, O^c)$. Let $\mu$ be a Radon measure, and define

$$C(d\mu)(x) = \sup_{x \in B} \frac{1}{|B|} \int_{T(B)} d\mu(x,t)$$

Note that if $\mu$ is a Carleson measure, then $C(d\mu)(x) \le C$, uniform in $x$. Then define

$$\mathcal{C} = \{d\mu > 0, \text{measures on } \mathbb{R}_+^{n+1} \text{ such that } C(d\mu(x)) \in L^\infty(\mathbb{R}^n)\}$$

This will behave like the dual to the $\mathcal{N}$ space.

**Theorem 69.** *For all $F \in \mathcal{N}$ and $\mu \in \mathcal{C}$, we have that*

$$\iint_{\mathbb{R}_+^{n+1}} F(x,t)\, d\mu(x,t) \le \int_{\mathbb{R}^n} F^*(x)\, C(d\mu(x))\, dx \le C_\mu \int_{\mathbb{R}^n} F^*(x)\, dx$$

**Corollary 70.** *If $F(x,t) = |P_t * f(x)|^2$, then $F^*(x) \le C(Mf(x))^2$,* and since *$f \in L^2$ implies that $Mf \in L^2$, we have that $F(x,t) \in \mathcal{N}$, and this implies the theorem concerning*

$$\int_{\mathbb{R}_+^{n+1}} |P_t * f|^2 d\mu \le C_\mu \|f\|_{L^2(\mathbb{R}^n)}^2$$

**Proof.** If $F \ge 0$, and $\mu \ge 0$, let $O = \{x \in \mathbb{R}^n : F^*(x) > \alpha\}$, which is an open set (maximal function properties). Then

$$\int_{\mathbb{R}_+^{n+1}} F(x,t)\, d\mu(x,t) = \int_0^\infty \mu\{(x,t) : F(x,t) > \alpha\}\, d\alpha$$

and

$$\int F^*(x)\, dx = \int_0^\infty |\{x : F^*(x) > \alpha\}|\, d\alpha$$

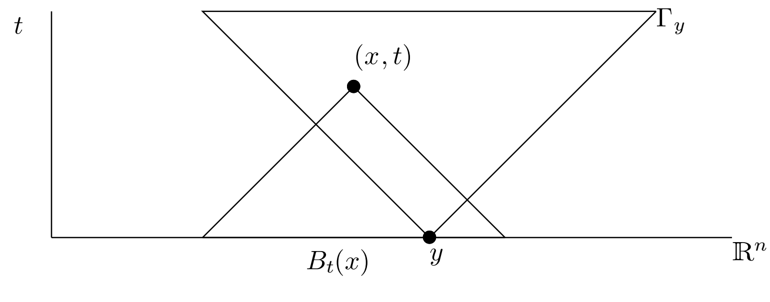
We have two observations:

- $\{(x,t) : F(x,t) > \alpha\} \subset T(O)$
- If $C(d\mu(x)) \le 1$ , i.e. $\mu(T(B)) \le |B|$ for all $B \subset \mathbb{R}^n$, then $\mu(T(O)) \le C|O|$

Given these two observations, we are done, since

$$\mu\{(x,t) : F(x,t) > \alpha\} \le \mu(T(O)) \le C\,|O| = C|\{x : F^*(x) > \alpha\}|$$

For the first observation, we note that if $y \in B_t(x)$, then $F^*(y) > \alpha$, since $(x,t) \in \Gamma_y$:




Thus $\{(x,t) : F(x,t) > \alpha\} \subset T(O)$ by taking union of all balls.

For the second observation, the 1 dimensional case is easy, since we can write any open set as a disjoint union of intervals $O = \cup_k I_k$, and thus $\mu(T(O)) = \sum_k \mu(T(I_k)) \leq \sum |I_k| = |O|$.

