A

Appendix

This appendix consists of nine parts. In the first eight, Sections A.1–A.8, we present results from linear algebra used throughout this book. In the last part, Section A.9, we address some numerical considerations. In all cases, our aim is to present a concise summary of pertinent results and not a full development of the subject on hand. For a more extensive exposition of the materials presented herein, refer to Antsaklis and Michel [1, Section 2.2] and to the other sources cited at the end of this appendix.

A.1 Vector Spaces

In defining vector space, we require the notion of a field.

A.1.1 Fields

Definition A.1. *Let F be a set containing more than one element, and let there be two operations "+" and "·" defined on F (i.e., "+" and "·" are mappings of $F \times F$ into F), called* addition *and* multiplication*, respectively. Then for each $\alpha, \beta \in F$ there is a unique element $\alpha + \beta \in F$, called the* sum *of α and β, and a unique element $\alpha\beta \triangleq \alpha \cdot \beta \in F$, called the* product *of α and β.* We say that $\{F; +, \cdot\}$ is a *field* provided that the following axioms are satisfied:

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ *and* $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$ *for all* $\alpha, \beta, \gamma \in F$ *(i.e., "+" and "·" are associative operations);*
(ii) $\alpha + \beta = \beta + \alpha$ *and* $\alpha \cdot \beta = \beta \cdot \alpha$ *for all* $\alpha, \beta \in F$ *(i.e., "+" and "·" are commutative operations);*
(iii) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ *for all all* $\alpha, \beta, \gamma \in F$ *(i.e., "·" is distributive over "+");*
(iv) There exists an element $0_F \in F$ *such that* $0_F + \alpha = \alpha$ *for all* $\alpha \in F$ *(i.e.,* 0_F *is the identity element of* F *with respect to "+");*

(v) There exists an element $1_F \in F, 1_F \neq 0_F$, such that $1_F \cdot \alpha = \alpha$ for all $\alpha \in F$ (i.e., 1_F is the identity element of F with respect to "$\cdot$");
(vi) For every $\alpha \in F$, there exists an element $-\alpha \in F$ such that $\alpha + (-\alpha) = 0_F$ (i.e., $-\alpha$ is the additive inverse of F);
(vii) For any $\alpha \neq 0_F$, there exists an $\alpha^{-1} \in F$ such that $\alpha \cdot (\alpha^{-1}) = 1_F$ (i.e., α^{-1} is the multiplicative inverse of F). ■

In the sequel, we will usually speak of a field F rather than of "a field $\{F; +, \cdot\}$."

Perhaps the most widely known fields are the *field of real numbers R* and the *field of complex numbers C*. Another field that we will encounter (see Example A.11) is the *field of rational functions* (i.e., rational fractions over polynomials).

As a third example, we let $F = \{0, 1\}$ and we define on F (binary) addition as $0 + 0 = 0 = 1 + 1$, $1 + 0 = 1 = 0 + 1$ and (binary) multiplication as $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$, $1 \cdot 1 = 1$. It is easily verified that $\{F; +, \cdot\}$ is a field.

As a fourth example, let P denote the set of polynomials with real coefficients and define addition "+" and multiplication "$\cdot$" on P in the usual manner. Then $\{F; +, \cdot\}$ is *not* a field since, e.g., axiom (vii) in Definition A.1 is violated (i.e., the *multiplicative* inverse of a polynomial $p \in P$ is not necessarily a polynomial).

A.1.2 Vector Spaces

Definition A.2. *Let V be a nonempty set, F a field, "+" a mapping of $V \times V$ into V, and "$\cdot$" a mapping of $F \times V$ into V. Let the members $x \in V$ be called* vectors*, let the elements $\alpha \in F$ be called* scalars*, let the operation "+" defined on V be called* vector addition*, and let the mapping "$\cdot$" be called* scalar multiplication *or* multiplication of vectors by scalars*. Then for each $x, y \in V$, there is a unique element, $x + y \in V$, called the* sum of x and y*, and for each $x \in V$ and $\alpha \in F$, there is a unique element, $\alpha x \triangleq \alpha \cdot x \in V$, called the* multiple of x by α. We say that the nonempty set V and the field F, along with the two mappings of vector addition and scalar multiplication, constitute a *vector space* or a *linear space* if the following axioms are satisfied:

(i) $x + y = y + x$ for every $x, y \in V$;
(ii) $x + (y + z) = (x + y) + z$ for every $x, y, z \in V$;
(iii) There is a unique vector in V, called the zero vector *or the* null vector *or the* origin*, that is denoted by 0_V and has the property that $0_V + x = x$ for all $x \in V$;*
(iv) $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in F$ and for all $x, y \in V$;
(v) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in F$ and for all $x \in V$;
(vi) $(\alpha\beta)x = \alpha(\beta x)$ for all $\alpha, \beta \in F$ and for all $x \in V$;
(vii) $0_F x = 0_V$ for all $x \in V$;
(viii) $1_F x = x$ for all $x \in V$. ■

When the meaning is clear from context, we will write 0 in place of 0_F, 1 in place of 1_F, and 0 in place of 0_V. To indicate the relationship between the set of vectors V and the underlying field F, we sometimes refer to a *vector space V over the field F*, and we signify this by writing (V, F). However, usually, when the field in question is clear from context, we simply speak of a vector space V. If F is the field of real numbers R, we call the space a *real vector space*. Similarly, if F is the field of complex numbers C, we speak of a *complex vector space*.

Examples of Vector Spaces

Example A.3. Let $V = F^n$ denote the set of all ordered n-tuples of elements from a field F. Thus, if $x \in F^n$, then $x = (x_1, \dots, x_n)^T$, where $x_i \in F$, $i = 1, \dots, n$. With $x, y \in F^n$ and $\alpha \in F$, let vector addition and scalar multiplication be defined as

$$\begin{aligned} x + y &= (x_1, \dots, x_n)^T + (y_1, \dots, y_n)^T \\ &\triangleq (x_1 + y_1, \dots, x_n + y_n)^T \end{aligned} \tag{A.1}$$

and

$$\alpha x = \alpha(x_1, \dots, x_n)^T \triangleq (\alpha x_1, \dots, \alpha x_n)^T. \tag{A.2}$$

In this case the null vector is defined as $0 = (0, \dots, 0)^T$ and the vector $-x$ is defined as $-x = -(x_1, \dots, x_n)^T = (-x_1, \dots, -x_n)^T$. Utilizing the properties of the field F, all axioms of Definition A.2 are readily verified, and therefore, F^n is a vector space. We call this space the *space F^n of n-tuples of elements of F*. If in particular we let $F = R$, we have R^n, *the n-dimensional real coordinate space*. Similarly, if we let $F = C$, we have C^n, *the n-dimensional complex coordinate space*.

We note that the set of points in R^2, (x_1, x_2), that satisfy the linear equation

$$x_1 + x_2 + c = 0, \quad c \neq 0,$$

with addition and multiplication defined as in (A.1) and (A.2), is *not* a vector space.

Example A.4. Let $V = R^\infty$ denote the set of all infinite sequences of real numbers,

$$x = \{x_1, x_2, \dots, x_k, \dots\} \triangleq \{x_i\},$$

let vector addition be defined similarly as in (A.1), and let scalar multiplication be defined similarly as in (A.2). It is again an easy matter to show that this space is a vector space.

On some occasions we will find it convenient to modify $V = R^\infty$ to consist of the set of all real infinite sequences $\{x_i\}$, $i \in Z$.

Example A.5. Let $1 \le p \le \infty$, and define $V = l_p$ by

$$l_p = \{x \in R^\infty : \sum_{i=1}^{\infty} |x_i|^p < \infty\}, \quad 1 \le p < \infty,$$
$$l_\infty = \{x \in R^\infty : \sup_i\{|x_i|\} < \infty\}. \tag{A.3}$$

Define vector addition and scalar multiplication on l_p as in (A.1) and (A.2), respectively. It can be verified that this space, called the l_p-*space*, is a vector space.

In proving that l_p, $1 \le p \le \infty$, is indeed a vector space, in establishing some properties of norms defined on the l_p-spaces, in defining linear transformations on l_p-spaces, and in many other applications, we make use of the *Hölder and Minkowski Inequalities for infinite sums*, given below. (These inequalities are of course also valid for *finite sums*.) For proofs of these results, refer, e.g., to Michel and Herget [9, pp. 268–270].

Hölder's Inequality states that if $p, q \in R$ are such that $1 < p < \infty$ and $1/p + 1/q = 1$, if $\{x_i\}$ and $\{y_i\}$ are sequences in either R or C, and if $\sum_{i=1}^{\infty} |x_i|^p < \infty$ and $\sum_{i=1}^{\infty} |y_i|^q < \infty$, then

$$\sum_{i=1}^{\infty} |x_i y_i| \le (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} (\sum_{i=1}^{\infty} |y_i|^q)^{1/q}. \tag{A.4}$$

Minkowski's Inequality states that if $p \in R$, where $1 \le p < \infty$, if $\{x_i\}$ and $\{y_i\}$ are sequences in either R or C, and if $\sum_{i=1}^{\infty} |x_i|^p < \infty$ and $\sum_{i=1}^{\infty} |y_i|^p < \infty$, then

$$(\sum_{i=1}^{\infty} |x_i \pm y_i|^p)^{1/p} \le (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} + (\sum_{i=1}^{\infty} |y_i|^p)^{1/p}. \tag{A.5}$$

If in particular $p = q = 2$, then (A.4) reduces to *the Schwarz Inequality for sums*.

Example A.6. Let $V = C([a, b], R)$. We note that $x = y$ if and only if $x(t) = y(t)$ for all $t \in [a, b]$, and that the null vector is the function that is zero for all $t \in [a, b]$. Let F denote the field of real numbers, let $\alpha \in F$, and let vector addition and scalar multiplication be defined pointwise by

$$(x + y)(t) = x(t) + y(t) \quad \text{for all } t \in [a, b] \tag{A.6}$$

and

$$(\alpha x)(t) = \alpha x(t) \quad \text{for all } t \in [a, b]. \tag{A.7}$$

Then clearly $x+y \in V$ whenever $x, y \in V, \alpha x \in V$, whenever $\alpha \in F$ and $x \in V$, and all the axioms of a vector space are satisfied. We call this space the *space of real-valued continuous functions on* $[a,b]$, and we frequently denote it simply by $C[a,b]$.

Example A.7. Let $1 \le p < \infty$, and let V denote the set of all real-valued functions x on the interval $[a,b]$ such that

$$\int_a^b |x(t)|^p dt < \infty. \tag{A.8}$$

Let $F = R$, and let vector addition and scalar multiplication be defined as in (A.6) and (A.7), respectively. It can be verified that this space is a vector space.

In this book we will usually assume that in (A.8), integration is in the Riemann sense. When integration in (A.8) is in the Lebesgue sense, then the vector space under discussion is called an L_p-space (or the space $L_p[a,b]$).

In proving that the L_p-spaces are indeed vector spaces, in establishing properties of norms defined on L_p-spaces, in defining linear transformations on L_p-spaces, and in many other applications, we make use of the *Hölder and Minkowski Inequalities for integrals*, given below. (These inequalities are valid when integration is in the Riemann and the Lebesgue senses.) For proofs of these results, refer, e.g., to Michel and Herget [9, pp. 268–270].

Hölder's Inequality states that if $p, q \in R$ are such that $1 < p < \infty$ and $1/p + 1/q = 1$, if $[a,b]$ is an interval on the real line, if $f, g : [a,b] \to R$, and if $\int_a^b |f(t)|^p dt < \infty$ and $\int_a^b |g(t)|^q dt < \infty$, then

$$\int_a^b |f(t)g(t)| dt \le (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |g(t)|^q dt)^{1/q}. \tag{A.9}$$

Minkowski's Inequality states that if $p \in R$, where $1 \le p < \infty$, if $f, g : [a,b] \to R$, and if $\int_a^b |f(t)|^p dt < \infty$ and $\int_a^b |g(t)|^p dt < \infty$, then

$$(\int_a^b |f(t) \pm g(t)|^p dt)^{1/p} \le (\int_a^b |f(t)|^p dt)^{1/p} + (\int_a^b |g(t)|^p dt)^{1/p}. \tag{A.10}$$

If in particular $p = q = 2$, then (A.9) reduces to *the Schwarz Inequality for integrals.*

Example A.8. Let V denote the set of all continuous real-valued functions on the interval $[a,b]$ such that

$$\max_{a \le t \le b} |x(t)| < \infty. \tag{A.11}$$

Let $F = R$, and let vector addition and scalar multiplication be defined as in (A.6) and (A.7), respectively. It can readily be verified that this space is a vector space.

In some applications it is necessary to expand the above space to the set of measurable real-valued functions on $[a, b]$ and to replace (A.11) by

$$ess \sup_{a \le t \le b} |x(t)| < \infty, \tag{A.12}$$

where $ess \sup$ denotes the essential supremum; i.e.,

$$ess \sup_{a \le t \le b} |x(t)| = \inf\{M : \mu\{t : |x(t)| > M\} = 0, \},$$

where μ denotes the Lebesgue measure. In this case, the vector space under discussion is called the L_∞-space.

A.2 Linear Independence and Bases

We now address the important concepts of linear independence of a set of vectors in general and bases in particular. We first require the notion of linear subspace.

A.2.1 Linear Subspaces

A nonempty subset W of a vector space V is called a *linear subspace* (or a *linear manifold*) in V if (i) $w_1 + w_2$ is in W whenever w_1 and w_2 are in W, and (ii) αw is in W whenever $\alpha \in F$ and $w \in W$. It is an easy matter to verify that a linear subspace W satisfies all the axioms of a vector space and may as such be regarded as a linear space itself.

Two trivial examples of linear subspaces include the null vector (i.e., the set $W = \{0\}$ is a linear subspace of V) and the vector space V itself. Another example of a linear subspace is the set of all real-valued polynomials defined on the interval $[a, b]$ that is a linear subspace of the vector space consisting of all real-valued continuous functions defined on the interval $[a, b]$ (refer to Example A.6).

As another example of a linear subspace (of R^2), we cite the set of all points on a straight line passing through the origin. On the other hand, a straight line that does not pass through the origin is *not* a linear subspace of R^2.

It is an easy matter to show that if W_1 and W_2 are linear subspaces of a vector space V, then $W_1 \cap W_2$, the intersection of W_1 and W_2, is also a linear subspace of V. A similar statement cannot be made, however, for the union of W_1 and W_2 (prove this). Note that to show that a set V is a vector space, it suffices to show that it is a linear subspace of some vector space.

A.2.2 Linear Independence

Throughout this section, we let $\{\alpha_1, \dots, \alpha_n\}, \alpha_i \in F$, denote an indexed set of scalars and we let $\{v^1, \dots, v^n\}, v^i \in V$, denote an indexed set of vectors.

Now let W be a set in a linear space V (W may be a finite set or an infinite set). We say that a vector $v \in V$ is a *finite linear combination of vectors* in W if there is a finite set of elements $\{w^1, \dots, w^n\}$ in W and a finite set of scalars $\{\alpha_1, \dots, \alpha_n\}$ in F such that

$$v = \alpha_1 w^1 + \cdots + \alpha_n w^n.$$

Now let W be a nonempty subset of a linear space V and let $S(W)$ be the set of all finite linear combinations of the vectors from W; i.e., $w \in S(W)$ if and only if there is some set of scalars $\{\alpha_1, \dots, \alpha_m\}$ and some finite subset $\{w^1, \dots, w^m\}$ of W such that $w = \alpha_1 w^1 + \cdots + \alpha_m w^m$, where m may be any positive integer. Then it is easily shown that $S(W)$ is a linear subspace of V, called the *linear subspace generated* by the set W.

