4

Stability

Dynamical systems, either occurring in nature or man made, usually function in some specified mode. The most common such modes are operating points that frequently turn out to be equilibria.

In this chapter we will concern ourselves primarily with the qualitative behavior of equilibria. Most of the time, we will be interested in the asymptotic stability of an equilibrium (operating point), which means that when the state of a given system is displaced (disturbed) from its desired operating point (equilibrium), the expectation is that the state will eventually return to the equilibrium. For example, in the case of an automobile under cruise control, traveling at the desired constant speed of 50 mph (which determines the operating point, or equilibrium condition), perturbations due to hill climbing (hill descending), will result in decreasing (increasing) speeds. In a properly designed cruise control system, it is expected that the car will return to its desired operating speed of 50 mph.

Another qualitative characterization of dynamical systems is the expectation that bounded system inputs will result in bounded system outputs, and that small changes in inputs will result in small changes in outputs. System properties of this type are referred to as input–output stability. Such properties are important for example in tracking systems, where the output of the system is expected to follow a desired input. Frequently, it is possible to establish a connection between the input–output stability properties and the Lyapunov stability properties of an equilibrium. In the case of linear systems, this connection is well understood. This will be addressed in Section 7.3.

4.1 Introduction

In this chapter we present a brief introduction to stability theory. We are concerned primarily with linear systems and systems that are a consequence of linearizations of nonlinear systems. As in the other chapters of this book, we consider finite-dimensional continuous-time systems and finite-dimensional discrete-time systems described by systems of first-order ordinary differential equations and systems of first-order ordinary difference equations, respectively.

In Section 4.2 we introduce the concept of equilibrium of dynamical systems described by systems of first-order ordinary differential equations, and in Section 4.3 we give definitions of various types of stability in the sense of Lyapunov (including stability, uniform stability, asymptotic stability, uniform asymptotic stability, exponential stability, and instability).

In Section 4.4 we establish conditions for the various Lyapunov stability and instability types enumerated in Section 4.3 for linear systems $\dot{x} = Ax$. Most of these results are phrased in terms of the properties of the state transition matrix for such systems.

In Section 4.5 we introduce the Second Method of Lyapunov, also called the Direct Method of Lyapunov, to establish necessary and sufficient conditions for various Lyapunov stability types of an equilibrium for linear systems $\dot{x} = Ax$. These results, which are phrased in terms of the system parameters [coefficients of the matrix A], give rise to the Lyapunov matrix equation.

In Section 4.6 we use the Direct Method of Lyapunov in deducing the asymptotic stability and instability of an equilibrium of nonlinear autonomous systems $\dot{x} = Ax + F(x)$ from the stability properties of their linearizations $\dot{w} = Aw$.

In Section 4.7 we establish necessary and sufficient conditions for the input–output stability (more precisely, for the bounded input/bounded output stability) of continuous-time, linear, time-invariant systems. These results involve the system impulse response matrix.

The stability results presented in Sections 4.2 through and including Section 4.7 pertain to continuous-time systems. In Section 4.8 we present analogous stability results for discrete-time systems.

4.2 The Concept of an Equilibrium

In this section we concern ourselves with systems of first-order autonomous ordinary differential equations,

$$\dot{x} = f(x), \tag{4.1}$$

where $x \in R^n$. When discussing global results, we shall assume that $f : R^n \to R^n$, while when considering local results, we may assume that $f : B(h) \to R^n$ for some $h > 0$, where $B(h) = \{x \in R^n :\| x \|< h\}$ and $\| \cdot \|$ denotes a norm on R^n. Unless otherwise stated, we shall assume that for every $(t_0, x_0), t_0 \in R^+$, the initial-value problem

$$\dot{x} = f(x), \quad x(t_0) = x_0 \tag{4.2}$$

possesses a unique solution $\phi(t, t_0, x_0)$ that exists for all $t \geq t_0$ and that depends continuously on the initial data (t_0, x_0). Refer to Section 1.5 for conditions that ensure that (4.2) has these properties. Since (4.1) is time-invariant, we may assume without loss of generality that $t_0 = 0$ and we will denote the solutions of (4.1) by $\phi(t, x_0)$ (rather than $\phi(t, t_0, x_0)$) with $x(0) = x_0$.

Definition 4.1. *A point $x_e \in R^n$ is called an* equilibrium point *of (4.1), or simply an* equilibrium *of (4.1), if*

$$f(x_e) = 0.$$

■

We note that

$$\phi(t, x_e) = x_e \quad \text{for all } t \geq 0;$$

i.e., the equilibrium x_e is the unique solution of (4.1) with initial data given by $\phi(0, x_e) = x_e$.