In multiple dimensions, we write an open set as a union of cubes with disjoint interior, $O = \cup_k Q_k$, where $\mathrm{diam}(Q_k) \approx d(Q_k, O^c)$ (Whitney's decomposition). Let $B_k \supset Q_k$, where $\mathrm{diam}\, B_k = \sqrt{n}\, \mathrm{diam}(Q_k)$. Then $T(O) \subset \cup_k T(B_k)$ so that

$$\begin{aligned}\mu(T(O)) &\leq \sum_k \mu(T(B_k)) \\ &\leq \sum_k |B_k| \\ &\leq C \sum_k |Q_k| \\ &= C\,|O|\end{aligned}$$

(i.e. even though the union is not disjoint, we can set it up so that there is bounded overlap, and we therefore get the same result). This proves both observations, from which the theorem follows. $\square$

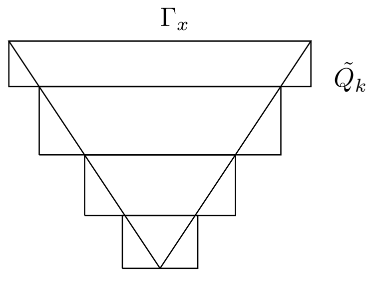
Now we return to $\frac{|Q_t f|^2}{t}$ with $\widehat{Q_t f} = \hat{\psi}(t\xi)\, \hat{f}(\xi)$. We are interested in

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|Q_t f|^2(x)}{t} dx\, dt = C_0 \int_0^\infty \int_{\mathbb{R}^n} \int_{\Gamma_x} \frac{|Q_t f|^2(x-y)}{t^{n+1}} dy\, dx\, dt$$

where the equality above holds from a computation involving Fubini. Then

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^n} \langle t\nabla P_t * g(x), \nabla P_t * f(x)\rangle\, dx\, dt \\ \leq\ & \int_0^\infty \int_{\mathbb{R}^n} \int_{\Gamma_x} \frac{|t\nabla P_t * g(x-y)|\,|\nabla P_t * f(x-y)|}{t^n} dy\, dx\, \frac{dt}{t} \\ =\ & \int_0^\infty \int_{\mathbb{R}^n} \sum_k \int_{\tilde{Q}_k} \frac{|t\nabla P_t * g(x-y)|\,|\nabla P_t * f(x-y)|}{t^n} dy\, dx\, \frac{dt}{t} \\ =\ & \int_0^\infty \int_{\mathbb{R}^n} \sum_k \left(\frac{1}{h_k^n} \int_{\tilde{Q}_k} t\,|\nabla P_t * g|^2\right)^{1/2} \left(\frac{1}{h_k^n} \int_{\tilde{Q}_k} |\nabla P_t * f|^2\right)^{1/2} dy\, dx\, \frac{dt}{t}\end{aligned}$$

The third line is a decomposition of the physical space, decomposing the cone $\Gamma_x$ into cubes $\tilde{Q}_k$ of length $h_k$:




The fourth line above applies Cauchy Schwarz. The first term involving $P_t * g$ is bounded since by assumption the integral is bounded by $B^2 h_k^n$. The second can be estimated by a harmonic function on a slightly larger domain (introducing bounded overlap). We have the standard PDE estimate

$$\left(\int_{B_{1/2}} |\nabla u|^2\right)^{1/2} \leq c\left(\int_{B_1} |u|\right)$$

This means that we can further bound the above by

$$\begin{aligned}
&\leq \int_0^\infty \int_{\mathbb{R}^n} C \int_{\Gamma_x} |F(t, x-y)| dy dx \frac{dt}{t} \\
&\leq C \int_{\mathbb{R}^n} F^*(x) dx
\end{aligned}$$

where $F(t,x) = P_t * f$. This finshes the proof of Theorem 68.

We conclude with the atomic decomposition for $\mathcal{N}$. Note that we did not prove the atomic decomposition for $\mathcal{H}^1$. It turns out the decomposition for $\mathcal{N}$ is a lot simpler, and using this we can complete the equivalences of $\mathcal{H}^1$ definitions... (recall that the idea for $\mathcal{H}^1$ atomic decomposition requires Calderon-Zygmund decomposition on a dyadic family of $\alpha_k$, with further adjustments...)