Now if U is a linear subspace of a vector space V and if there exists a set of vectors $W \subset V$ such that the linear space $S(W)$ generated by W is U, then we say that W *spans* U. It is easily shown that $S(W)$ is the smallest linear subspace of a vector space V containing the subset W of V. Specifically, if U is a linear subspace of V and if U contains W, then U also contains $S(W)$.

As an example, in the space (R^2, R), the set $S_1 = \{e^1\} = \{(1,0)^T\}$ spans the set consisting of all vectors of the form $(a, 0)^T, a \in R$, whereas the set $S_2 = \{e^1, e^2\}, e^2 = (0,1)^T$ spans all of R^2.

We are now in a position to introduce the notion of linear dependence.

Definition A.9. *Let $S = \{v^1, \dots, v^m\}$ be a finite nonempty set in a linear space V.* If there exist scalars $\alpha_1, \dots, \alpha_m$, not all zero, such that

$$\alpha_1 v^1 + \cdots + \alpha_m v^m = 0, \tag{A.13}$$

then the set S is said to be *linearly dependent* (over F). *If a set is not linearly dependent, then it is said to be* linearly independent. *In this case relation (A.13) implies that $\alpha_1 = \cdots = \alpha_m = 0$. An infinite set of vectors W in V is said to be linearly independent if every finite subset of W is linearly independent.* ■

Example A.10. Consider the linear space (R^n, R) (see Example A.3), and let $e^1 = (1, 0, \dots, 0)^T, e^2 = (0, 1, 0, \dots, 0)^T, \dots, e^n = (0, \dots, 0, 1)^T$. Clearly, $\sum_{i=1}^n \alpha_i e^i = 0$ implies that $\alpha_i = 0, i = 1, \dots, n$. Therefore, the set $S = \{e^1, \dots, e^n\}$ is a linearly independent set of vectors in R^n over the field of real numbers R.

Example A.11. Let V be the set of 2-tuples whose entries are complex-valued rational functions over the field of complex-valued rational functions. Let

$$v^1 = \begin{bmatrix} 1/(s+1) \\ 1/(s+2) \end{bmatrix}, \quad v^2 = \begin{bmatrix} (s+2)/[(s+1)(s+3)] \\ 1/(s+3) \end{bmatrix},$$

and let $\alpha_1 = -1$, $\alpha_2 = (s+3)/(s+2)$. Then $\alpha_1 v^1 + \alpha_2 v^2 = 0$, and therefore, the set $S = \{v^1, v^2\}$ is *linearly dependent over the field of rational functions.* On the other hand, since $\alpha_1 v^1 + \alpha_2 v^2 = 0$ when $\alpha_1, \alpha_2 \in R$ is true if and only if $\alpha_1 = \alpha_2 = 0$, it follows that S is linearly independent over the field of real numbers (which is a subset of the field of rational functions). This shows that *linear dependence* of a set of vectors in V depends on the field F.

A.2.3 Linear Independence of Functions of Time

Example A.12. Let $V = C((a,b), R^n)$, let $F = R$, and for $x, y \in V$ and $\alpha \in F$, define addition of elements in V and multiplication of elements in V by elements in F by $(x+y)(t) = x(t)+y(t)$ for all $t \in (a,b)$ and $(\alpha x)(t) = \alpha x(t)$ for all $t \in (a,b)$. Then, as in Example A.6, we can easily show that (V, F) is a vector space. An interesting question that arises is whether for this space, linear dependence (and linear independence) of a set of vectors can be phrased in some testable form. The answer is affirmative. Indeed, it can readily be verified that for the present vector space (V, F), *linear dependence of a set of vectors $S = \{\phi_1, \dots, \phi_k\}$ in $V = C((a,b), R^n)$ over $F = R$ is equivalent to the requirement that there exist scalars $\alpha_i \in F$, $i = 1, \dots, k$, not all zero, such that*

$$\alpha_1 \phi_1(t) + \cdots + \alpha_k \phi_k(t) = 0 \quad \text{for all } t \in (a,b).$$

Otherwise, S is *linearly independent.*

To see how the above example applies to specific cases, let $V = C((-\infty, \infty), R^2)$, and consider the vectors $\phi_1(t) = [1, t]^T, \phi_2(t) = [1, t^2]^T$. To show that the set $S = \{\phi_1, \phi_2\}$ is linearly independent (over $F = R$), assume for purposes of contradiction that S is linearly dependent. Then there must exist scalars α_1 and α_2, not both zero, such that $\alpha_1[1, t]^T + \alpha_2[1, t^2]^T = [0, 0]^T$ for all $t \in (-\infty, \infty)$. But in particular, for $t = 2$, the above equation is satisfied if and only if $\alpha_1 = \alpha_2 = 0$, which contradicts the assumption. Therefore, $S = \{\phi_1, \phi_2\}$ is linearly independent.

As another specific case of the above example, let $V = C((-\infty, \infty), R^2)$ and consider the set $S = \{\phi_1, \phi_2, \phi_3, \phi_4\}$, where $\phi_1(t) = [1, t]^T, \phi_2(t) = [1, t^2], \phi_3(t) = [0, 1]^T$, and $\phi_4(t) = [e^{-t}, 0]$. The set S is clearly independent over R since $\alpha_1 \phi_1(t) + \alpha_2 \phi_2(t) + \alpha_3 \phi_3(t) + \alpha_4 \phi_4(t) = 0$ for all $t \in (-\infty, \infty)$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

A.2.4 Bases

We are now in a position to introduce another important concept.

Definition A.13. A set W in a linear space V is called a *basis* for V if

(i) W is linearly independent, and
(ii) the span of W is the linear space V itself; i.e., $S(W) = V$. ■

An immediate consequence of the above definition is that if W is a linearly independent set in a vector space V, then W is a basis for $S(W)$.

To introduce the notion of dimension of a vector space, it is shown that if a linear space V is generated by a finite number of linearly independent elements, then this number of elements must be unique. The following results lead up to this.

Let $\{v^1, \dots, v^n\}$ be a basis for a linear space V. Then it is easily shown that for each vector $v \in V$, there exist *unique* scalars $\alpha_1, \dots, \alpha_n$ such that

$$v = \alpha_1 v^1 + \cdots + \alpha_n v^n.$$

Furthermore, if $u^1, \dots, u^m$ is any linearly independent set of vectors in V, then $m \leq n$. Moreover, any other basis of V consists of exactly n elements. These facts allow the following definitions.

If a linear space V has a basis consisting of a finite number of vectors, say, $\{v^1, \dots, v^n\}$, then V is said to be a *finite-dimensional vector space* and the *dimension of* V is n, abbreviated $\dim V = n$. In this case we speak of an *n-dimensional vector space*. If V is not a finite-dimensional vector space, it is said to be an *infinite-dimensional vector space*.

By convention, the linear space consisting of the null vector is finite-dimensional with dimension equal to zero.

An alternative to the above definition of dimension of a (finite-dimensional) vector space is given by the following result, which is easily verified: Let V be a vector space that contains n linearly independent vectors. If every set of $n+1$ vectors in V is linearly dependent, then V is finite-dimensional and $\dim V = n$.

The preceding results enable us now to introduce the concept of coordinates of a vector. We let $\{v^1, \dots, v^n\}$ be a basis of a vector space V, and we let $v \in V$ be represented by

$$v = \xi_1 v^1 + \cdots + \xi_n v^n.$$

The *unique* scalars $\xi_1, \dots, \xi_n$ are called the *coordinates of v with respect to the basis* $\{v^1, \dots, v^n\}$.

Example A.14. For the linear space (R^n, R), let $S = \{e^1, \dots, e^n\}$, where the $e^i \in R^n$, $i = 1, \dots, n$, were defined earlier (in Example A.10). Then

S is clearly a basis for (R^n, R) since it is linearly independent and since given any $v \in R^n$, there exist unique real scalars $\alpha_i, i = 1, \ldots, n$, such that $v = \sum_{i=1}^{n} \alpha_i e^i = (\alpha_1, \ldots, \alpha_n)^T$; i.e., S spans R^n. It follows that with every vector $v \in R^n$, we can associate a *unique* n-tuple of scalars

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \text{or} \quad (\alpha_1, \ldots, \alpha_n)$$

relative to the basis $\{e^1, \ldots, e^n\}$, the *coordinate representation* of the vector $v \in R^n$ with respect to the basis $S = \{e^1, \ldots, e^n\}$. Henceforth, we will refer to the basis S of this example as the *natural basis for R^n*.

Example A.15. We note that the vector space of all (complex-valued) polynomials with real coefficients of degree less than n is an n-dimensional vector space over the field of real numbers. A basis for this space is given by $S = \{1, s, \ldots, s^{n-1}\}$ where s is a complex variable. Associated with a given element of this vector space, say $p(s) = \alpha_0 + \alpha_1 s + \cdots + \alpha_{n-1} s^{n-1}$, we have the *unique* n-tuple given by $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})^T$, which constitutes the coordinate representation of $p(s)$ with respect to the basis S given above.

Example A.16. We note that the space (V, R), where $V = C([a, b], R)$, given in Example A.6 is an infinite-dimensional vector space.

A.3 Linear Transformations

Definition A.17. *A mapping $\mathcal{T}$ of a linear space V into a linear space W, where V and W are vector spaces over the same field F, is called a* linear transformation *or a* linear operator *provided that*

(L-i) $\quad \mathcal{T}(x + y) = \mathcal{T}(x) + \mathcal{T}(y)$ *for all $x, y \in V$, and*

(L-ii) $\quad \mathcal{T}(\alpha x) = \alpha \mathcal{T}(x)$ *for all $x \in V$ and $\alpha \in F$.* ■

In the following discussion, we consider three specific examples of linear transformations.

Example A.18. Let $(V, R) = (R^n, R)$ and $(W, R) = (R^m, R)$ be vector spaces defined as in Example A.3, let $A = [a_{ij}] \in R^{m \times n}$, and let $\mathcal{T} : V \to W$ be defined by the equation

$$y = Ax, \quad y \in R^m, \quad x \in R^n,$$

where Ax denotes multiplication of the matrix A and the vector y. It is easily verified using the properties of matrices that $\mathcal{T}$ is a linear transformation.

Example A.19. Let $(V, R) = (l_p, R)$ be the vector space defined in Example A.5 (modified to consist of sequences $\{x_i\}, i \in Z$, in place of $\{x_i\}$, $i = 1, 2, \dots$). Let $h : Z \times Z \to R$ be a function having the property that for each $x \in V$, the infinite sum

$$\sum_{k=-\infty}^{\infty} h(n, k)x(k)$$

exists and defines a function of n on Z. Let $\mathcal{T} : V \to V$ be defined by

$$y(n) = \sum_{k=-\infty}^{\infty} h(n, k)x(k).$$

It is easily verified that $\mathcal{T}$ is a linear transformation.

The existence of the above sum is ensured under appropriate assumptions. For example, by using the Hölder Inequality, it is readily shown that if, e.g., for fixed n, $\{h(n, k)\} \in l_2$ and $\{x(k)\} \in l_2$, then the above sum is well defined. The above sum exists also if, e.g., $\{x(k)\} \in l_\infty$ and $\{h(n, k)\} \in l_1$ for fixed n.

Example A.20. Let (V, R) denote the vector space given in Example A.7, and let $k \in C([a, b] \times [a, b], R)$ have the property that for each $x \in V$, the Riemann integral

$$\int_a^b k(s, t)x(t)dt$$

exists and defines a continuous function of s on $[a, b]$. Let $\mathcal{T} : V \to V$ be defined by

$$(\mathcal{T}x)(s) = y(s) = \int_a^b k(s, t)x(t)dt.$$

It is readily verified that $\mathcal{T}$ is a linear transformation of V into V.

Henceforth, if $\mathcal{T}$ is a linear transformation from a vector space V (over a field F) into a vector space W (over the same field F) we will write $\mathcal{T} \in L(V, W)$ to express this. In the following discussion, we will identify some of the important properties of linear transformations.

A.3.1 Linear Equations

With $\mathcal{T} \in L(V, W)$ we define the *null space of* $\mathcal{T}$ as the set

$$\mathcal{N}(\mathcal{T}) = \{v \in V : \mathcal{T}v = w = 0\}$$

and the *range space of* $\mathcal{T}$ as the set

$$\mathcal{R}(\mathcal{T}) = \{w \in W : w = \mathcal{T}v, v \in V\}.$$

Note that since $\mathcal{T}0 = 0, \mathcal{N}(\mathcal{T})$ and $\mathcal{R}(\mathcal{T})$ are never empty. It is easily verified that $\mathcal{N}(\mathcal{T})$ is a linear subspace of V and that $\mathcal{R}(\mathcal{T})$ is a linear subspace of W. If V is finite-dimensional (of dimension n), then it is easily shown that $\dim \mathcal{R}(\mathcal{T}) \leq n$. Also, if V is finite-dimensional and if $\{w^1, \dots, w^n\}$ is a basis for $\mathcal{R}(\mathcal{T})$ and v^i is defined by $\mathcal{T}v^i = w^i, i = 1, \dots, n$, then it is readily proved that the vectors $v^1, \dots, v^n$ are linearly independent.

One of the important results of linear algebra, called the *fundamental theorem of linear equations*, states that for $\mathcal{T} \in L(V, W)$ with V finite-dimensional, we have

$$\dim \mathcal{N}(\mathcal{T}) + \dim \mathcal{R}(\mathcal{T}) = \dim V.$$

For the proof of this result, refer to any of the references on linear algebra cited at the end of Chapters 4, 6, 8, and 9.

The above result gives rise to the notions of the *rank*, $\rho(\mathcal{T})$, of a linear transformation $\mathcal{T}$ of a finite-dimensional vector space V into a vector space W, which we define as the dimension of the range space $\mathcal{R}(\mathcal{T})$, and the *nullity*, $\nu(\mathcal{T})$, of $\mathcal{T}$, which we define as the dimension of the null space $\mathcal{N}(\mathcal{T})$.

With the above machinery in place, it is now easy to establish the following important results concerning *linear equations*.

Let $\mathcal{T} \in L(V, W)$, where V is finite-dimensional, let $s = \dim \mathcal{N}(\mathcal{T})$, and let $\{v^1, \dots, v^s\}$ be a basis for $\mathcal{N}(\mathcal{T})$. Then it is easily verified that

(i) a vector $v \in V$ satisfies the equation $\mathcal{T}v = 0$ if and only if $v = \sum_{i=1}^{s} \alpha_i v^i$ for some set of scalars $\{\alpha_1, \dots, \alpha_s\}$, and furthermore, for each $v \in V$ such that $\mathcal{T}v = 0$ is true, the set of scalars $\{\alpha_1, \dots, \alpha_s\}$ is unique;
(ii) if $w^0 \in W$ is a fixed vector, then $\mathcal{T}v = w^0$ holds for at least one vector $v \in V$ (called the *solution* of the equation $\mathcal{T}v = w^0$) if and only if $w^0 \in \mathcal{R}(\mathcal{T})$; and
(iii) if w^0 is any fixed vector in W and if v^0 is some vector in V such that $\mathcal{T}v^0 = w^0$ (i.e., v^0 is a solution of the equation $\mathcal{T}v^0 = w^0$), then a vector $v \in V$ satisfies $\mathcal{T}v = w^0$ if and only if $v = v^0 + \sum_{i=1}^{s} \beta_i v^i$ for some set of scalars $\{\beta_1, \dots, \beta_s\}$, and furthermore, for each $v \in V$ such that $\mathcal{T}v = w_0$, the set of scalars $\{\beta_1, \dots, \beta_s\}$ is unique.