We will usually assume that in a given discussion, unless otherwise stated, the equilibrium of interest is located at the origin of R^n. This assumption can be made without loss of generality by noting that if $x_e \neq 0$ is an equilibrium point of (4.1), i.e., $f(x_e) = 0$, then by letting $w = x - x_e$, we obtain the transformed system

$$\dot{w} = F(w) \tag{4.3}$$

with $F(0) = 0$, where

$$F(w) = f(w + x_e). \tag{4.4}$$

Since the above transformation establishes a one-to-one correspondence between the solutions of (4.1) and (4.3), we may assume henceforth that the equilibrium of interest for (4.1) is located at the origin. This equilibrium, $x = 0$, will be referred to as the *trivial solution* of (4.1).

Before concluding this section, it may be fruitful to consider some specific cases.

Example 4.2. In Example 1.4 we considered the simple pendulum given in Figure 1.7. Letting $x_1 = x$ and $x_2 = \dot{x}$ in (1.37), we obtain the system of equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -k \sin x_1, \end{aligned} \tag{4.5}$$

where $k > 0$ is a constant. *Physically*, the pendulum has two equilibrium points: one where the mass M is located vertically at the bottom of the figure (i.e., at 6 o'clock) and the other where the mass is located vertically at the top of the figure (i.e., at 12 o'clock). The *model* of this pendulum, however, described by (4.5), has countably infinitely many equilibrium points that are located in R^2 at the points $(\pi n, 0)^T, n = 0, \pm 1, \pm 2, \ldots$.

Example 4.3. The linear, autonomous, homogenous system of ordinary differential equations

$$\dot{x} = Ax \tag{4.6}$$

has a unique equilibrium that is at the origin if and only if A is nonsingular. Otherwise, (4.6) has nondenumerably many equilibria. [Refer to Chapter 1 for the definitions of symbols in (4.6).]

Example 4.4. Assume that for

$$\dot{x} = f(x), \tag{4.7}$$

f is continuously differentiable with respect to all of its arguments, and let

$$J(x_e) = \frac{\partial f}{\partial x}(x)\bigg|_{x=x_e},$$

where $\partial f/\partial x$ denotes the $n \times n$ *Jacobian matrix* defined by

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f_i}{\partial x_j}\right].$$

If $f(x_e) = 0$ and $J(x_e)$ is nonsingular, then x_e is an equilibrium of (4.7).

Example 4.5. The system of ordinary differential equations given by

$$\begin{aligned}\dot{x}_1 &= k + \sin(x_1 + x_2) + x_1,\\ \dot{x}_2 &= k + \sin(x_1 + x_2) - x_1,\end{aligned}$$

with $k > 1$, has no equilibrium points at all.

4.3 Qualitative Characterizations of an Equilibrium

In this section we consider several qualitative characterizations that are of fundamental importance in systems theory. These characterizations are concerned with various types of stability properties of an equilibrium and are referred to in the literature as *Lyapunov stability*.

Throughout this section, we consider systems of equations

$$\dot{x} = f(x), \tag{4.8}$$

and we assume that (4.8) possesses an equilibrium at the origin. We thus have $f(0) = 0$.

Definition 4.6. *The equilibrium $x = 0$ of (4.8) is said to be* stable *if for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$\| \phi(t, x_0) \| < \epsilon \text{ for all } t \geq 0 \tag{4.9}$$

whenever

$$\| x_0 \| < \delta(\epsilon). \tag{4.10}$$

■

In Definition 4.6, $\| \cdot \|$ denotes any one of the equivalent norms on R^n, and (as in Chapters 1 and 2) $\phi(t, x_0)$ denotes the solution of (4.8) with initial condition $x(0) = x_0$. The notation $\delta(\epsilon)$ indicates that δ depends on the choice of ϵ.

In words, Definition 4.6 states that by choosing the initial points in a sufficiently small spherical neighborhood, when the equilibrium $x = 0$ of (4.8) is stable, we can force the graph of the solution for $t \geq 0$ to lie entirely inside a given cylinder. This is depicted in Figure 4.1 for the case $x \in R^2$.

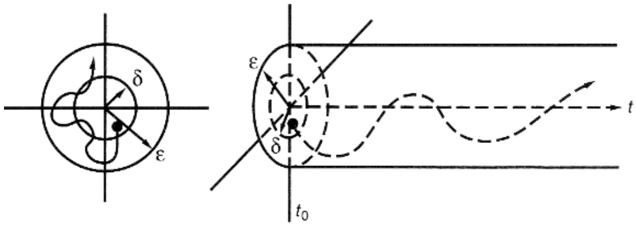

Figure 4.1. Stability of an equilibrium

Definition 4.7. *The equilibrium $x = 0$ of (4.8) is said to be* asymptotically stable *if*

(i) it is stable,
(ii) there exists an $\eta > 0$ such that $\lim_{t\to\infty} \phi(t, x_0) = 0$ whenever $\| x_0 \| < \eta$. ■

The set of all $x_0 \in R^n$ such that $\phi(t, x_0) \to 0$ as $t \to \infty$ is called the *domain of attraction* of the equilibrium $x = 0$ of (4.8). Also, if for (4.8) condition (ii) is true, then the equilibrium $x = 0$ is said to be *attractive*.