Let $B \subset \mathbb{R}^n$. An atom associated with $B$ is a measurable function $a$ supported in $T(B) \subset \mathbb{R}^{n+1}_+$ such that $\|a\|_{L^\infty} \leq \frac{1}{|B|}$. Note $a^*(x) \leq \frac{1}{|B|}$ when $x \in B$ and $a^*(x) = 0$ for $x \notin B$. This implies that $a \in \mathcal{N}$ and $\|a\|_{\mathcal{N}} = \|a^*\|_{L^1} \leq 1$.

**Observation.** If $a_k \in \mathcal{N}$ are atoms and $\lambda_k \geq 0$ such that $\sum \lambda_k < \infty$, then $\sum_k \lambda_k a_k \in \mathcal{N}$, $\big(\sum_k \lambda_k a_k\big)^* \leq \sum |\lambda_k| a_k^* \in L^1$ and $\|\sum_k \lambda_k a_k\|_{\mathcal{N}} \leq \sum_k \lambda_k$.

**Theorem 71.** *If $F \in \mathcal{N}$ then $F$ can be written as $F = \sum_k \lambda_k a_k$ where $a_k$ are atoms and $\lambda_k \geq 0$, $\sum_k \lambda_k \leq C\|F\|_{\mathcal{N}}$.*

**Proof.** Let $O_j = \{x: F^*(x) > 2^j\}$, $j \in \mathbb{Z}$. Note that $O_j \supset O_{j+1}$ so that $T(O_j) \supset T(O_{j+1})$, and $\cup_{j \in \mathbb{Z}} T(O_j) \supset \operatorname{supp}(F)$. Apply Whitney's decomposition to each $O_j = \cup_k Q_j^k$, where $\operatorname{diam}(Q_j^k) \approx d(Q_j^k, O_j^c)$. Let $B_j^k$ be balls containing $Q_j^k$ (while introducing bounded overlap. Then

$$T(O_j) \subset \bigcup_k T(B_j^k) \cap (Q_j^k \times [0, \infty))$$

Set $\Delta_j^k = T(B_j^k) \cap Q_j^k \times [0, \infty) \cap T(O_j) - T(O_{j+1})$. Then $\operatorname{supp} F \subset \cup_{j,k} \Delta_j^k$ (on $\Delta_j^k$ $F$ has values between $2^j$ and $2^{j+1}$) which are mutually disjoint. Now let $F_{j,k}(x,\, t) = F(x,\, t)\mathbf{1}_{\Delta_j^k}$. Then we have that $F = \sum_{j,k} F_{j,k}(x,\, t)$. Write $F_{j,k} = \lambda_{j,k} a_{j,k}$ where $\lambda_{j,k} = 2^{j+1}|B_j^k|$ and $a_{j,k} = 2^{-j-1}|B_j^k|^{-1} F_{j,k}$. Note that $|F_{j,k}| \leq 2^{j+1}$ and $\operatorname{supp} a_{j,k} \subset T(B_j^k)$, and $|a_{j,k}|(x) \leq \frac{1}{|B_j^k|}$. Then it remains to verify that

$$\sum \lambda_{j,k} = \sum_{j,k} 2^{j+1}|B_j^k| \approx C_0 \sum_{j,k} 2^{j+1}|Q_j^k|$$

(bounded overlap). Then for $k = 1, 2, \ldots, |Q_j^k|$ is a Whitney decomposition of $O_j = \{x: F^*(x) > 2^j\}$. This implies that summing over $j$ gives

$$= c_0 \sum_k 2^{j+1}|O_j| = C_0 \int F^*(x) = C_0\|F\|_{\mathcal{N}}$$

where we may need to use $O_j = \{2^{j-1} < F^* < 2^j\}$ instead to make the middle equality work out...

$\square$