A.3.2 Representation of Linear Transformations by Matrices

In the following discussion, we let (V, F) and (W, F) be vector spaces over the *same* field and we let $\mathcal{A} : V \to W$ denote a linear mapping. We let $\{v^1, \dots, v^n\}$ be a basis for V, and we set $\bar{v}^1 = \mathcal{A}v^1, \dots, \bar{v}^n = \mathcal{A}v^n$. Then it is an easy matter to show that if v is any vector in V and if $(\alpha_1, \dots, \alpha_n)$ are the coordinates of v with respect to $\{v^1, \dots, v^n\}$, then $\mathcal{A}v = \alpha_1 \bar{v}^1 + \cdots + \alpha_n \bar{v}^n$. Indeed, we have $\mathcal{A}v = \mathcal{A}(\alpha_1 v^1 + \cdots + \alpha_n v^n) = \alpha_1 \mathcal{A}v^1 + \cdots + \alpha_n \mathcal{A}v^n = \alpha_1 \bar{v}^1 + \cdots + \alpha_n \bar{v}^n$.

Next, we let $\{\bar{v}^1, \dots, \bar{v}^n\}$ be any set of vectors in W. Then it can be shown that there exists a unique linear transformation $\mathcal{A}$ from V into W such that $\mathcal{A}v^1 = \bar{v}^1, \dots, \mathcal{A}v^n = \bar{v}^n$. To show this, we first observe that for each $v \in V$ we have unique scalars $\alpha_1, \dots, \alpha_n$ such that

$$v = \alpha_1 v^1 + \cdots + \alpha_n v^n.$$

Now define a mapping $\mathcal{A} : V \to W$ as

$$\mathcal{A}(v) = \alpha_1 \bar{v}^1 + \cdots + \alpha_n \bar{v}^n.$$

Clearly, $\mathcal{A}(v^i) = \bar{v}^i$, $i = 1, \dots, n$. We first must show that $\mathcal{A}$ is linear and, then, that $\mathcal{A}$ is unique. Given $v = \alpha_1 v^1 + \cdots + \alpha_n v^n$ and $w = \beta_1 v^1 + \cdots + \beta_n v^n$, we have $\mathcal{A}(v+w) = \mathcal{A}[(\alpha_1+\beta_1)v^1 + \cdots + (\alpha_n+\beta_n)v^n] = (\alpha_1+\beta_1)\bar{v}^1 + \cdots + (\alpha_n + \beta_n)\bar{v}^n$. On the other hand, $\mathcal{A}(v) = \alpha_1\bar{v}^1 + \cdots + \alpha_n\bar{v}^n, \mathcal{A}(w) = \beta_1\bar{v}^1 + \cdots + \beta_n\bar{v}^n$. Thus, $\mathcal{A}(v) + \mathcal{A}(w) = (\alpha_1\bar{v}^1 + \cdots + \alpha_n\bar{v}^n) + (\beta_1\bar{v}^1 + \cdots + \beta_n\bar{v}^n) = (\alpha_1+\beta_1)\bar{v}^1 + \cdots + (\alpha_n+\beta_n)\bar{v}^n = \mathcal{A}(v+w)$. In a similar manner, it is easily established that $\alpha\mathcal{A}(v) = \mathcal{A}(\alpha v)$ for all $\alpha \in F$ and $v \in V$. Therefore, $\mathcal{A}$ is linear. Finally, to show that $\mathcal{A}$ is unique, suppose there exists a linear transformation $\mathcal{B} : V \to W$ such that $\mathcal{B}v^i = \bar{v}^i$, $i = 1, \dots, n$. It follows that $(\mathcal{A} - \mathcal{B})v^i = 0, i = 1, \dots, n$, and, therefore, that $\mathcal{A} = \mathcal{B}$.

These results show that *a linear transformation is completely determined by knowing how it transforms the basis vectors in its domain, and that this linear transformation is uniquely determined in this way.* These results enable us to represent linear transformations defined on finite-dimensional spaces in an unambiguous way by means of matrices. We will use this fact in the following development.

Let (V, F) and (W, F) denote n-dimensional and m-dimensional vector spaces, respectively, and let $\{v^1, \dots, v^n\}$ and $\{w^1, \dots, w^m\}$ be bases for V and W, respectively. Let $\mathcal{A} : V \to W$ be a linear transformation, and let $\bar{v}^i = \mathcal{A}v^i$, $i = 1, \dots, n$. Since $\{w^1, \dots, w^m\}$ is a basis for W, there are unique scalars $\{a_{ij}\}$, $i = 1, \dots, m$, $j = 1, \dots, n$, such that

$$\begin{aligned} \mathcal{A}v^1 &= \bar{v}^1 = a_{11}w^1 + a_{21}w^2 + \cdots + a_{m1}w^m, \\ \mathcal{A}v^2 &= \bar{v}^2 = a_{12}w^1 + a_{22}w^2 + \cdots + a_{m2}w^m, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathcal{A}v^n &= \bar{v}^n = a_{1n}w^1 + a_{2n}w^2 + \cdots + a_{mn}w^m. \end{aligned} \tag{A.14}$$

Next, let $v \in V$. Then v has the unique representation $v = \alpha_1 v^1 + \alpha_2 v^2 + \cdots + \alpha_n v^n$ with respect to the basis $\{v^1, \dots, v^n\}$. In view of the result given at the beginning of this subsection, we now have

$$\mathcal{A}v = \alpha_1\bar{v}^1 + \cdots + \alpha_n\bar{v}^n. \tag{A.15}$$

Since $\mathcal{A}v \in W, \mathcal{A}v$ has a unique representation with respect to the basis $\{w^1, \dots, w^m\}$, say,

$$\mathcal{A}v = \gamma_1 w^1 + \gamma_2 w^2 + \cdots + \gamma_m w^m. \tag{A.16}$$

Combining (A.14) and (A.16), and rearranging, in view of the uniqueness of the representation in (A.16), we have

$$\begin{aligned} \gamma_1 &= a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n, \\ \gamma_2 &= a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_m &= a_{m1}\alpha_1 + a_{m2}\alpha_2 + \cdots + a_{mn}\alpha_n, \end{aligned} \tag{A.17}$$

where $(\alpha_1, \dots, \alpha_n)^T$ and $(\gamma_1, \dots, \gamma_m)^T$ are coordinate representations of $v \in V$ and $\mathcal{A}v \in W$ with respect to the bases $\{v^1, \dots, v^n\}$ of V and $\{w^1, \dots, w^m\}$ of W, respectively. This set of equations enables us to represent the linear transformation $\mathcal{A}$ from the linear space V into the linear space W by the unique scalars $\{a_{ij}\}, i = 1, \dots, m, j = 1, \dots, n$. For convenience we let

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \tag{A.18}$$

We see that once the bases $\{v^1, \dots, v^n\}, \{w^1, \dots, w^m\}$ *are fixed, we can represent the linear transformation* $\mathcal{A}$ *by the array of scalars in (A.18) that are uniquely determined by (A.14).* Note that the jth column of A is the coordinate representation of the vector $Av^j \in W$ with respect to the basis $\{w^1, \dots, w^m\}$.

The converse to the preceding statement also holds. Specifically, with the bases for V and W still fixed, the array given in (A.18) is uniquely associated with the linear transformation $\mathcal{A}$ of V into W. The above discussion gives rise to the following important definition.

Definition A.21. *The array given in (A.18) is called the* matrix A *of the linear transformation* $\mathcal{A}$ *from a linear space* V *into a linear space* W *(over* F*) with respect to the basis* $\{v^1, \dots, v^n\}$ *of* V *and the basis* $\{w^1, \dots, w^m\}$ *of* W*.* ■

If in Definition A.21, $V = W$, and if for both V and W the same basis $\{v^1, \dots, v^n\}$ is used, then we simply speak of the *matrix* A *of the linear transformation* $\mathcal{A}$ *with respect to the basis* $\{v^1, \dots, v^n\}$.

In (A.18) the scalars $(a_{i1}, a_{i2}, \dots, a_{in})$ form the ith *row* of A and the scalars $(a_{1j}, a_{2j}, \dots, a_{mj})^T$ form the jth *column* of A. The scalar a_{ij} refers to that element of matrix A that can be found in the ith row and jth column of A. The array in (A.18) is said to be an $m \times n$ *matrix.* If $m = n$, we speak of a *square matrix.* Consistent with the above discussion, an $n \times 1$ matrix is called a *column vector, column matrix*, or n*-vector*, and a $1 \times n$ matrix is called a *row vector.* Finally, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices, then

$A = B$; i.e., A and B are *equal* if and only if $a_{ij} = b_{ij}$ for all $i = 1, \ldots, m$, and for all $j = 1, \ldots, n$. Furthermore, we call $A^T = [a_{ij}]^T = [a_{ji}]$ the *transpose* of A.

The preceding discussion shows in particular that if $\mathcal{A}$ is a linear transformation of an n-dimensional vector space V into an m-dimensional vector space W,

$$w = \mathcal{A}v, \tag{A.19}$$

if $\gamma = (\gamma_1, \ldots, \gamma_m)^T$ denotes the coordinate representation of w with respect to the basis $\{w^1, \ldots, w^m\}$, if $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ denotes the coordinate representation of v with respect to the basis $\{v^1, \ldots, v^n\}$, and if A denotes the matrix of $\mathcal{A}$ with respect to the bases $\{v^1, \ldots, v^n\}, \{w^1, \ldots, w^m\}$, then

$$\gamma = A\alpha, \tag{A.20}$$

or equivalently,

$$\gamma_i = \sum_{j=1}^{n} a_{ij}\alpha_j, \quad i = 1, \ldots, m \tag{A.21}$$

which are alternative ways to write (A.17).

The Rank of a Matrix

Let A denote the matrix representation of a linear transformation $\mathcal{A}$. The rank of A, $\rho(A)$, is defined as the rank of $\mathcal{A}$, $\rho(\mathcal{A})$. It can be shown that the *rank* $\rho(A)$ *of an* $m \times n$ *matrix* A is the largest number of linearly independent columns of A. The rank is also equal to the largest numbers of linearly independent rows of A. It also equals the dimension of the largest nonzero minor of A.

A.3.3 Solving Linear Algebraic Equations

Now consider the linear system of equations given by

$$A\alpha = \gamma, \tag{A.22}$$

where $A \in R^{m\times n}$ and $\gamma \in R^m$ are given and $\alpha \in R^n$ is to be determined.

1. For a given γ, a solution α of (A.22) exists (not necessarily unique) if and only if $\gamma \in \mathcal{R}(A)$, or equivalently, if and only if

$$\rho([A, \gamma]) = \rho(A). \tag{A.23}$$

2. Every solution α of (A.22) can be expressed as a sum

$$\alpha = \alpha_p + \alpha_h, \tag{A.24}$$

where α_p is a specific solution of (A.22) and α_h satisfies $A\alpha_h = 0$. This result allows us to span the space of all solutions of (A.22). Note that there are

$$\dim \mathcal{N}(A) = n - \rho(A) \tag{A.25}$$

linearly independent solutions of the system of equations $A\beta = 0$.

3. $A\alpha = \gamma$ has a unique solution if and only if (A.23) is satisfied and

$$\rho(A) = n \leq m. \tag{A.26}$$

4. A solution α of (A.22) exists for any γ if and only if

$$\rho(A) = m. \tag{A.27}$$

If (A.27) is satisfied, a solution of (A.22) can be found by using the relation

$$\alpha = A^T (AA^T)^{-1} \gamma. \tag{A.28}$$

When in (A.22), $\rho(A) = m = n$, then $A \in R^{n \times n}$ and is nonsingular and the unique solution of (A.28) is given by

$$\alpha = A^{-1} \gamma. \tag{A.29}$$

Example A.22. Consider

$$A\alpha = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \alpha = \gamma. \tag{A.30}$$

It is easily verified that $\{(0,1,0)^T\}$ is a basis for $\mathcal{R}(A)$. Since a solution of (A.30) exists if and only if $\gamma \in \mathcal{R}(A)$, γ must be of the form $\gamma = (0, k, 0)^T$, $k \in R$. Note that

$$\rho(A) = 1 = \rho([A, \gamma]) = \operatorname{rank} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 0 \end{array}\right],$$

as expected. To determine all solutions of (A.30), we need to determine an α_p and an α_h [see (A.24)]. In particular, $\alpha_p = (0\,0\,k)^T$ will do. To determine α_h, we consider $A\beta = 0$. There are $\dim \mathcal{N}(A) = 2$ linearly independent solutions of $A\beta = 0$. In particular, $\{(1,0,0)^T, (0,1,0)^T\}$ is a basis for $\mathcal{N}(A)$. Therefore, any solution of (A.30) can be expressed as

$$\alpha = \alpha_p + \alpha_h = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

where c_1, c_2 are appropriately chosen real numbers.

A.4 Equivalence and Similarity

From our previous discussion it is clear that a linear transformation $\mathcal{A}$ of a finite-dimensional vector space V into a finite-dimensional vector space W can be represented by means of different matrices, depending on the particular choice of bases in V and W. The choice of bases may in different cases result in matrices that are easy or hard to utilize. Many of the resulting "standard" forms of matrices, called *canonical forms*, arise because of practical considerations. Such canonical forms often exhibit inherent characteristics of the underlying transformation $\mathcal{A}$.

Throughout this section, V and W are finite-dimensional vector spaces over the same field $F, \dim V = n$, and $\dim W = m$.

A.4.1 Change of Bases: Vector Case

Our first aim will be to consider the change of bases in the coordinate representation of vectors. Let $\{v^1, \dots, v^n\}$ be a basis for V, and let $\{\bar{v}^1, \dots, \bar{v}^n\}$ be a set of vectors in V given by

$$\bar{v}^i = \sum_{j=1}^{n} p_{ji} v^j, \quad i = 1, \dots, n, \tag{A.31}$$

where $p_{ij} \in F$ for all $i, j = 1, \dots, n$. It is easily verified that the set $\{\bar{v}^1, \dots, \bar{v}^n\}$ forms a basis for V if and only if the $n \times n$ matrix $P = [p_{ij}]$ is nonsingular (i.e., $\det P \neq 0$). We call P *the matrix of the basis* $\{\bar{v}^1, \dots, \bar{v}^n\}$ *with respect to the basis* $\{v^1, \dots, v^n\}$. Note that the ith column of P is the coordinate representation of $\bar{v}^i$ with respect to the basis $\{v^1, \dots, v^n\}$.

Continuing the above discussion, let $\{v^1, \dots, v^n\}$ and $\{\bar{v}^1, \dots, \bar{v}^n\}$ be two bases for V and let P be the matrix of the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$ with respect to the basis $\{v^1, \dots, v^n\}$. Then it is easily shown that P^{-1} is the matrix of the basis $\{v^1, \dots, v^n\}$ with respect to the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$.

Next, let the sets of vectors $\{v^1, \dots, v^n\}, \{\bar{v}^1, \dots, \bar{v}^n\}$, and $\{\tilde{v}^1, \dots, \tilde{v}^n\}$ be bases for V. If P is the matrix of the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$ with respect to the basis $\{v^1, \dots, v^n\}$ and if Q is the matrix of the basis $\{\tilde{v}^1, \dots, \tilde{v}^n\}$ with respect to the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$, then it is easily verified that PQ is the matrix of the basis $\{\tilde{v}^1, \dots, \tilde{v}^n\}$ with respect to the basis $\{v^1, \dots, v^n\}$.