Definition 4.8. *The equilibrium $x = 0$ of (4.8) is* exponentially stable *if there exists an $\alpha > 0$, and for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that*

$$\| \phi(t, x_0) \| \leq \epsilon e^{\alpha t} \text{ for all } t \geq 0$$

whenever $||x_0|| < \delta(\epsilon)$. ■

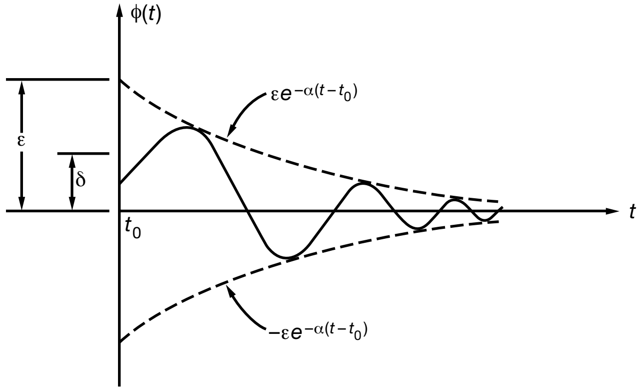

Figure 4.2. An exponentially stable equilibrium

Figure 4.2 shows the behavior of a solution in the vicinity of an exponentially stable equilibrium $x = 0$.

Definition 4.9. *The equilibrium $x = 0$ of (4.8) is* unstable *if it is not stable. In this case, there exists an $\epsilon > 0$, and a sequence $x_m \to 0$ of initial points and a sequence $\{t_m\}$ such that $\| \phi(t_m, x_m) \| \geq \epsilon$ for all $m, t_m \geq 0$.* ■

If $x = 0$ is an unstable equilibrium of (4.8), then it still can happen that all the solutions tend to zero with increasing t. This indicates that instability and attractivity of an equilibrium are compatible concepts. We note that the equilibrium $x = 0$ of (4.8) is necessarily unstable if every neighborhood of the origin contains initial conditions corresponding to unbounded solutions (i.e., solutions whose norm grows to infinity on a sequence $t_m \to \infty$). However, it can happen that a system (4.8) with unstable equilibrium $x = 0$ may have only bounded solutions.

The concepts that we have considered thus far pertain to *local* properties of an equilibrium. In the following discussion, we consider *global* characterizations of an equilibrium.

Definition 4.10. *The equilibrium $x = 0$ of (4.8) is* asymptotically stable in the large *if it is stable and if every solution of (4.8) tends to zero as $t \to \infty$.* ■

When the equilibrium $x = 0$ of (4.8) is asymptotically stable in the large, its domain of attraction is all of R^n. Note that in this case, $x = 0$ is the *only* equilibrium of (4.8).

Definition 4.11. *The equilibrium $x = 0$ of (4.8) is* exponentially stable in the large *if there exists $\alpha > 0$ and for any $\beta > 0$, there exists $k(\beta) > 0$ such that*

$$\| \phi(t, x_0) \| \leq k(\beta) \| x_0 \| e^{-\alpha t} \quad \textit{for all } t \geq 0$$

whenever $\| x_0 \| < \beta$. ■

We conclude this section with a few specific cases.

The scalar differential equation

$$\dot{x} = 0 \tag{4.11}$$

has for any initial condition $x(0) = x_0$ the solution $\phi(t, x_0) = x_0$; i.e., all solutions are equilibria of (4.11). The trivial solution is stable; however, it is not asymptotically stable.

The scalar differential equation

$$\dot{x} = ax \tag{4.12}$$

has for every $x(0) = x_0$ the solution $\phi(t, x_0) = x_0 e^{at}$, and $x = 0$ is the only equilibrium of (4.12). If $a > 0$, this equilibrium is unstable, and when $a < 0$, this equilibrium is exponentially stable in the large.

As mentioned earlier, a system

$$\dot{x} = f(x) \tag{4.13}$$

can have all solutions approaching an equilibrium, say, $x = 0$, without this equilibrium being asymptotically stable. An example of this type of behavior is given by the *nonlinear* system of equations

$$\begin{aligned}
\dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}, \\
\dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)[1 + (x_1^2 + x_2^2)^2]}.
\end{aligned}$$

For a detailed discussion of this system, refer to [6], pp. 191–194, cited at the end of this chapter.