Continuing further, let $\{v^1, \dots, v^n\}$ and $\{\bar{v}^1, \dots, \bar{v}^n\}$ be two bases for V and let P be the matrix of the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$ with respect to the basis $\{v^1, \dots, v^n\}$. Let $a \in V$, and let $\alpha^T = (\alpha_1, \dots, \alpha_n)$ denote the coordinate representation of a with respect to the basis $\{v^1, \dots, v^n\}$ (i.e., $a = \sum_{i=1}^n \alpha_i v^i$). Let $\bar{\alpha}^T = (\bar{\alpha}_1, \dots, \bar{\alpha}_n)$ denote the coordinate representation of a with respect to the basis $\{\bar{v}^1, \dots, \bar{v}^n\}$. Then it is readily verified that

$$P\bar{\alpha} = \alpha.$$

Example A.23. Let $V = R^3, F = R$, and let $a = (1,2,3)^T \in R^3$ be given. Let $\{v^1, v^2, v^3\} = \{e^1, e^2, e^3\}$ denote the natural basis for R^3; i.e., $e^1 = (1,0,0)^T, e^2 = (0,1,0)^T, e^3 = (0,0,1)^T$. Clearly, the coordinate representation α of a with respect to the natural basis is $(1,2,3)^T$.

Now let $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ be another basis for R^3, given by $\bar{v}^1 = (1,0,1)^T, \bar{v}^2 = (0,1,0)^T, \bar{v}^3 = (0,1,1)^T$. From the relation

$$(1,0,1)^T = \bar{v}^1 = p_{11}v^1 + p_{21}v^2 + p_{31}v^3 = p_{11}\begin{bmatrix}1\\0\\0\end{bmatrix} + p_{21}\begin{bmatrix}0\\1\\0\end{bmatrix} + p_{31}\begin{bmatrix}0\\0\\1\end{bmatrix},$$

we conclude that $p_{11} = 1, p_{21} = 0$, and $p_{31} = 1$. Similarly, from

$$(0,1,0)^T = \bar{v}^2 = p_{12}v^1 + p_{22}v^2 + p_{32}v^3 = p_{12}\begin{bmatrix}1\\0\\0\end{bmatrix} + p_{22}\begin{bmatrix}0\\1\\0\end{bmatrix} + p_{32}\begin{bmatrix}0\\0\\1\end{bmatrix},$$

we conclude that $p_{12} = 0, p_{22} = 1$, and $p_{32} = 0$. Finally, from the relation

$$(0,1,1)^T = \bar{v}^3 = p_{13}\begin{bmatrix}1\\0\\0\end{bmatrix} + p_{23}\begin{bmatrix}0\\1\\0\end{bmatrix} + p_{33}\begin{bmatrix}0\\0\\1\end{bmatrix},$$

we obtain that $p_{13} = 0, p_{23} = 1$, and $p_{33} = 1$.

The matrix $P = [p_{ij}]$ of the basis $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ with respect to the basis $\{v^1, v^2, v^3\}$ is therefore determined to be

$$P = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix},$$

and the coordinate representation of a with respect to the basis $\{\bar{v}^1, \bar{v}^2, \bar{v}^3\}$ is given by $\bar{\alpha} = P^{-1}\alpha$, or

$$\bar{\alpha} = \begin{bmatrix}1&0&0\\0&1&1\\1&0&1\end{bmatrix}^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&-1\\-1&0&0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}.$$

A.4.2 Change of Bases: Matrix Case

Having addressed the relationship between the coordinate representations of a given vector with respect to different bases, we next consider the relationship between the matrix representations of a given linear transformation relative to different bases. To this end, let $\mathcal{A} \in L(V, W)$ and let $\{v^1, \ldots, v^n\}$ and

$\{w^1, \dots, w^m\}$ be bases for V and W, respectively. Let A be the matrix of $\mathcal{A}$ with respect to the bases $\{v^1, \dots, v^n\}$ and $\{w^1, \dots, w^m\}$. Let $\{\bar{v}^1, \dots, \bar{v}^n\}$ be another basis for V, and let the matrix of $\{\bar{v}^1, \dots, \bar{v}^n\}$ with respect to $\{v^1, \dots, v^n\}$ be P. Let $\{\bar{w}^1, \dots, \bar{w}^m\}$ be another basis for W, and let Q be the matrix of $\{w^1, \dots, w^m\}$ with respect to $\{\bar{w}^1, \dots, \bar{w}^m\}$. Let $\bar{A}$ be the matrix of $\mathcal{A}$ with respect to the bases $\{\bar{v}^1, \dots, \bar{v}^n\}$ and $\{\bar{w}^1, \dots, \bar{w}^m\}$. Then it is readily verified that

$$\bar{A} = QAP. \tag{A.32}$$

This result is depicted schematically in Figure A.1.

$$V \xrightarrow{\mathcal{A}} W$$

$$\begin{array}{ccc}
\boxed{\begin{array}{c}\{v^1, \dots, v^n\} \\ \nu = P\bar{\nu}\end{array}} & \xrightarrow{A} & \boxed{\begin{array}{c}\{w^1, \cdots, w^m\} \\ \omega = A\nu\end{array}} \\
P \uparrow & & \downarrow Q \\
\boxed{\begin{array}{c}\{\bar{v}^1, \dots, \bar{v}^n\} \\ \bar{\nu}\end{array}} & \xrightarrow{\bar{A}} & \boxed{\begin{array}{c}\{\bar{w}^1, \dots, \bar{w}^m\} \\ \bar{\omega} = Q\omega\end{array}}
\end{array}$$

Figure A.1. Schematic diagram of the equivalence of two matrices

A.4.3 Equivalence and Similarity of Matrices

The preceding discussion motivates the following definition.

Definition A.24. *An $m \times n$ matrix $\bar{A}$ is said to be* equivalent *to an $m \times n$ matrix A if there exists an $m \times m$ nonsingular matrix Q and an $n \times n$ nonsingular matrix P such that (A.32) is true. If $\bar{A}$ is equivalent to A, we write $\bar{A} \sim A$.* ■

Next, let $V = W$, let $\mathcal{A} \in L(V, V)$, let $\{v^1, \dots, v^n\}$ be a basis for V, and let A be the matrix of $\mathcal{A}$ with respect to $\{v^1, \dots, v^n\}$. Let $\{\bar{v}^1, \dots, \bar{v}^n\}$ be another basis for V whose matrix with respect to $\{v^1, \dots, v^n\}$ is P. Let $\bar{A}$ be the matrix of $\mathcal{A}$ with respect to $\{\bar{v}^1, \dots, \bar{v}^n\}$. Then it follows immediately from (A.32) that

$$\bar{A} = P^{-1}AP. \tag{A.33}$$

The meaning of this result is depicted schematically in Figure A.2. The above discussion motivates the following definition.

Definition A.25. *An $n \times n$ matrix $\bar{A}$ is said to be* similar *to an $n \times n$ matrix A if there exists an $(n \times n)$ nonsingular matrix P such that*

$$\begin{array}{ccc} V & \xrightarrow{\mathcal{A}} & V \\ \{v^1,\ldots,v^n\} & \xrightarrow{A} & \{v^1,\ldots,v^n\} \\ \uparrow P & & \downarrow P^{-1} \\ \{\bar{v}^1,\ldots,\bar{v}^n\} & \xrightarrow{\bar{A}} & \{\bar{v}^1,\ldots,\bar{v}^n\} \end{array}$$

Figure A.2. Schematic diagram of the similarity of two matrices

$$\bar{A} = P^{-1}AP.$$

If $\bar{A}$ is similar to A, we write $\bar{A} \sim A$. We call P a similarity transformation. ■

It is easily verified that if $\bar{A}$ is similar to A [i.e., (A.33) is true], then A is similar to $\bar{A}$; i.e.,

$$A = P\bar{A}P^{-1}. \tag{A.34}$$

In view of this, there is no ambiguity in saying "two matrices are similar," and we could just as well have used (A.34) [in place of (A.33)] to define similarity of matrices. To sum up, if two matrices A and $\bar{A}$ represent the same linear transformation $\mathcal{A} \in L(V, V)$, possibly with respect to two different bases for V, then A and $\bar{A}$ are similar matrices.

A.5 Eigenvalues and Eigenvectors

Definition A.26. *Let A be an $n \times n$ matrix whose elements belong to the field F. If there exist $\lambda \in F$ and a nonzero vector $\alpha \in F^n$ such that*

$$A\alpha = \lambda\alpha, \tag{A.35}$$

then λ is called an eigenvalue *of A and α is called an* eigenvector *of A corresponding to the eigenvalue λ.* ■

We note that if α is an eigenvector of A, then any nonzero multiple of α is also an eigenvector of A.

A.5.1 Characteristic Polynomial

Let $A \in C^{n\times n}$. Then

$$\det(A - \lambda I) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n \tag{A.36}$$

[note that $\alpha_0 = \det(A)$ and $\alpha_n = (-1)^n$]. The eigenvalues of A are precisely the roots of the equation

$$\det(A - \lambda I) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \cdots + \alpha_n\lambda^n = 0 \tag{A.37}$$

and A has at most n distinct eigenvalues.

We call (A.36) the *characteristic polynomial of A*, and we call (A.37) *the characteristic equation of A*.

Remarks

The above definition of characteristic polynomial is the one usually used in texts on linear algebra and matrix theory (refer, e.g., to some of the books on this subject cited at the end of this chapter). An alternative to the above definition is given by the expression

$$\alpha(\lambda) \triangleq \det(\lambda I - A) = (-1)^n \det(A - \lambda I).$$

Now consider

$$\det(A - \lambda I) = (\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2} \cdots (\lambda_p - \lambda)^{m_p}, \tag{A.38}$$

where $\lambda_i, i = 1, \ldots, p$, are the distinct roots of (A.37) (i.e., $\lambda_i \neq \lambda_j$, if $i \neq j$). In (A.38), m_i is called the *algebraic multiplicity* of the root λ_i. The m_i are positive integers, and $\sum_{i=1}^{p} m_i = n$.

The reader should make note of the distinction between the concept of *algebraic multiplicity* m_i of λ_i, given above, and the *(geometric) multiplicity* l_i of an eigenvalue λ_i, given by $l_i = n - \rho(\lambda_i I - A)$. In general these need not be the same.

A.5.2 The Cayley–Hamilton Theorem and Applications

We now state and prove a result that is very important in linear systems theory.

Theorem A.27. (Cayley–Hamilton Theorem) *Every square matrix satisfies its characteristic equation. More specifically, if A is an $n \times n$ matrix and $p(\lambda) = \det(A - \lambda I)$ is the characteristic polynomial of A, then $p(A) = O$.*

Proof. Let the characteristic polynomial for A be $p(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$, and let $B(\lambda) = [b_{ij}(\lambda)]$ be the classical adjoint of $(A - \lambda I)$. (For a nonsingular matrix C with inverse $C^{-1} = \frac{1}{\det(C)}\,\mathrm{adj}(C)$, $\mathrm{adj}(C)$ is called the *classical adjoint of C*.) Since the $b_{ij}(\lambda)$ are cofactors of the matrix $A - \lambda I$, they are polynomials in λ of degree not more than $n - 1$. Thus, $b_{ij}(\lambda) = \beta_{ij0} + \beta_{ij1}\lambda + \cdots + \beta_{ij(n-1)}\lambda^{n-1}$. Letting $B_k = [\beta_{ijk}]$ for $k = 0, 1, \ldots, n-1$, we have $B(\lambda) = B_0 + \lambda B_1 + \cdots + \lambda^{n-1}B_{n-1}$ and $(A - \lambda I)B(\lambda) = [\det(A - \lambda I)]I$. Thus, $(A - \lambda I)[B_0 + \lambda B_1 + \cdots + \lambda^{n-1}B_{n-1}] = (\alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n)I$. Expanding the left-hand side of this equation and equating like powers of λ, we have $-B_{n-1} = \alpha_n I, AB_{n-1} - B_{n-2} = \alpha_{n-1}I, \ldots, AB_1 - B_0 = \alpha_1 I, AB_0 = \alpha_0 I$. Premultiplying the above matrix equations by $A^n, A^{n-1}, \ldots, A, I$, respectively, we have $-A^n B_{n-1} = \alpha_n A^n, A^n B_{n-1} - A^{n-1}B_{n-2} = \alpha_{n-1}A^{n-1}, \ldots, A^2 B_1 - AB_0 = \alpha_1 A, AB_0 = \alpha_0 I$. Adding these matrix equations, we obtain $O = \alpha_0 I + \alpha_1 A + \cdots + \alpha_n A^n = p(A)$, which was to be shown. ∎

As an immediate consequence of the Cayley–Hamilton Theorem, we have the following results: Let A be an $n \times n$ matrix with characteristic polynomial given by (A.37). Then (i) $A^n = (-1)^{n+1}[\alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1}]$; and (ii) if $f(\lambda)$ is any polynomial in λ, then there exist $\beta_0, \beta_1, \ldots, \beta_{n-1} \in F$ such that

$$f(A) = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1}A^{n-1}. \tag{A.39}$$

Part (i) follows from the Cayley–Hamilton Theorem and from the fact that $\alpha_n = (-1)^n$. To prove part (ii), let $f(\lambda)$ be any polynomial in λ and let $p(\lambda)$ denote the characteristic polynomial of A. From a result for polynomials (called the *division algorithm*), we know that there exist two unique polynomials $g(\lambda)$ and $r(\lambda)$ such that

$$f(\lambda) = p(\lambda)g(\lambda) + r(\lambda), \tag{A.40}$$

where the degree of $r(\lambda) \le n-1$. Now since $p(A) = 0$, we have that $f(A) = r(A)$ and the result follows.

The Cayley–Hamilton Theorem can also be used to express $n \times n$ matrix-valued power series (as well as other kinds of functions) as matrix polynomials of degree $n-1$. Consider in particular the matrix exponential e^{At} defined by

$$e^{At} = \sum_{k=0}^{\infty}(t^k/k!)A^k, t \in (-a, a). \tag{A.41}$$

In view of the Cayley–Hamilton Theorem, we can write

$$f(A) = e^{At} = \sum_{i=0}^{n-1}\alpha_i(t)A^i. \tag{A.42}$$

In the following discussion, we present a method to determine the coefficients $\alpha_i(t)$ in (A.42) [or β_i in (A.39)].

In accordance with (A.38), let $p(\lambda) = \det(A - \lambda I) = \prod_{i=1}^{p}(\lambda_i - \lambda)^{m_i}$ be the characteristic polynomial of A. Also, let $f(\lambda)$ and $g(\lambda)$ be two analytic functions. Now if

$$f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i), \quad l = 0, \ldots, m_i - 1, \quad i = 1, \ldots, p, \tag{A.43}$$

where $f^{(l)}(\lambda_i) = \frac{d^l f}{d\lambda^l}(\lambda)|_{\lambda=\lambda_i}$, $\sum_{i=1}^{p} m_i = n$, then $f(A) = g(A)$. To see this, we note that condition (A.43) written as $(f-g)^l(\lambda_i) = 0$ implies that $f(\lambda) - g(\lambda)$ has $p(\lambda)$ as a factor; i.e., $f(\lambda) - g(\lambda) = w(\lambda)p(\lambda)$ for some analytic function $w(\lambda)$. From the Cayley–Hamilton Theorem we have that $p(A) = O$ and therefore $f(A) - g(A) = O$.

Example A.28. As a specific application of the Cayley–Hamilton Theorem, we evaluate the matrix A^{37}, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$. Since $n = 2$, we assume, in view of (A.39), that A^{37} is of the form $A^{37} = \beta_0 I + \beta_1 A$. The characteristic polynomial of A is $p(\lambda) = (1-\lambda)(2-\lambda)$, and the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. In this case, $f(\lambda) = \lambda^{37}$ and $r(\lambda)$ in (A.40) is $r(\lambda) = \beta_0 + \beta_1\lambda$. To determine β_0 and β_1 we use the fact that $p(\lambda_1) = p(\lambda_2) = 0$ to conclude that $f(\lambda_1) = r(\lambda_1)$ and $f(\lambda_2) = r(\lambda_2)$. Therefore, we have that $\beta_0 + \beta_1 = 1^{37} = 1$ and $\beta_0 + 2\beta_1 = 2^{37}$. Hence, $\beta_1 = 2^{37} - 1$ and $\beta_0 = 2 - 2^{37}$. Therefore, $A^{37} = (2 - 2^{37})I + (2^{37} - 1)A$ or $A^{37} = \begin{bmatrix} 1 & 0 \\ 2^{37} - 1 & 2^{37} \end{bmatrix}$.

Example A.29. Let $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, and let $f(A) = e^{At}, f(\lambda) = e^{\lambda t}$, and $g(\lambda) = \alpha_1\lambda + \alpha_0$. The matrix A has an eigenvalue $\lambda = \lambda_1 = \lambda_2 = 0$ with multiplicity $m_1 = 2$. Conditions (A.43) are given by $f(\lambda_1) = g(\lambda_1) = 1$ and $f^{(1)}(\lambda_1) = g^{(1)}(\lambda_1)$, which imply that $\alpha_0 = 1$ and $\alpha_1 = t$. Therefore,

$$e^{At} = f(A) = g(A) = \alpha_1 A + \alpha_0 I = \begin{bmatrix} -\alpha_1 + \alpha_0 & \alpha_1 \\ -\alpha_1 & \alpha_1 + \alpha_0 \end{bmatrix} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}.$$

A.5.3 Minimal Polynomials

For purposes of motivation, consider the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 4 & -2 \\ 0 & 3 & -1 \end{bmatrix}.$$

The characteristic polynomial of A is $p(\lambda) = (1-\lambda)^2(2-\lambda)$, and we know from the Cayley–Hamilton Theorem that

$$p(A) = O. \tag{A.44}$$

Now let us consider the polynomial $m(\lambda) = (1-\lambda)(2-\lambda) = 2 - 3\lambda + \lambda^2$. Then

$$m(A) = 2I - 3A + A^2 = O. \tag{A.45}$$

Thus, matrix A satisfies (A.45), which is of lower degree than (A.44), the characteristic equation of A.

More generally, it can be shown that for an $n \times n$ matrix A, there exists a unique polynomial $m(\lambda)$ such that (i) $m(A) = O$, (ii) $m(\lambda)$ is *monic* (i.e., if m is an nth-order polynomial in λ, then the coefficient of λ^n is unity), and (iii) if $m'(\lambda)$ is any other polynomial such that $m'(A) = O$, then the degree of $m(\lambda)$ is less or equal to the degree of $m'(\lambda)$ [i.e., $m(\lambda)$ is of the lowest degree such that $m(A) = O$]. The polynomial $m(\lambda)$ is called the *minimal polynomial of A*.

Let $f(\lambda)$ be any polynomial such that $f(A) = O$ (e.g., the characteristic polynomial). Then it is easily shown that $m(\lambda)$ divides $f(\lambda)$ [i.e., there is a polynomial $q(\lambda)$ such that $f(\lambda) = q(\lambda)m(\lambda)$]. In particular, the minimal polynomial of A, $m(\lambda)$, divides the characteristic polynomial of A, $p(\lambda)$. Also, it can be shown that $p(\lambda)$ divides $[m(\lambda)]^n$.

Next, let $p(\lambda)$ be given by

$$p(\lambda) = (\lambda_1 - \lambda)^{m_1}(\lambda_2 - \lambda)^{m_2} \cdots (\lambda_p - \lambda)^{m_p}, \tag{A.46}$$

where $m_1, \ldots, m_p$ are the algebraic multiplicities of the distinct eigenvalues $\lambda_1, \ldots, \lambda_p$ of A, respectively. It can be shown that

$$m(\lambda) = (\lambda - \lambda_1)^{\mu_1}(\lambda - \lambda_2)^{\mu_2} \cdots (\lambda - \lambda_p)^{\mu_p}, \tag{A.47}$$

where $1 \leq \mu_i \leq m_i$, $i = 1, \ldots, p$.

It can also be shown that $(\lambda - \lambda_i)^{\mu_i}$ is the minimal polynomial of the A_i diagonal block in the *Jordan canonical form* of A, which we discuss in the next section. When A has all n distinct eigenvalues, the Jordan canonical form has n diagonal blocks and, therefore, $\mu_i = 1$ and $p(\lambda) = m(\lambda)$. The Jordan canonical form is described in Section A.6 and in [1, Section 2.2].

A.6 Diagonal and Jordan Canonical Form of Matrices

Let A be an $n \times n$ matrix $A \in C^{n \times n}$. The following developement follows [10]. To begin with, let us assume that A has *distinct eigenvalues* $\lambda_1, \ldots, \lambda_n$. Let v_i be an eigenvector of A corresponding to λ_i, $i = 1, \ldots, n$. Then it can be easily shown that the set of vectors $\{v_1, \ldots, v_n\}$ is linearly independent over C, and as such, it can be used as a basis for C^n. Now let $\widetilde{A}$ be the representation of A with respect to the basis $\{v_1, \ldots, v_n\}$. Since the ith column of $\widetilde{A}$ is the representation of $Av_i = \lambda_i v_i$ with respect to the basis $\{v_1, \ldots, v_n\}$, it follows that

$$\widetilde{A} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \triangleq \operatorname{diag}(\lambda_1, \ldots, \lambda_n). \tag{A.48}$$

Since A and $\widetilde{A}$ are matrix representations of the same linear transformation, it follows that A and $\widetilde{A}$ are similar matrices. Indeed, this can be checked by computing

$$\widetilde{A} = P^{-1}AP, \tag{A.49}$$

where $P = [v_1, \ldots, v_n]$ and where the v_i are eigenvectors corresponding to λ_i, $i = 1, \ldots, n$. Note that $AP = \widetilde{A}P$ is true because the ith column of AP is Av_i, which equals $\lambda_i v_i$, the ith column of $\widetilde{A}P$.

When a matrix $\widetilde{A}$ is obtained from a matrix A via a similarity transformation P, we say that matrix A *has been diagonalized*. Now if the matrix A has repeated eigenvalues, then it is not always possible to diagonalize it. In generating a "convenient" basis for C^n in this case, we introduce the concept of *generalized eigenvector*. Specifically, a vector v is called a *generalized eigenvector of rank* k of A, associated with an eigenvalue λ if and only if

$$(A - \lambda I_n)^k v = 0 \quad \text{and} \quad (A - \lambda I_n)^{k-1} v \neq 0, \tag{A.50}$$

where I_n denotes the $n \times n$ identity matrix. Note that when $k = 1$, this definition reduces to the preceding defintion of eigenvector.

Now let v be a generalized eigenvector of rank k associated with the eigenvalue λ. Define

$$\begin{aligned}
v_k &= v, \\
v_{k-1} &= (A - \lambda I_n)v = (A - \lambda I_n)v_k, \\
v_{k-2} &= (A - \lambda I_n)^2 v = (A - \lambda I_n)v_{k-1}, \\
&\vdots \\
v_1 &= (A - \lambda I_n)^{k-1} v = (A - \lambda I_n)v_2.
\end{aligned} \tag{A.51}$$

Then for each i, $1 \leq i \leq k$, v_i is a generalized eigenvector of rank i. We call the set of vectors $\{v_1, \ldots, v_k\}$ a *chain of generalized eigenvectors*.

For generalized eigenvectors, we have the following results:

(i) The generalized eigenvectors $\{v_1, \ldots, v_k\}$ defined in (A.51) are linearly independent.

(ii) The generalized eigenvectors of A associated with different eigenvalues are linearly independent.

(iii) If u and v are generalized eigenvectors of rank k and l, respectively, associated with the same eigenvalue λ, and if u_i and v_j are defined by

$$\begin{aligned}
u_i &= (A - \lambda I_n)^{k-i} u, \quad i = 1, \ldots, k, \\
v_j &= (A - \lambda I_n)^{l-j} v, \quad j = 1, \ldots, l,
\end{aligned}$$

and if u_1 and v_1 are linearly independent, then the generalized eigenvectors $u_1, \ldots, u_k$, $v_1, \ldots, v_l$ are linearly independent.

These results can be used to construct a new basis for C^n such that the matrix representation of A with respect to this new basis is in the *Jordan canonical form* J. We characterize J in the following result: For every complex $n \times n$ matrix A, there exists a nonsingular matrix P such that the matrix

$$J = P^{-1} A P$$

is in the canonical form

$$J = \begin{bmatrix} J_0 & & & 0 \\ & J_1 & & \\ & & \ddots & \\ 0 & & & J_s \end{bmatrix}, \tag{A.52}$$

where J_0 is a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_k$ (not necessarily distinct), i.e.,

$$J_0 = \mathrm{diag}(\lambda_1, \ldots, \lambda_k),$$

and each J_p is an $n_p \times n_p$ matrix of the form

$$J_p = \begin{bmatrix} \lambda_{k+p} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{k+p} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda_{k+p} \end{bmatrix}, \quad p = 1, \ldots, s,$$

where λ_{k+p} need not be different from λ_{k+q} if $p \neq q$ and $k + n_1 + \cdots + n_s = n$. The numbers λ_i, $i = 1, \ldots, k+s$, are the eigenvalues of A. If λ_i is a simple eigenvalue of A, it appears in the block J_0. The blocks $J_0, J_1, \ldots, J_s$ are called *Jordan blocks*, and J is called the *Jordan canonical form.*

Note that a matrix may be similar to a diagonal matrix without having distinct eigenvalues. The identity matrix I is such an example. Also, it can be shown that any real symmetric matrix A has only real eigenvalues (which may be repeated) and is similar to a diagonal matrix.

We now give a procedure for computing a set of basis vectors that yield the Jordan canonical form J of an $n \times n$ matrix A and the required nonsingular transformation P that relates A to J:

1. Compute the eigenvalues of A. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A with multiplicities $n_1, \ldots, n_m$, respectively.
2. Compute n_1 linearly independent generalized eigenvectors of A associated with λ_1 as follows: Compute $(A - \lambda_1 I_n)^i$ for $i = 1, 2, \ldots$ until the rank of $(A - \lambda_1 I_n)^k$ is equal to the rank of $(A - \lambda_1 I_n)^{k+1}$. Find a generalized eigenvector of rank k, say u. Define $u_i = (A - \lambda_1 I_n)^{k-i} u$, $i = 1, \ldots, k$. If $k = n_1$, proceed to step 3. If $k < n_1$, find another linearly independent generalized eigenvector with the largest possible rank; i.e., try to find another generalized eigenvector with rank k. If this is not possible, try $k - 1$, and so forth, until n_1 linearly independent generalized eigenvectors are determined. Note that if $\rho(A - \lambda_1 I_n) = r$, then there are totally $(n - r)$ chains of generalized eigenvectors associated with λ_1.
3. Repeat step 2 for $\lambda_2, \ldots, \lambda_m$.
4. Let $u_1, \ldots, u_k, \ldots$ be the new basis. Observe, from (A.51), that

$$
\begin{aligned}
Au_1 &= \lambda_1 u_1 = [u_1 u_2 \cdots u_k \cdots][\lambda_1, 0, \ldots, 0]^T, \\
Au_2 &= u_1 + \lambda_1 u_2 = [u_1 u_2 \cdots u_k \cdots][1, \lambda_1, 0, \ldots, 0]^T, \\
&\ \ \vdots \\
Au_k &= u_{k-1} + \lambda_1 u_k = [u_1 u_2 \cdots u_k \cdots][0, \ldots, 0, 1, \lambda_1, 0, \ldots, 0]^T,
\end{aligned}
$$

with λ_1 in the kth position, which yields the representation J in (A.52) of A with respect to the new basis, where the $k \times k$ matrix J_1 is given by

$$
J_1 = \begin{bmatrix} \lambda_1 & 1 & \cdots & 0 \\ 0 & \lambda_1 & & \vdots \\ & & \ddots & 1 \\ 0 & & \cdots & \lambda_1 \end{bmatrix}.
$$

Note that each chain of generalized eigenvectors generates a Jordan block whose order equals the length of the chain.

5. The similarity transformation that yields $J = Q^{-1}AQ$ is given by $Q = [u_1, \ldots, u_k, \ldots]$.
6. Rearrange the Jordan blocks in the desired order to yield (A.52) and the corresponding similarity transformation P.

Example A.30. The characteristic equation of the matrix

$$
A = \begin{bmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
$$

is given by

$$
\det(A - \lambda I) = (\lambda - 2)^5 \lambda = 0.
$$

Thus, A has eigenvalue $\lambda_2 = 2$ with multiplicity 5 and eigenvalue $\lambda_1 = 0$ with multiplicity 1.

Now compute $(A - \lambda_2 I)^i$, $i = 1, 2, \ldots,$ as follows:

$$
(A - 2I) = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \rho(A - 2I) = 4,
$$

$$(A-2I)^2 = \begin{bmatrix} 0&0&2&2&0&0 \\ 0&0&2&2&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&2&-2 \\ 0&0&0&0&-2&2 \end{bmatrix} \qquad \text{and} \quad \rho(A-2I)^2 = 2,$$

$$(A-2I)^3 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&-4&4 \\ 0&0&0&0&4&-4 \end{bmatrix} \qquad \text{and} \quad \rho(A-2I)^3 = 1,$$

$$(A-2I)^4 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&8&-8 \\ 0&0&0&0&-8&8 \end{bmatrix} \qquad \text{and} \quad \rho(A-2I)^4 = 1.$$

Since $\rho(A-2I)^3 = \rho(A-2I)^4$, we stop at $(A-2I)^3$. It can be easily verified that if $u = [0\ 0\ 1\ 0\ 0\ 0]^T$, then $(A-2I)^3u = 0$ and $(A-2I)^2u = [2\ 2\ 0\ 0\ 0\ 0]^T \neq 0$. Therefore, u is a generalized eigenvector of rank 3. So we define

$$\begin{aligned} u_1 &\triangleq (A-2I)^2u = \begin{bmatrix} 2\ 2\ 0\ 0\ 0\ 0 \end{bmatrix}^T, \\ u_2 &\triangleq (A-2I)u = \begin{bmatrix} 1\ -1\ 0\ 0\ 0\ 0 \end{bmatrix}^T, \\ u_3 &\triangleq u = \begin{bmatrix} 0\ 0\ 1\ 0\ 0\ 0 \end{bmatrix}^T. \end{aligned}$$

Since we have only three generalized eigenvectors for $\lambda_2 = 2$ and since the multiplicity of $\lambda_2 = 2$ is five, we have to find two more linearly independent eigenvectors for $\lambda_2 = 2$. So let us try to find a generalized eigenvector of rank 2. Let $v = [0\ 0\ 1\ \ -1\ 1\ 1]^T$. Then $(A-2I)v = [0\ 0\ 2\ \ -2\ 0\ 0]^T \neq 0$ and $(A-2I)^2v = 0$. Moreover, $(A-2I)v$ is linearly independent of u_1, and hence, we have another linearly independent generalized eigenvector of rank 2. Define

$$v_2 \triangleq v = \begin{bmatrix} 0\ 0\ 1\ -1\ 1\ 1 \end{bmatrix}^T$$

and

$$v_1 = (A-2I)v = \begin{bmatrix} 0\ 0\ 2\ -2\ 0\ 0 \end{bmatrix}^T.$$

Next, we compute an eigenvector associated with $\lambda_1 = 0$. Since $w = [0\ 0\ 0\ 0\ 1\ \ -1]^T$ is a solution of $(A-\lambda_1 I)w = 0$, the vector w will do.

Finally, with respect to the basis $w_1, u_1, u_2, u_3, v_1, v_2$, the Jordan canonical form of A is given by

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \tag{A.53}$$

and

$$P = \begin{bmatrix} w_1 \; u_1 \; u_2 \; u_3 \; v_1 \; v_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{A.54}$$

The correctness of P is easily checked by computing $PJ = AP$.