Before proceeding any further, a few comments are in order concerning the reasons for considering equilibria and their stability properties as well as other types of stability that we will encounter. To this end we consider linear time-invariant systems given by

$$\dot{x} = Ax + Bu, \tag{4.14a}$$

$$y = Cx + Du, \tag{4.14b}$$

where all symbols in (4.14) are defined as in (2.7). The usual qualitative analysis of such systems involves two concepts, *internal stability* and *input–output stability*.

In the case of *internal stability*, the output equation (4.14b) plays no role whatsoever, the system input u is assumed to be identically zero, and the focus of the analysis is concerned with the qualitative behavior of the solutions of linear time-invariant systems

$$\dot{x} = Ax \tag{4.15}$$

near the equilibrium $x = 0$. This is accomplished by making use of the various types of Lyapunov stability concepts introduced in this section. In other words, internal stability of system (4.14) concerns the Lyapunov stability of the equilibrium $x = 0$ of system (4.15).

In the case of *input–output stability*, we view systems as operators determined by (4.14) that relate outputs y to inputs u and the focus of the analysis is concerned with qualitative relations between system inputs and system outputs. We will address this type of stability in Section 4.7.

4.4 Lyapunov Stability of Linear Systems

In this section we first study the stability properties of the equilibrium $x = 0$ of linear autonomous homogeneous systems

$$\dot{x} = Ax, \quad t \geq 0. \tag{4.16}$$

Recall that $x = 0$ is always an equilibrium of (4.16) and that $x = 0$ is the only equilibrium of (4.16) if A is nonsingular. Recall also that the solution of (4.16) for $x(0) = x_0$ is given by

$$\begin{aligned} \phi(t, x_0) &= \Phi(t, 0)x_0 = \Phi(t - 0, 0)x_0 \\ &\triangleq \Phi(t)x_0 = e^{At}x_0, \end{aligned}$$

where in the preceding equation, a slight abuse of notation has been used.

We first consider some of the basic properties of system (4.16).

Theorem 4.12. *The equilibrium $x = 0$ of (4.16) is* stable *if and only if the solutions of (4.16) are bounded, i.e., if and only if*

$$\sup_{t \geq t_0} \| \Phi(t) \| \triangleq k < \infty,$$

where $\| \Phi(t) \|$ denotes the matrix norm induced by the vector norm used on R^n and k denotes a constant.

Proof. Assume that the equilibrium $x = 0$ of (4.16) is stable. Then for $\epsilon = 1$ there is a $\delta = \delta(1) > 0$ such that $\| \phi(t, x_0) \| < 1$ for all $t \geq 0$ and all x_0 with $\| x_0 \| \leq \delta$. In this case

$$\| \phi(t, x_0) \| = \| \Phi(t)x_0 \| = \| [\Phi(t)(x_0\delta)/ \| x_0 \|] \| (\| x_0 \| /\delta) < \| x_0 \| /\delta$$

for all $x_0 \neq 0$ and all $t \geq 0$. Using the definition of matrix norm [refer to Section A.7], it follows that

$$\| \Phi(t) \| \leq \delta^{-1}, \quad t \geq 0.$$

We have proved that if the equilibrium $x = 0$ of (4.16) is stable, then the solutions of (4.16) are bounded.

Conversely, suppose that all solutions $\phi(t, x_0) = \Phi(t)x_0$ are bounded. Let $\{e_1, \ldots, e_n\}$ denote the natural basis for n-space, and let $\| \phi(t, e_j) \| < \beta_j$ for all $t \geq 0$. Then for any vector $x_0 = \sum_{j=1}^n \alpha_j e_j$ we have that

$$\begin{aligned} \| \phi(t, x_0) \| &= \| \sum_{j=1}^n \alpha_j \phi(t, e_j) \| \leq \sum_{j=1}^n |\alpha_j| \beta_j \\ &\leq (\max_j \beta_j) \sum_{j=1}^n |\alpha_j| \leq k \| x_0 \| \end{aligned}$$

for some constant $k > 0$ for $t \geq 0$. For given $\epsilon > 0$, we choose $\delta = \epsilon/k$. Thus, if $\| x_0 \| < \delta$, then $\| \phi(t, x_0) \| < k \| x_0 \| < \epsilon$ for all $t \geq 0$. We have proved that if the solutions of (4.16) are bounded, then the equilibrium $x = 0$ of (4.16) is stable. ■

Theorem 4.13. *The following statements are equivalent.*

(i) The equilibrium $x = 0$ of (4.16) is asymptotically stable.
(ii) The equilibrium $x = 0$ of (4.16) is asymptotically stable in the large.
(iii) $\lim_{t\to\infty} \| \Phi(t) \| = 0$.