A.7 Normed Linear Spaces

In the following discussion, we require for (V, F) that F be either the field of real numbers R or the field of complex numbers C. For such linear spaces we say that a function $\| \cdot \|: V \to R^+$ is a *norm* if

(N-i) $\| x \| \geq 0$ for every *vector* $x \in V$ and $\| x \| = 0$ if and only if x is the null vector (i.e., $x = 0$);

(N-ii) for every scalar $\alpha \in F$ and for every vector $x \in V$, $\| \alpha x \| = |\alpha| \, \| x \|$, where $|\alpha|$ denotes the absolute value of α when $F = R$ and the modulus when $F = C$; and

(N-iii) for every x and y in V, $\| x + y \| \leq \| x \| + \| y \|$. (This inequality is called the *triangle inequality*.)

We call a vector space on which a norm has been defined a *normed vector space* or a *normed linear space.*

Example A.31. On the linear space (R^n, R), we define for every $x = (x_1, \ldots, x_n)^T$,

$$\| x \|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}, \quad 1 \leq p < \infty \tag{A.55}$$

and

$$\| x \|_\infty = \max\{|x_i| : \; 1 \leq i \leq n\}. \tag{A.56}$$

Using Minkowski's Inequality for finite sums, see (A.5), it is an easy matter to show that for every p, $1 \leq p \leq \infty$, $\| \cdot \|_p$ is a norm on R^n. In addition to $\| \cdot \|_\infty$, of particular interest to us will be the cases $p = 1$ and $p = 2$; i.e.,

$$\| x \|_1 = \sum_{i=1}^{n} |x_i| \tag{A.57}$$

and

$$\| x \|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} . \tag{A.58}$$

The norm $\| \cdot \|_1$ is sometimes referred to as the *taxicab norm* or *Manhattan norm*, whereas, $\| \cdot \|_2$ is called the *Euclidean norm*. The linear space (R^n, R) with norm $\| \cdot \|_2$ is called a *Euclidean vector space*.

The foregoing norms are related by the inequalities

$$\| x \|_\infty \leq \| x \|_1 \leq n \| x \|_\infty, \tag{A.59}$$

$$\| x \|_\infty \leq \| x \|_2 \leq \sqrt{n} \| x \|_\infty, \tag{A.60}$$

$$\| x \|_2 \leq \| x \|_1 \leq \sqrt{n} \| x \|_2 . \tag{A.61}$$

Also, for $p = 2$, we obtain from the Hölder Inequality for finite sums, (A.4), the *Schwarz Inequality*

$$|x^T y| = |\sum_{i=1}^{n} x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} |y_i|^2 \right)^{1/2} \tag{A.62}$$

for all $x, y \in R^n$.

The assertions made in the above example turn out to be also true for the space (C^n, C). We ask the reader to verify these relations.

Example A.32. On the space l_p given in Example A.5, let

$$\| x \|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} , \quad 1 \leq p < \infty,$$

and

$$\| x \|_\infty = \sup_i |x_i|.$$

Using Minkowski's Inequality for infinite sums, (A.5), it is an easy matter to show that $\| \cdot \|_p$ is a norm for every p, $1 \leq p \leq \infty$.

Example A.33. On the space given in Example A.7, let

$$\| x \|_p = \left(\int_a^b |x(t)|^p dt \right)^{1/p} , \quad 1 \leq p < \infty.$$

Using Minkowski's Inequality for integrals, (A.10), see Example A.7, it can readily be verified that $\| \cdot \|_p$ is a norm for every p, $1 \leq p < \infty$. Also, on the space of continuous functions given in Example A.8, assume that (A.11) holds. Then

$$\| x \|_\infty = \max_{a \leq t \leq b} |x(t)|$$

is easily shown to determine a norm. Furthermore, expression (A.12) can also be used to determine a norm.

Example A.34. We can also define the norm of a matrix. To this end, consider the set of real $m \times n$ matrices, $R^{m \times n} = V$ and $F = R$. It is easily verified that $(V, F) = (R^{m \times n}, R)$ is a vector space, where vector addition is defined as matrix addition and multiplication of vectors by scalars is defined as multiplication of matrices by scalars.

For a given norm $\| \cdot \|_u$ on R^n and a given norm $\| \cdot \|_v$ on R^m, we define $\| \cdot \|_{vu}$: $R^{m \times n} \to R^+$ by

$$\| A \|_{vu} = \sup\{\| Ax \|_v : x \in R^n \text{ with } \| x \|_u = 1\}. \tag{A.63}$$

It is easily verified that

(M-i) $\| Ax \|_v \leq \| A \|_{vu} \| x \|_u$ for any $x \in R^n$,
(M-ii) $\| A + B \|_{vu} \leq \| A \|_{vu} + \| B \|_{vu}$,
(M-iii) $\| \alpha A \|_{vu} = |\alpha| \, \| A \|_{vu}$ for all $\alpha \in R$,
(M-iv) $\| A \|_{vu} \geq 0$ and $\| A \|_{vu} = 0$ if and only if A is the zero matrix (i.e., $A = 0$),
(M-v) $\| A \|_{vu} \leq \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|$ for any p-vector norms defined on R^n and R^m.

Properties (M-ii) to (M-iv) clearly show that $\| \cdot \|_{vu}$ defines a norm on $R^{m \times n}$ and justifies the use of the term *matrix norm.* Since the matrix norm $\| \cdot \|_{vu}$ depends on the choice of the vector norms, $\| \cdot \|_u$, and $\| \cdot \|_v$, defined on $U \triangleq R^n$ and $V \triangleq R^m$, respectively, we say that the matrix norm $\| \cdot \|_{uv}$ is *induced* by the vector norms $\| \cdot \|_u$ and $\| \cdot \|_v$. In particular, if $\| \cdot \|_u = \| \cdot \|_p$ and $\| \cdot \|_v = \| \cdot \|_p$, then the notation $\| A \|_p$ is frequently used to denote the norm of A.

As a specific case, let $A = [a_{ij}] \in R^{m \times n}$. Then it is easily verified that

$$\| A \|_1 = \max_j \left(\sum_{i=1}^m |a_{ij}| \right),$$

$$\| A \|_2 = [\max \lambda(A^T A)]^{1/2},$$

where $\max \lambda(A^T A)$ denotes the largest eigenvalue of $A^T A$ and

$$\| A \|_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right).$$

When it is clear from context which vector spaces and vector norms are being used, the indicated subscripts on the matrix norms are usually not used. For example, if $A \in R^{m\times n}$ and $B \in R^{n\times k}$, it can be shown that

(M-vi) $\| AB \| \leq \| A \| \, \| B \|$.

In (M-vi) we have omitted subscripts on the matrix norms to indicate inducing vector norms.

We conclude by noting that it is possible to define norms on $(R^{m\times n}, R)$ that need not be induced by vector norms. Furthermore, the entire discussion given in Example A.34 holds also for norms defined on complex spaces, e.g., $(C^{m\times n}, C)$.

A.8 Some Facts from Matrix Algebra

Determinants

We recall that the determinant of a matrix $A = [a_{ij}] \in R^{n\times n}$, $\det A$, can be evaluated by the relation

$$\det A = \sum_j a_{ij} d_{ij} \quad \text{for any} \quad i = 1, 2, \ldots, n,$$

where $d_{ij} = (-1)^{i+j} \det A_{ij}$ and A_{ij} is the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A. The term d_{ij} is the *cofactor of A corresponding to a_{ij}* and $\det A_{ij}$ is the *ijth minor of the matrix*. The *principal minors* of A are obtained by letting $i = j$, i, $j = 1, \ldots n$.

If any column (or row) of A is multiplied by a scalar k, then the determinant of the new matrix is $k \det A$. If every entry is multiplied by k, then the determinant of the new matrix is $k^n \det A$. Also,

$$\det A^T = \det A \quad \text{where } A^T \text{ is the transpose of } A.$$

Determinants of Products

$\det AB = \det A \det B$ when A and B are square matrices, and
$\det[I_m - AB] = \det[I_n - BA]$ where $A \in R^{m\times n}$ and $B \in R^{n\times m}$.

Determinants of Block Matrices

$$\det \begin{bmatrix} \overset{m\times m}{A} & \overset{m\times n}{B} \\ \underset{n\times m}{C} & \underset{n\times n}{D} \end{bmatrix} = \det A \det[D - CA^{-1}B], \quad \det A \neq 0$$

$$= \det D \det[A - BD^{-1}C], \quad \det D \neq 0.$$

Inverse A^{-1} of A

If $A \in R^{n \times n}$ and if A is nonsingular (i.e., $\det A \neq 0$), then $AA^{-1} = A^{-1}A = I$.

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A),$$

where $\operatorname{adj}(A) = [d_{ij}]^T$ is the *adjoint* of A, where d_{ij} is the cofactor of A corresponding to a_{ij}. When

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix, then

$$A^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $A \in R^{m \times m}$ and $C \in R^{n \times n}$, if A and C are nonsingular, and if $B \in R^{m \times n}$ and $D \in R^{n \times m}$, then

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}.$$

For example

$$\left[I + C(sI - A)^{-1}B\right]^{-1} = I - C(sI - A + BC)^{-1}B.$$

When $A \in R^{m \times n}$ and $B \in R^{n \times m}$, then

$$(I_m + AB)^{-1} = I_m - A(I_n + BA)^{-1}B.$$

Sylvester Rank Inequality

If $X \in R^{p \times n}$ and $Y \in R^{n \times m}$, then

$$\operatorname{rank} X + \operatorname{rank} Y - n \leq \operatorname{rank}(xy) \leq \min\{\operatorname{rank} X, \operatorname{rank} Y\}.$$

A.9 Numerical Considerations

Computing the rank of the controllability matrix $[B, AB, \ldots, A^{n-1}B]$, the eigenvalues of A, or the zeros of the system $\{A, B, C, D\}$ typically requires the use of a digital computer. When this is the case, one must deal with the selection of an algorithm and interpret numerical results. In doing so, two issues arise that play important roles in numerical computations using a computer, namely, the *numerical stability or instability* of the computational method used, and how *well or ill conditioned* the problem is numerically.

An example of a problem that can be ill conditioned is the problem of calculating the roots of a polynomial, given its coefficients. This is so because for certain polynomials, small variations in the values of the coefficients, introduced say via round-off errors, can lead to great changes in the roots of the polynomial. That is to say, the roots of a polynomial can be very sensitive to changes in its coefficients. Note that ill conditioning is a property of the problem to be solved and does not depend on the floating-point system used in the computer, nor on the particular algorithm being implemented.

A computational method is numerically stable if it yields a solution that is near the true solution of a problem with slightly changed data. An example of a numerically unstable method to compute the roots of $ax^2 + 2bx + c = 0$ is the formula $(-b \pm \sqrt{(b^2 - ac)})/a$, which for certain parameters a, b, c may give erroneous results in finite arithmetic. This instability is caused by the subtraction of two approximately equal large numbers in the numerator when $b^2 >> ac$. Note that the roots may be calculated in a numerically stable way, using the mathematically equivalent, but numerically very different, expression $c/(-b \mp \sqrt{(b^2 - ac)})$.

We would like of course, to always use numerically stable methods, and we would prefer to have well-conditioned problems. In what follows, we discuss briefly the problem of solving a set of algebraic equations given by $Ax = b$. We will show that a measure of how ill conditioned a given problem is, is the size of the *condition number* (to be defined) of the matrix A. There are many algorithms to numerically solve $Ax = b$, and we will briefly discuss numerically stable ones. Singular values, singular value decomposition, and the least-squares problem are also discussed.

A.9.1 Solving Linear Algebraic Equations

Consider the set of linear algebraic equations given by

$$Ax = b, \tag{A.64}$$

where $A \in R^{m \times n}$, $b \in R^m$ and $x \in R^n$ is to be determined.

Existence and Uniqueness of Solutions

See also Sec. A.3, (A.22)–(A.29). Given (A.64), for a given b, a solution x exists if and only if $b \in \mathcal{R}(A)$, or equivalently, if and only if

$$\rho\left([A, b]\right) = \rho(A). \tag{A.65}$$

Every solution of (A.64) can be expressed as a sum

$$x = x_p + x_h, \tag{A.66}$$

where x_p is a specific solution and x_h satisfies $Ax_h = 0$. There are

$$\dim \mathcal{N}(A) = n - \rho(A) \tag{A.67}$$

linearly independent solutions of the systems of equations $Ax = 0$.

$Ax = b$ has a unique solution if and only if (A.65) is satisfied and

$$\rho(A) = n \leq m. \tag{A.68}$$

A solution exists for any b if and only if $\rho(A) = m$. In this case, a solution may be found using

$$x = A^T(AA^T)^{-1}b.$$

When $\rho(A) = m = n$, then A is nonsingular and the unique solution is

$$x = A^{-1}b. \tag{A.69}$$

It is of interest to know the effects of small variations of A and b to the solution x of this system of equations. Note that such variations may be introduced, for example, by rounding errors when calculating a solution or by noisy data.

Condition Number

Let $A \in R^{n \times n}$ be nonsingular. If A is known exactly and b has some uncertainty Δb associated with it, then $A(x + \Delta x) = b + \Delta b$. It can then be shown that the variation in the solution x is bounded by

$$\frac{\| \Delta x \|}{\| x \|} \leq \text{cond}(A) \frac{\| \Delta b \|}{\| b \|}, \tag{A.70}$$

where $\| \cdot \|$ denotes any vector norm (and consistent matrix norm) and $\text{cond}(A)$ denotes the *condition number of* A, where $\text{cond}(A) \triangleq \| A \| \, \| A^{-1} \|$. Note that

$$\text{cond}(A) = \sigma_{\max}(A)/\sigma_{\min}(A), \tag{A.71}$$

where $\sigma_{\max}(A)$ and $\sigma_{\min}(A)$ are the maximum and minimum singular values of A, respectively (see the next section). From the property of matrix norms, $\| AA^{-1} \| \leq \| A \| \, \| A^{-1} \|$, it follows that $\text{cond}(A) \geq 1$. This also follows from the expression involving singular values. If $\text{cond}(A)$ is small, then A is said to be *well conditioned* with respect to the problem of solving linear equations. If $\text{cond}(A)$ is large, then A is *ill conditioned* with respect to the problem of solving linear equations. In this case the relative uncertainty in the solution ($\| \Delta x \| / \| x \|$) can be many times the relative uncertainty in b ($\| \Delta b \| / \| b \|$). This is of course undesirable. Similar results can be derived when variations in both b and A are considered, i.e., when b and A become $b + \Delta b$ and $A + \Delta A$. Note that the conditioning of A, and of the given problem, is independent of the algorithm used to determine a solution.

The condition number of A provides a measure of the distance of A to the set of singular (reduced rank) matrices. In particular, if $\| \Delta A \|$ is the norm of the smallest perturbation ΔA such that $A + \Delta A$ is singular, and is denoted by

$d(A)$, then $dA/\parallel A \parallel = 1/\operatorname{cond}(A)$. Thus, a large condition number indicates a short distance to a singularity and it is not surprising that this implies great sensitivity of the numerical solution x of $Ax = b$ to variations in the problem data.

The condition number of A plays a similar role in the case when A is not square. It can be determined in terms of the singular values of A defined in the next subsection.

Computational Methods

The system of equations $Ax = b$ is easily solved if A has some special form (e.g., if it is diagonal or triangular). Using the method of *Gaussian elimination*, any nonsingular matrix A can be reduced to an upper triangular matrix U. These operations can be represented by premultiplication of A by a sequence of lower triangular matrices. It can then be shown that A can be represented as

$$A = LU, \tag{A.72}$$

where L is a lower triangular matrix with all diagonal elements equal to 1 and U is an upper triangular matrix. The solution of $Ax = b$ is then reduced to the solution of two systems of equations with triangular matrices, $Ly = b$ and $Ux = y$. This method of solving $Ax = b$ is based on the decomposition (A.72) of A, which is called the *LU decomposition* of A.