Proof. Assume that statement (i) is true. Then there is an $\eta > 0$ such that when $\| x_0 \| \leq \eta$, then $\phi(t, x_0) \to 0$ as $t \to \infty$. But then we have for any $x_0 \neq 0$ that

$$\phi(t, x_0) = \phi(t, \eta x_0 / \| x_0 \|)(\| x_0 \| / \eta) \to 0$$

as $t \to \infty$. It follows that statement (ii) is true.

Next, assume that statement (ii) is true. For any $\epsilon > 0$, there must exist a $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$ we have that $\| \phi(t, x_0) \| = \| \Phi(t)x_0 \| < \epsilon$. To see this, let $\{e_1, \ldots, e_n\}$ be the natural basis for R^n. Thus, for some fixed constant $k > 0$, if $x_0 = (\alpha_1, \ldots, \alpha_n)^T$ and if $\| x_0 \| \leq 1$, then $x_0 = \sum_{j=1}^n \alpha_j e_j$ and $\sum_{j=1}^n |\alpha_j| \leq k$. For each j, there is a $T_j(\epsilon)$ such that $\| \Phi(t)e_j \| < \epsilon/k$ and $t \geq T_j(\epsilon)$. Define $T(\epsilon) = \max\{T_j(\epsilon) : j = 1, \ldots, n\}$. For $\| x_0 \| \leq 1$ and $t \geq T(\epsilon)$, we have that

$$\| \Phi(t)x_0 \| = \| \sum_{j=1}^n \alpha_j \Phi(t) e_j \| \leq \sum_{j=1}^n |\alpha_j| (\epsilon/k) \leq \epsilon.$$

By the definition of the matrix norm [see the appendix], this means that $\| \Phi(t) \| \leq \epsilon$ for $t \geq T(\epsilon)$. Therefore, statement (iii) is true.

Finally, assume that statement (iii) is true. Then $\| \Phi(t) \|$ is bounded in t for all $t \geq 0$. By Theorem 4.12, the equilibrium $x = 0$ is stable. To prove asymptotic stability, fix $\epsilon > 0$. If $\| x_0 \| < \eta = 1$, then $\| \phi(t, x_0) \| \leq \| \Phi(t) \| \| x_0 \| \to 0$ as $t \to \infty$. Therefore, statement (i) is true. This completes the proof. ■

Theorem 4.14. *The equilibrium $x = 0$ of (4.16) is asymptotically stable if and only if it is exponentially stable.*

Proof. The exponential stability of the equilibrium $x = 0$ implies the asymptotic stability of the equilibrium $x = 0$ of systems (4.13) in general and, hence, for systems (4.16) in particular.

Conversely, assume that the equilibrium $x = 0$ of (4.16) is asymptotically stable. Then there is a $\delta > 0$ and a $T > 0$ such that if $\| x_0 \| \leq \delta$, then

$$\| \Phi(t+T)x_0 \| < \delta/2$$

for all $t \geq 0$. This implies that

$$\| \Phi(t+T) \| \leq \frac{1}{2} \text{ if } t \geq 0. \tag{4.17}$$

From Theorem 3.9 (iii) we have that $\Phi(t-\tau) = \Phi(t-\sigma)\Phi(\sigma-\tau)$ for any t, σ, and τ. Therefore,

$$\| \Phi(t+2T) \| = \| \Phi(t+2T-t-T)\Phi(t+T) \| \leq \frac{1}{4},$$

in view of (4.17). By induction, we obtain for $t \geq 0$ that

$$\| \Phi(t+nT) \| \leq 2^{-n}. \tag{4.18}$$

Now let $\alpha = (ln2)/T$. Then (4.18) implies that for $0 \leq t < T$ we have that

$$\begin{aligned} \| \phi(t+nT, x_0) \| &\leq 2 \| x_0 \| 2^{-(n+1)} = 2 \| x_0 \| e^{-\alpha(n+1)T} \\ &\leq 2 \| x_0 \| e^{-\alpha(t+nT)}, \end{aligned}$$

which proves the result. ∎

Even though the preceding results require knowledge of the state transition matrix $\Phi(t)$ of (4.16), they are quite useful in the qualitative analysis of linear systems. In view of the above results, we can state the following equivalent definitions.

The equilibrium $x = 0$ of (4.16) is *stable* if and only if there exists a finite positive constant γ, such that for any x_0, the corresponding solution satisfies the inequality

$$\| \phi(t, x_0) \| \leq \gamma \| x_0 \|, \quad t \geq 0.$$

Furthermore, in view of the above results, if the equilibrium $x = 0$ of (4.16) is asymptotically stable, then in fact it must be globally asymptotically stable, and exponentially stable in the large. In this case there exist finite constants $\gamma \geq 1$ and $\lambda > 0$ such that

$$\| \phi(t, x_0) \| \leq \gamma e^{-\lambda t} \| x_0 \|$$

for $t \geq 0$ and $x_0 \in R^n$.