If A is a symmetric positive definite matrix, then it may be represented as

$$A = U^T U, \tag{A.73}$$

where U is an upper triangular matrix. This is known as the *Cholesky decomposition* of a positive definite matrix. It can be obtained using a variant of Gaussian elimination. Note that this method requires half of the operations necessary for Gaussian elimination on an arbitrary nonsingular matrix A, since A is symmetric.

Now consider the system of equations $Ax = b$, where $A \in R^{m\times n}$, and let $\operatorname{rank} A = n (\leq m)$. Then

$$A = Q \begin{bmatrix} R \\ O \end{bmatrix} = [Q_1, Q_2] \begin{bmatrix} R \\ O \end{bmatrix} = Q_1 R, \tag{A.74}$$

where Q is an *orthogonal matrix* ($Q^T = Q^{-1}$) and $R \in R^{n\times n}$ is an upper triangular matrix of full rank n. Expression (A.74) is called the *QR decomposition* of A. When $\operatorname{rank} A = r$, the QR decomposition of A is expressed as

$$AP = Q \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}, \tag{A.75}$$

where Q is orthogonal, $R_1 \in R^{r\times r}$ is nonsingular and upper triangular, and P is a permutation matrix that represents the moving of the columns during the reduction (in $Q^T AP$).

The QR decomposition can be used to determine solutions of $Ax = b$. In particular, consider $A \in R^{m\times n}$ with $\operatorname{rank} A = n (\leq m)$ and assume that a solution exists. First, determine the QR decomposition of A given in (A.74). Then $Q^T Ax = Q^T b$ or $\begin{bmatrix} R \\ 0 \end{bmatrix} x = Q^T b$ (since $Q^T = Q^{-1}$) or $Rx = c$. Solve this system of equations, where R is triangular and $c = [I_n, 0]Q^T b$. In the general case when $\operatorname{rank}(A) = r \leq \min(n, m)$, determine the QR decomposition of A (2.7) and assume that a solution exists. The solutions are given by $x = P\begin{bmatrix} R_1^{-1}(c - R_2 y) \\ y \end{bmatrix}$, $c = [I_r, 0]Q^T b$, where $y \in R^{m-r}$ is arbitrary.

A related problem is the *linear least-squares problem* where a solution x of the system of equations $Ax = b$ is to be found that minimizes $\| b - Ax \|_2$. This is a more general problem than simply solving $Ax = b$, since solving it provides the "best" solution in the above sense, even when an exact solution does not exist. The least-squares problem is discussed further in a subsequent subsection.

A.9.2 Singular Values and Singular Value Decomposition

The singular values of a matrix and the *Singular Value Decomposition Theorem* play a significant role in a number of problems of interest in the area of systems and control, from the computation of solutions of linear systems of equations, to computations of the norm of transfer matrices at specified frequencies, to model reduction, and so forth. In what follows, we provide a brief description of some basic results and we introduce some terminology.

Consider $A \in \mathcal{C}^{n\times n}$, and let $A^* = \bar{A}^T$; i.e., the complex conjugate transpose of A. $A \in \mathcal{C}^{n\times n}$ is said to be *Hermitian* if $A^* = A$. If $A \in R^{n\times n}$, then $A^* = A^T$ and if $A = A^T$, then A is *symmetric*. $A \in \mathcal{C}^{n\times n}$ is *unitary* if $A^* = A^{-1}$. In this case $A^*A = AA^* = I_n$. If $A \in R^{n\times n}$, then $A^* = A^T$ and if $A^T = A^{-1}$, i.e., if $A^T A = AA^T = I_n$, then A is *orthogonal*.

Singular Values

Let $A \in \mathcal{C}^{m\times n}$, and consider $AA^* \in \mathcal{C}^{m\times m}$. Let λ_i, $i = 1, \dots, m$ denote the eigenvalues of AA^*, and note that these are all real and nonnegative numbers. Assume that $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r \geq \cdots \geq \lambda_m$. Note that if $r = \operatorname{rank} A = \operatorname{rank}(AA^*)$, then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_m = 0$. *The singular values* σ_i *of* A are the positive square roots of λ_i, $i = 1, \dots, \min(m, n)$. In fact, the nonzero singular values of A are

$$\sigma_i = (\lambda_i)^{1/2}, \quad i = 1, \dots, r, \tag{A.76}$$

where $r = \operatorname{rank} A$, whereas the remaining $(\min(m, n) - r)$ of the singular values are zero. Note that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_{\min(m,n)} = 0$. The singular values could also have been found as the square roots of the eigenvalues of $A^*A \in \mathcal{C}^{n\times n}$ (instead of $AA^* \in \mathcal{C}^{m\times m}$). To see this, consider the following result.

Lemma A.35. *Let $m \geq n$. Then*

$$|\lambda I_m - AA^*| = \lambda^{m-n}|\lambda I_n - A^*A|; \tag{A.77}$$

i.e., all eigenvalues of A^*A are eigenvalues of AA^* that also has $m-n$ additional eigenvalues at zero. Thus $AA^* \in \mathcal{C}^{m\times m}$ and $A^*A \in \mathcal{C}^{n\times n}$ have precisely the same r nonzero eigenvalues ($r = \operatorname{rank} A$); their remaining eigenvalues, $(m-r)$ for AA^* and $(n-r)$ for A^*A, are all at zero. Therefore, either AA^* or A^*A can be used to determine the r nonzero singular values of A. All remaining singular values are zero.

Proof. The proof is based on Schur's formula for determinants. In particular, we have

$$\begin{aligned} D(\lambda) &= \begin{vmatrix} \lambda^{1/2} I_m & A \\ A^* & \lambda^{1/2} I_n \end{vmatrix} = |\lambda^{1/2} I_m|\,|\lambda^{1/2} I_n - A^*\lambda^{-1/2} I_m A| \\ &= |\lambda^{1/2} I_m|\,|\lambda^{-1/2} I_n|\,|\lambda I_n - A^*A| \\ &= \lambda^{\frac{m-n}{2}} \cdot |\lambda I_n - A^*A|, \end{aligned} \tag{A.78}$$

where Schur's formula was applied to the $(1,1)$ block of the matrix. If it is applied to the $(2,2)$ block, then

$$D(\lambda) = \lambda^{\frac{n-m}{2}} \cdot |\lambda I_m - AA^*|. \tag{A.79}$$

Equating (A.78) and (A.79) we obtain $|\lambda I_m - AA^*| = \lambda^{m-n}|\lambda I_n - A^*A|$, which is (A.78). ∎

Example A.36. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in R^{2\times 3}$. Here $\operatorname{rank} A = r = 1$, $\lambda_i(AA^*) = \lambda_i\left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \lambda_i\left(\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\right) = \{5, 0\}$ and $\lambda_1 = 5, \lambda_2 = 0$. Also, $\lambda_i(A^*A) = \lambda_i\left(\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \lambda_i\left(\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$, and $\lambda_1 = 5, \lambda_2 = 0$, and $\lambda_3 = 0$. The only nonzero singular value is $\sigma_1 = \sqrt{\lambda_1} = +\sqrt{5}$. The remaining singular values are zero.

There is an important relation between the singular values of A and its induced Hilbert or 2-norm, also called the spectral norm $\| A \|_2 = \| A \|_s$. In particular,

$$\| A \|_2 \ (= \| A \|_s) = \sup_{\|x\|_2=1} \| Ax \|_2 = \max_i \{(\lambda_i(A^* A))^{1/2}\} = \bar{\sigma}(A), \quad \text{(A.80)}$$

where $\bar{\sigma}(A)$ denotes the largest singular value of A. Using the inequalities that are axiomatically true for induced norms, it is possible to establish relations between singular values of various matrices that are useful in MIMO control design. The significance of the singular values of a gain matrix $A(jw)$ is discussed later in this section.

There is an interesting relation between the eigenvalues and the singular values of a (square) matrix. Let λ_i, $i = 1, \dots, n$ denote the eigenvalues of $A \in R^{n\times n}$, let $\underline{\lambda}(A) = \min_i |\lambda_i|$, and let $\overline{\lambda}(A) = \max_i |\lambda_i|$. Then

$$\underline{\sigma}(A) \le \underline{\lambda}(A) \le \overline{\lambda}(A) \le \overline{\sigma}(A). \quad \text{(A.81)}$$

Note that the ratio $\bar{\sigma}(A)/\underline{\sigma}(A)$, i.e., the ratio of the largest and smallest singular values of A, is called the *condition number of* A, and is denoted by cond(A). This is a very useful measure of how well conditioned a system of linear algebraic equations $Ax = b$ is (refer to the discussion of the previous section). The singular values provide a reliable way of determining how far a square matrix is from being singular, or a nonsquare matrix is from losing rank. This is accomplished by examining how close to zero $\underline{\sigma}(A)$ is. In contrast, the eigenvalues of a square matrix are not a good indicator of how far the matrix is from being singular, and a typical example in the literature to illustrate this point is an $n \times n$ lower triangular matrix A with -1's on the diagonal and $+1$'s everywhere else. In this case, $\underline{\sigma}(A)$ behaves as $1/2^n$ and the matrix is nearly singular for large n, whereas all of its eigenvalues are at -1. In fact, it can be shown that by adding $1/2^{n-1}$ to every element in the first column of A results in an exactly singular matrix (try it for $n = 2$).

Singular Value Decomposition

Let $A \in C^{m\times n}$ with $\operatorname{rank} A = r \le \min(m, n)$. Let $A^* = \bar{A}^T$, the complex conjugate transpose of A.

Theorem A.37. *There exist unitary matrices $U \in C^{m\times n}$ and $V \in C^{n\times n}$ such that*

$$A = U\Sigma V^*, \quad \text{(A.82)}$$

where $\Sigma = \left[\begin{array}{c|c} \Sigma_r & 0_{r\times(n-r)} \\ \hline 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{array}\right]$ *with* $\Sigma_r = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \in R^{r\times r}$ *selected so that* $\sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_r > 0$.

Proof. For the proof, see for example, Golub and Van Loan [7], and Patel et al. [11]. ∎

Let $U = [U_1, U_2]$ with $U_1 \in C^{m\times r}, U_2 \in C^{m\times(m-r)}$ and $V = [V_1, V_2]$ with $V_1 \in C^{n\times r}, V_2 \in C^{n\times(n-r)}$. Then

$$A = U\Sigma V^* = U_1 \Sigma_r V_1^*. \tag{A.83}$$

Since U and V are unitary, we have

$$U^*U = \begin{bmatrix} U_1^* \\ U_2^* \end{bmatrix} [U_1,\ U_2] = I_m, U_1^* U_1 = I_r \tag{A.84}$$

and

$$V^*V = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} [V_1,\ V_2] = I_n, V_1^* V_1 = I_r. \tag{A.85}$$

Note that the columns of U_1 and V_1 determine orthonormal bases for $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. Now

$$AA^* = (U_1\Sigma_r V_1^*)(V_1\Sigma_r U_1^*) = U_1 \Sigma_r^2 U_1^*, \tag{A.86}$$

from which we have

$$AA^*U_1 = U_1\Sigma_r^2 U_1^* U_1 = U_1 \Sigma_r^2. \tag{A.87}$$

If u_i, $i = 1, \ldots, r$, is the ith column of U_1, i.e., $U_1 = [u_1, u_2, \ldots, u_r]$, then

$$AA^* u_i = \sigma_i^2 u_i, \quad i = 1, \ldots, r. \tag{A.88}$$

This shows that the σ_i^2 are the r nonzero eigenvalues of AA^*; i.e., σ_i, $i = 1, \ldots, r$, are the nonzero singular values of A. Furthermore, u_i, $i = 1, \ldots, r$, are the eigenvectors of AA^* corresponding to σ_i^2. They are the *left singular vectors of* A. Note that the u_i are orthonormal vectors (in view of $U_1^* U_1 = I_r$). Similarly,

$$A^*A = (V_1\Sigma_r U_1^*)(U_1\Sigma_r V_1^*) = V_1 \Sigma_r^2 V_1^*, \tag{A.89}$$

from which we obtain

$$A^*AV_1 = V_1\Sigma_r^2 V_1^* V_1 = V_1 \Sigma_r^2. \tag{A.90}$$

If v_i, $i = 1, \ldots, r$, is the ith column of V_1, i.e., $V_1 = [v_1, v_2, \ldots, v_r]$, then

$$A^*Av_i = \sigma_i^2 v_i, \quad i = 1, 2, \ldots, r. \tag{A.91}$$

The vectors v_i are the eigenvectors of A^*A corresponding to the eigenvalues σ_i^2. They are the *right singular vectors* of A. Note that the v_i are orthonormal vectors (in view of $V_1^* V_1 = I_r$).

The singular values are unique, whereas the singular vectors are *not*. To see this, consider

$$\widehat{V}_1 = V_1 \,\mathrm{diag}(e^{j\theta_i}) \quad \text{and} \quad \widehat{U}_1 = U_1 \,\mathrm{diag}(e^{-j\theta_i}).$$

Their columns are also singular vectors of A.

Note also that $A = U_1 \Sigma_r V_1^*$ implies that

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* . \tag{A.92}$$

The significance of the singular values of a gain matrix $A(jw)$ is now briefly discussed. This is useful in the control theory of MIMO systems. Consider the relation between signals y and v, given by $y = Av$. Then

$$\max_{\|v\|_2 \neq 0} \frac{\| y \|_2}{\| v \|_2} = \max_{\|v\|_2 \neq 0} \frac{\| Av \|_2}{\| v \|_2} = \bar{\sigma}(A)$$

or

$$\max_{\|v\|_2 = 1} \| y \|_2 = \max_{\|v\|_2 = 1} \| Av \|_2 = \bar{\sigma}(A). \tag{A.93}$$

Thus, $\bar{\sigma}(A)$ yields the maximum amplification, in energy terms (2-norm), when the transformation A operates on a signal v. Similarly,

$$\min_{\|v\|_2 = 1} \| y \|_2 = \min_{\|v\|_2 = 1} \| Av \|_2 = \underline{\sigma}(A). \tag{A.94}$$

Therefore,

$$\underline{\sigma}(A) \leq \frac{\| Av \|_2}{\| v \|_2} \leq \bar{\sigma}(A), \tag{A.95}$$

where $\| v \|_2 \neq 0$. Thus the gain (energy amplification) is bounded from above and below by $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$, respectively. The exact value depends on the direction of v.

To determine the particular directions of vectors v for which these (max and min) gains are achieved, consider (A.92) and write

$$y = Av = \sum_{i=1}^{r} \sigma_i u_i v_i^* v. \tag{A.96}$$

Notice that $|v_i^* v| \leq ||v_i|| ||v|| = ||v||$, since $||v_i|| = 1$, with equality holding only when $v = \alpha v_i$, $\alpha \in C$. Therefore, to maximize, consider v along the singular value directions v_i and let $v = \alpha v_i$ with $|\alpha| = 1$ so that $||v|| = 1$. Then in view of $v_i^* v_j = 0$, $i \neq j$ and $v_i^* v_j = 1$, $i = j$, we have that $y = Av = \alpha A v_i = \alpha \sigma_i u_i$ and $||y||_2 = ||Av||_2 = \sigma_i$, since $||u_i||_2 = 1$. Thus, the maximum possible gain is σ_1; i.e., $\max_{\|v\|_2=1} ||y||_2 = \max_{\|v\|_2=1} ||Av||_2 = \sigma_1 (= \bar{\sigma}(A))$, as was shown above. This maximum gain occurs when v is along the right singular vector v_1. Then $Av = Av_1 = \sigma_1 u_1 = y$ in view of (A.92); i.e., the projection is along the left singular vector u_1, also of the same singular value σ_1. Similarly, for the minimum gain, we have $\sigma_r = \underline{\sigma}(A) = \min_{\|v\|_2=1} ||y||_2 = \min_{\|v\|_2=1} ||Av||_2$; in which case, $Av = Av_r = \sigma_r u_r = y$.