We now continue our investigation of system (4.16) by referring to the discussion in Subsection 3.3.2 [refer to (3.23) to (3.39)] concerning the use of the Jordan canonical form to compute $\exp(At)$. We let $J = P^{-1}AP$ and define $x = Py$. Then (4.16) yields

$$\dot{y} = P^{-1}APy = Jy. \tag{4.19}$$

It is easily verified (the reader is asked to do so in the Exercises section) that the equilibrium $x = 0$ of (4.16) is stable (resp., asymptotically stable or unstable) if and only if $y = 0$ of (4.19) is stable (resp., asymptotically stable or unstable). In view of this, we can assume without loss of generality that the matrix A in (4.16) is in Jordan canonical form, given by

$$A = \operatorname{diag}[J_0, J_1, \dots, J_s],$$

where

$$J_0 = \operatorname{diag}[\lambda_1, \dots, \lambda_k] \quad \text{and} \quad J_i = \lambda_{k+i} I_i + N_i$$

for the Jordan blocks $J_1, \dots, J_s$.

As in (3.33), (3.34), (3.38), and (3.39), we have

$$e^{At} = \begin{bmatrix} e^{J_0 t} & & & 0 \\ & e^{J_1 t} & & \\ & & \ddots & \\ 0 & & & e^{J_s t} \end{bmatrix},$$

where

$$e^{J_0 t} = \operatorname{diag}[e^{\lambda_1 t}, \dots, e^{\lambda_k t}] \tag{4.20}$$

and

$$e^{J_i t} = e^{\lambda_{k+i} t} \begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{n_i - 1}/(n_i - 1)! \\ 0 & 1 & t & \cdots & t^{n_i - 2}/(n_i - 2)! \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{4.21}$$

for $i = 1, \dots, s$.

Now suppose that $Re\lambda_i \leq \beta$ for all $i = 1, \dots, k$. Then it is clear that $\lim_{t\to\infty}(\| e^{J_0 t} \| / e^{\beta t}) < \infty$, where $\| e^{J_0 t} \|$ is the matrix norm induced by one of the equivalent vector norms defined on R^n. We write this as $\| e^{J_0 t} \| = \mathcal{O}(e^{\beta t})$. Similarly, if $\beta = Re\lambda_{k+i}$, then for any $\epsilon > 0$ we have that $\| e^{J_i t} \| = \mathcal{O}(t^{n_i - 1} e^{\beta t}) = \mathcal{O}(e^{(\beta + \epsilon)t})$.

From the foregoing it is now clear that $\| e^{At} \| \leq K$ for some $K > 0$ if and only if all eigenvalues of A have nonpositive real parts, and the eigenvalues with zero real part occur in the Jordan form only in J_0 and not in any of the Jordan blocks J_i, $1 \leq i \leq s$. Hence, by Theorem 4.12, the equilibrium $x = 0$ of (4.16) is under these conditions stable.

Now suppose that all eigenvalues of A have negative real parts. From the preceding discussion it is clear that there is a constant $K > 0$ and an $\alpha > 0$ such that $\| e^{At} \| \leq Ke^{-\alpha t}$, and therefore, $\| \phi(t, x_0) \| \leq Ke^{-\alpha t} \| x_0 \|$ for all $t \geq 0$ and for all $x_0 \in R^n$. It follows that the equilibrium $x = 0$ is asymptotically stable in the large, in fact exponentially stable in the large. Conversely, assume that there is an eigenvalue λ_i with a nonnegative real part. Then either one term in (4.20) does not tend to zero, or else a term in (4.21) is unbounded as $t \to \infty$. In either case, $e^{At}x(0)$ will not tend to zero when the initial condition $x(0) = x_0$ is properly chosen. Hence, the equilibrium $x = 0$ of (4.16) cannot be asymptotically stable (and, hence, it cannot be exponentially stable).

Summarizing the above, we have proved the following result.

Theorem 4.15. *The equilibrium $x = 0$ of (4.16) is* stable*, if and only if all eigenvalues of A have nonpositive real parts, and every eigenvalue with zero real part has an associated Jordan block of order one. The equilibrium $x = 0$ of (4.16) is* asymptotically stable in the large*, in fact* exponentially stable in the large*, if and only if all eigenvalues of A have negative real parts.* ■

A direct consequence of the above result is that the equilibrium $x = 0$ of (4.16) is *unstable* if and only if at least one of the eigenvalues of A has either positive real part or has zero real part that is associated with a Jordan block of order greater than one.