Additional interesting properties include

$$\mathcal{R}(A) = \mathcal{R}(U_1) = \operatorname{span}\{u_1, \ldots, u_r\}, \tag{A.97}$$

$$\mathcal{N}(A) = \mathcal{R}(V_2) = \operatorname{span}\{v_{r+1}, \ldots, v_n\}, \tag{A.98}$$

where $U = [u_1, \ldots, u_r, u_{r+1}, \ldots, u_m] = [U_1, U_2]$, $V = [v_1, \ldots, v_r, v_{r+1}, \ldots, v_n] = [V_1, V_2]$.

A.9.3 Least-Squares Problem

Consider now *the least-squares problem* where a solution x to the system of linear equations $Ax = b$ is to be determined that minimizes $\| b - Ax \|_2$. Write $\min_x \| b - Ax \|_2^2 = \min_x (b - Ax)^T(b - Ax) = \min_x (x^T A^T Ax - 2b^T Ax + b^T b)$. Then $\nabla_x (x^T A^T Ax - 2b^T Ax + b^T b) = 2A^T Ax - 2A^T b = 0$ implies that the x, which minimizes $\| b - Ax \|_2$, is a solution of

$$A^T Ax = A^T b. \tag{A.99}$$

Rewrite this as $V_1 \Sigma_r^2 V_1^T x = (U_1 \Sigma_r V_1^T)^T b = V_1 \Sigma_r U_1^T b$ in view of (A.89) and (A.83). Now $x = V_1 \Sigma_r^{-1} U_1^T b$ is a solution. To see this, substitute and note that $V_1^T V_1 = I_r$. In view of the fact that $\mathcal{N}(A^T A) = \mathcal{N}(A) = \mathcal{R}(V_2) = \operatorname{span}\{v_{r+1}, \ldots, v_n\}$, the complete solution is given by

$$x_w = V_1 \Sigma_r^{-1} U_1^T b + V_2 w \tag{A.100}$$

for some $w \in R^{m-r}$. Since $V_1 \Sigma_r^{-1} U_1^T b$ is orthogonal to $V_2 w$ for all w,

$$x_0 = V_1 \Sigma_r^{-1} U_1^T b \tag{A.101}$$

is the optimal solution that minimizes $\| b - Ax \|_2$.

The *Moore–Penrose pseudo-inverse* of $A \in R^{m \times n}$ can be shown to be

$$A^+ = V_1 \Sigma_r^{-1} U_1^T. \tag{A.102}$$

We have seen that $x = A^+ b$ is the solution to the least-squares problem. It can be shown that this pseudo-inverse minimizes $\| AA^+ - I_m \|_F$, where $\| A \|_F$ denotes the *Frobenius norm* of A, which is equal to the square root of $\operatorname{trace}[AA^T] = \sum_{i=1}^m \lambda_i(AA^T) = \sum_{i=1}^m \sigma_i^2(A)$. It is of interest to note that the Moore–Penrose pseudo-inverse of A is defined as the unique matrix that satisfies the conditions (i) $AA^+A = A$, (ii) $A^+AA^+ = A^+$, (iii) $(AA^+)^T = AA^+$, and (iv) $(A^+A)^T = A^+A$.

Note that if $\operatorname{rank} A = m \leq n$ then it can be shown that $A^+ = A^T(AA^T)^{-1}$; this is, in fact, the *right inverse of* A, since $A(A^T(AA^T)^{-1}) = I_m$. Similarly, if $\operatorname{rank} A = n \leq m$, then $A^+ = (A^T A)^{-1} A^T$, the *left inverse of* A, since $((A^T A)^{-1} A^T)A = I_n$.

Singular values and singular value decomposition are discussed in a number of references. See for example, Golub and Van Loan [7], Patel et al. [11], Petkov et al. [12], and DeCarlo [5].

A.10 Notes

Standard references on linear algebra and matrix theory include Birkhoff and MacLane [4], Halmos [8], and Gantmacher [6]. Our presentation in this appendix follows Michel and Herget [9]. Conditioning and numerical stability of a problem are key issues in the area of numerical analysis. Our aim in Section A.9 was to make the reader aware that depending on the problem, the numerical considerations in the calculation of a solution may be nontrivial. These issues are discussed at length in many textbooks on numerical analysis. Examples of good books in this area include Golub and Van Loan [7] and Stewart [13] where matrix computations are emphasized. Also, see Petkov et al. [12] and Patel et al. [11] for computational methods with emphasis on system and control problems. For background on the theory of algorithms, optimization algorithms, and their numerical properties, see Bazaraa et al. [2] and Bertsekas and Tsitsiklis [3].

References

1. P.J. Antsaklis and A.N. Michel, *Linear Systems*, Birkhäuser, Boston, MA, 2006.
2. M.S. Bazaraa, H.D. Sherali, and C.M. Shetty, *Nonlinear Programming Theory and Algorithms*, 2nd edition, Wiley, New York, NY, 1993.
3. D.P. Bertsekas and J.N. Tsitsiklis, *Parallel and Distributed Computation–Numerical Methods*, Prentice Hall, Englewood Cliffs, NJ, 1989.
4. G. Birkhoff and S. MacLane, *A Survey of Modern Algebra*, Macmillan, New York, NY, 1965.
5. R.A. DeCarlo, *Linear Systems*, Prentice Hall, Englewood Cliffs, NJ, 1989.
6. F.R. Gantmacher, *Theory of Matrices*, Vols. I and II, Chelsea Publications, New York, NY, 1959.
7. G.H. Golub and C.F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, 1983.
8. P.R. Halmos, *Finite Dimensional Vector Spaces*, D. Van Nostrand, Princeton, NJ, 1958.
9. A.N. Michel and C.J. Herget, *Applied Algebra and Functional Analysis*, Dover, New York, NY, 1993.
10. R.K. Miller and A.N. Michel, *Ordinary Differential Equations*, Academic Press, New York, NY, 1982.
11. R.V. Patel, A.J. Laub, and P.M. VanDooren, Eds., *Numerical Linear Algebra Techniques for Systems and Control*, IEEE Press, Piscataway, NJ, 1993.
12. P.Hr. Petkov, N.D. Christov, and M.M. Konstantinov, *Computational Methods for Linear Control Systems*, Prentice Hall-International Series, Englewood Cliffs, NJ, 1991.
13. G.W. Stewart, *Introduction to Matrix Computations*, Academic Press, New York, NY, 1973.

Solutions to Selected Exercises

Exercises of Chapter 1

1.10 $\Delta\dot{x} = (k_1k_2/L)\,\Delta x + k_2\Delta u$, $\Delta y = (k_1/L)\,\Delta x$ where $L = 2\sqrt{k} - k_1k_2t$.

1.12 (a) $\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$.
(b) $x_1 = x, x_2 = \dot{x}$
$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -3\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix} + \begin{bmatrix}0\\ -1\end{bmatrix}\Delta u.$

1.13 $x_1 = \phi, x_2 = \dot{\phi}, x_3 = s, x_4 = \dot{s}$
$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\\ \Delta\dot{x}_3\\ \Delta\dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ \frac{g}{L'} & 0 & 0 & \frac{F}{L'M}\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -\frac{F}{M}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{L'M}\\ 0\\ \frac{1}{M}\end{bmatrix}\Delta\mu.$$

Exercises of Chapter 2

2.4 (b) Not linear; time-invariant; causal.

2.5 Causal; linear; not time-invariant.

2.6 Noncausal; linear; time-invariant.

2.7 Not linear.

2.8 Noncausal; time-invariant.

2.9 Noncausal; nonlinear (affine).

2.10 $y(n) = \sum_{k=-\infty}^{\infty} u(l)[s(n,l) - s(n,l-1)]$, where $s(n,l) = \sum_{l=-\infty}^{\infty} h(n,k)$ $p(k-l)$ is the unit step response of the system.

Exercises of Chapter 3

3.1 (a) $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 5)$. (b) $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) = (s, 2, s + 1/s)$.

3.2 $a = [1, 0, -2]^T$, $\bar{a} = [0, 1/2, 1/2]$.

3.3 A basis is $\{(1, \alpha)^T\}$, $\alpha \in R$.

3.4 It is a vector space of dimension n^2. The set of nonsingular matrices is not a vector space since closure of matrix addition is violated.

3.5 Dependent over the field of rational functions; independent over the field of reals.

3.6 (a) Rank is 1 over complex numbers. (b) 2 over reals. (c) 2 over rational functions. (d) 1 over rational functions.

3.8 Directly, from the series definition of e^{At} or using system concepts.

3.9 See also Subsection 6.4.1.

3.11 (λ_i^k, v^i) is an (eigenvalue, eigenvector) pair of A^k. Then $f(A)v^i = f(\lambda_i)v^i$.

3.13 Substitute $x(t) = \Phi(t, t_0)z(t)$ into $\dot{x} = A(t)x + B(t)u$.

3.14 Take derivatives of both sides of $\Phi(t, \tau)\Phi(\tau, t) = I$ with respect to τ.

3.19 Verify that $\Phi(t, 0) = e^{At}$ is the solution of $\Phi(t, 0) = A\Phi(t, 0)$, $\Phi(0, 0) = I$.

3.21 Use Exercise 3.19. $x_1^2 + x_2^2 = 2 = (\sqrt{2})^2$, so trajectory is a circle.

3.22 $x(0) = [1, 1]^T$ is colinear to the eigenvector of eigenvalue 1, and so e^t is the only mode that appears in the solution.

3.23 (a) Take $t = 0$ in the expression for e^{At}.

3.25 (a) $x(0) = [-1,\ 1,\ 0]^T$.

3.30 $(I - A)x(0) = Bu(0)$; $u(0) = 2$.

Exercises of Chapter 4

4.1 Set of equilibria is $\{(-4v, v, 5v)^T : v \in R\}$.

4.2 Set of equilibria is $\{(\frac{1}{k\pi}, 0)^T : k \in \mathcal{N}\backslash\{0\}\} \cup \{(0, 0)^T\}$.

4.3 $x = 0$ is uniformly asymptotically stable; $x = 1$ is unstable.

4.5 $A > 0$.

4.7 $x = 0$ is exponentially stable; $x = 1$ is unstable.

4.9 Uniformly BIBO stable.

4.10 (a) Set of equilibria $\{(\alpha, -\alpha)^T \in \mathcal{R}^2 : \alpha \geq 0\}$. (b) No equilibrium.

4.12 $x = 0$ is stable.

4.13 $x = 0$ is stable.

4.14 $x = 0$ is not stable.

4.15 $x = 0$ is not stable.

4.18 (a) $x = 0$ is stable. (b) $x = 0$ is stable.

4.21 $x = 0$ is unstable.

4.22 Not BIBO stable. Theorem cannot be applied.

Exercises of Chapter 5

5.2 (a) Controllable from u, observable from y. (b) when $u_1 = 0$, controllable from u_2; when $u_2 = 0$, not controllable from u_1. (c) not observable from y_1; observable from y_2.

5.3 (a) Use $u(t) = B^T e^{A^T(T-t)} W_r^{-1}(0, T)(x_1 - e^{AT} x_0)$.

5.4 (a) It can be reached in two steps, with $u(0) = 3$, $u(1) = -1$.
(b) Any $x = (b, a, a)^T$ will be reachable. $a, b \in R$.
(c) $x = (0, 0, a)^T$ unobservable. $a \in R$.

5.10 (a) $u(0) = -1$, $u(1) = 2$; (b) $y(1) = (1, 2)^T u(0)$.

5.11 $x_0 = [1\ 0\ \alpha]^T$, $\alpha \in R$.

Exercises of Chapter 6

6.2 (b) $\lambda = 3$ uncontrollable (first pair); $\lambda = -1$ uncontrollable (second pair).

6.3 Controllability indices are 1, 3.

6.7 Use controller form or Sylvester's Rank Inequality.

6.13 (a) It is controllable. (b) It is controllable from f_1 only. (c) It is observable.

Exercises of Chapter 7

7.1 Use the standard form for uncontrollable systems.

7.5 $\lambda_1 = 1$ is both controllable and observable, $\lambda_2 = -\frac{1}{2}$ is uncontrollable but observable, and $\lambda_3 = -\frac{1}{2}$ is controllable but unobservable.

7.6 (a) $\lambda = 2$ is uncontrollable and unobservable. (b) $H(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.
(c) It is not asymptotically stable, but it is BIBO stable.

7.7 (a) $p_H(s) = s^2(s+2)$, $m_H(s) = s(s+2)$. (b) $z_H(s) = 1$.

7.8 (a) $p_H(s) = s^3 = m_H(s)$. (b) $z_H(s) = 1$.

7.10 (a) They are rc. (b) They are not lc; a glcd is $\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$.

7.13 (a) It is uncontrollable and unobservable.
(b) $H(s) = \frac{2}{s+1}\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$.

Exercises of Chapter 8

8.4 $p_H = s^2 - 1$; McMillan degree is 2.

8.6 (a) $p_H(s) = s(s+1)(s+3)$; McMillan degree is 3.
(b) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

8.10 (a) $p_H(s) = s^2(s+1)^2$ so the order of any minimal realization is 4.
(b) Take $u_1 = 0$, and find the McMillan degree, which is 2. So in a fourth order realization, system will not be controllable from u_2 only. System will be observable from y_1.

8.12 $p_H(s) = s^3$, and so 3 is the order of any minimal realization. A minimal realization is $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Exercises of Chapter 9

9.4 $F = gf$, $g = (0,1)^T$, $f = (-11 \quad -19 \quad -12; -10)$ (after reducing the system to single-input controllable).

9.6 (a) $G = \frac{1}{2}$, $F = [-2, -\frac{7}{2}, -\frac{5}{2}]$. (b) controllable but unobservable.

9.9 (a) Let $x_1 = \theta, x_2 = \dot{\theta}$. Then $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.
(b) $F = [-1, -1]$.

9.11 $F = [-3.7321, -6, 4641]$ minimizes J_1.

9.12 (a) $F = \left[\frac{3}{2}, 1\right]$. (b) $K = [2\alpha, \alpha^2 + 2]^T$.

9.13 (a) $\mathcal{O}E = 0$. (b) $E = \alpha(1, -1)^T$.

9.14 $F = -\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.

9.15 (a) $x_0 = -(\lambda I - A)^{-1}B$; If λ is a zero of $H(s)$, a pole-zero cancellation will occur. (b) $x_0 = \alpha v$, where v is the eigenvector corresponding to λ.

Exercises of Chapter 10

10.1 (a) For $X_1 = \widetilde{X}_1 = 0$, $Y_1 = \widetilde{Y}_1 = 1$, $H_2 = (1 + s^2 K)/K$, where K is any stable rational function. Alternatively, for $X_1' = \widetilde{X}_1' = (s^2 + 8s + 24)/(s+2)^2$, $Y_1' = \widetilde{Y}_1' = (32s + 16)/(s+2)^2$, $N_1' = \widetilde{N}_1' = 1/(s+2)^2$, $D_1' = \widetilde{D}_1' = s^2/(s+2)^2$, and $H_2 = (32s + 16 - s^2 K')/(s^2 + 8s + 24 - K')$, where K' is any proper and stable rational function.
(b) In general, it is not easy (or may be impossible) to restrict appropriately K or K', so H_2 is of specific order. Here let $H_2 = (b_1 s + b_0)/(s + a_0)$ and establish conditions on the parameters so that the closed-loop system is stable.

10.4 (b) $N'^{-1}T = X'$, a proper and stable function, implies conditions on n_i, d_i of T.

10.5 (a) In view of the hint, $N^{-1}T = 1/(s+2)^2 = 1/G^{-1}D_F$, from which $G = 2$, $F(s) = FS(s) = [-6, -11][1, s]^T$.

Index

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