At this point, it may be appropriate to take note of certain conventions concerning matrices that are used in the literature. It should be noted that some of these are not entirely consistent with the terminology used in Theorem 4.15. Specifically, a real $n \times n$ matrix A is called *stable* or a *Hurwitz matrix* if all its eigenvalues have negative real parts. If at least one of the eigenvalues has a positive real part, then A is called *unstable*. A matrix A, which is neither stable nor unstable, is called *critical*, and the eigenvalues with zero real parts are called *critical eigenvalues*.

We conclude our discussion concerning the stability of (4.16) by noting that the results given above can also be obtained by directly using the facts established in Subsection 3.3.3, concerning modes and asymptotic behavior of time-invariant systems.

Example 4.16. We consider the system (4.16) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm j$. According to Theorem 4.15, the equilibrium $x = 0$ of this system is stable. This can also be verified by computing the solution of this system for a given set of initial data $x(0)^T = (x_1(0), x_2(0))$,

$$\phi_1(t, x_0) = x_1(0) \cos t + x_2(0) \sin t,$$
$$\phi_2(t, x_0) = -x_1(0) \sin t + x_2(0) \cos t,$$

$t \geq 0$, and then applying Definition 4.6.

Example 4.17. We consider the system (4.16) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 0, \lambda_2 = 0$. According to Theorem 4.15, the equilibrium $x = 0$ of this system is unstable. This can also be verified by computing the solution of this system for a given set of initial data $x(0)^T = (x_1(0), x_2(0))$,

$$\begin{aligned} \phi_1(t, x_0) &= x_1(0) + x_2(0)t, \\ \phi_2(t, x_0) &= x_2(0), \end{aligned}$$

$t \geq 0$, and then applying Definition 4.9. (Note that in this example, the entire x_1-axis consists of equilibria.)

Example 4.18. We consider the system (4.16) with

$$A = \begin{bmatrix} 2.8 & 9.6 \\ 9.6 & -2.8 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm 10$. According to Theorem 4.15, the equilibrium $x = 0$ of this system is unstable.

Example 4.19. We consider the system (4.16) with

$$A = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = -1, -2$. According to Theorem 4.15, the equilibrium $x = 0$ of this system is exponentially stable.

4.5 The Lyapunov Matrix Equation

In Section 4.4 we established a variety of stability results that require explicit knowledge of the solutions of (4.16). In this section we will develop stability criteria for (4.16) with *arbitrary* matrix A. In doing so, we will employ *Lyapunov's Second Method* (also called *Lyapunov's Direct Method*) for the case of linear systems (4.16). This method utilizes auxiliary real-valued functions

$v(x)$, called *Lyapunov functions*, that may be viewed as *generalized energy functions* or *generalized distance functions* (from the equilibrium $x = 0$), and the stability properties are then deduced directly from the properties of $v(x)$ and its time derivative $\dot{v}(x)$, evaluated along the solutions of (4.16).

A logical choice of Lyapunov function is $v(x) = x^T x = \| x \|^2$, which represents the square of the Euclidean distance of the state from the equilibrium $x = 0$ of (4.16). The stability properties of the equilibrium are then determined by examining the properties of $\dot{v}(x)$, the time derivative of $v(x)$ along the solutions of (4.16), which we repeat here,

$$\dot{x} = Ax. \tag{4.22}$$

This derivative can be determined *without explicitly solving for the solutions of (4.22)* by noting that

$$\begin{aligned}\dot{v}(x) &= \dot{x}^T x + x^T \dot{x} = (Ax)^T x + x^T (Ax) \\ &= x^T (A^T + A)x.\end{aligned}$$

If the matrix A is such that $\dot{v}(x)$ is negative for all $x \neq 0$, then it is reasonable to expect that the distance of the state of (4.22) from $x = 0$ will decrease with increasing time, and that the state will therefore tend to the equilibrium $x = 0$ of (4.22) with increasing time t.

It turns out that the Lyapunov function used in the above discussion is not sufficiently flexible. In the following discussion, we will employ as a "generalized distance function" the quadratic form given by

$$v(x) = x^T P x, \quad P = P^T, \tag{4.23}$$

where P is a real $n \times n$ matrix. The time derivative of $v(x)$ along the solutions of (4.22) is determined as

$$\begin{aligned}\dot{v}(x) &= \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x \\ &= x^T (A^T P + PA)x;\end{aligned}$$

i.e.,

$$\dot{v} = x^T C x, \tag{4.24}$$

where

$$C = A^T P + PA. \tag{4.25}$$

Note that C is real and $C^T = C$. The system of equations given in (4.25) is called the *Lyapunov Matrix Equation*.

We recall that since P is real and symmetric, all its eigenvalues are real. Also, we recall that P is said to be *positive definite* (resp., *positive semidefinite*) if all its eigenvalues are positive (resp., nonnegative), and it is called *indefinite* if P has eigenvalues of opposite sign. The concepts of *negative definite* and *negative semidefinite* (for P) are similarly defined. Furthermore, we recall that the *function* $v(x)$ given in (4.23) is said to be *positive definite, positive semidefinite, indefinite*, and so forth, if P has the corresponding definiteness properties.

Instead of solving for the eigenvalues of a real symmetric matrix to determine its definiteness properties, there are more efficient and direct methods of accomplishing this. We now digress to discuss some of these.

Let $G = [g_{ij}]$ be a real $n \times n$ matrix (not necessarily symmetric). Recall that the *minors* of G are the matrix itself and the matrix obtained by removing successively a row and a column. The *principal minors* of G are G itself and the matrices obtained by successively removing an ith row and an ith column, and the *leading principal minors* of G are G itself and the minors obtained by successively removing the last row and the last column. For example, if $G = [g_{ij}] \in R^{3\times 3}$, then the principal minors are

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}, \quad \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \quad [g_{11}],$$
$$\begin{bmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{bmatrix}, \quad \begin{bmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{bmatrix}, \quad [g_{22}], \quad [g_{33}].$$

The first three matrices above are the leading principal minors of G. On the other hand, the matrix

$$\begin{bmatrix} g_{21} & g_{22} \\ g_{31} & g_{32} \end{bmatrix}$$

is a minor but not a principal minor.

The following results, due to Sylvester, allow efficient determination of the definiteness properties of a *real, symmetric* matrix.

Proposition 4.20. *(i) A real symmetric matrix $P = [p_{ij}] \in R^{n\times n}$ is* positive definite *if and only if the determinants of its* leading principal minors *are positive, i.e., if and only if*

$$p_{11} > 0, \quad \det \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} > 0, \ldots, \det P > 0.$$

(ii) A real symmetric matrix P is positive semidefinite *if and only if the determinants of* all of its principal minors *are nonnegative.* ■

Still digressing, we consider next the quadratic form

$$v(w) = w^T G w, \quad G = G^T,$$

where $G \in R^{n\times n}$. Now recall that there exists an orthogonal matrix Q such that the matrix P defined by

$$P = Q^{-1} G Q = Q^T G Q$$

is diagonal. Therefore, if we let $w = Qx$, then

$$v(Qx) \triangleq v(x) = x^T Q^T G Q x = x^T P x,$$

where P is in the form given by

$$P = \operatorname{diag}[\Lambda_i] \quad i = 1, \ldots, p,$$

where $\Lambda_i = \operatorname{diag} \lambda_i$. From this, we immediately obtain the following useful result.

Proposition 4.21. *Let $P = P^T \in R^{n \times n}$, let $\lambda_M(P)$ and $\lambda_m(P)$ denote the largest and smallest eigenvalues of P, respectively, and let $\| \cdot \|$ denote the Euclidean norm. Then*

$$\lambda_m(P) \| x \|^2 \leq v(x) = x^T P x \leq \lambda_M(P) \| x \|^2 \tag{4.26}$$

for all $x \in R^n$ (refer to [1]). ■

Let $c_1 \triangleq \lambda_m(P)$ and $c_2 = \lambda_M(P)$. Clearly, $v(x)$ is positive definite if and only if $c_2 \geq c_1 > 0, v(x)$ is positive semidefinite if and only if $c_2 \geq c_1 \geq 0, v(x)$ is indefinite if and only if $c_2 > 0, c_1 < 0$, and so forth.

We are now in a position to prove several results.

Theorem 4.22. *The equilibrium $x = 0$ of (4.22) is* stable *if there exists a real, symmetric, and positive definite $n \times n$ matrix P such that the matrix C given in (4.25) is negative semidefinite.*

Proof. Along any solution $\phi(t, x_0) \triangleq \phi(t)$ of (4.22) with $\phi(0, x_0) = \phi(0) = x_0$, we have

$$\phi(t)^T P \phi(t) = x_0^T P x_0 + \int_0^t \frac{d}{d\eta} \phi(\eta)^T P \phi(\eta) d\eta = x_0^T P x_0 + \int_0^t \phi(\eta)^T C \phi(\eta) d\eta$$

for all $t \geq 0$. Since P is positive definite and C is negative semidefinite, we have

$$\phi(t)^T P \phi(t) - x_0^T P x_0 \leq 0$$

for all $t \geq 0$, and there exist $c_2 \geq c_1 > 0$ such that

$$c_1 \| \phi(t) \|^2 \leq \phi(t)^T P \phi(t) \leq x_0^T P x_0 \leq c_2 \| x_0 \|^2$$

for all $t \geq 0$. It follows that

$$\| \phi(t) \| \leq (c_2/c_1)^{1/2} \| x_0 \|$$

for all $t \geq 0$ and for any $x_0 \in R^n$. Therefore, the equilibrium $x = 0$ of (4.22) is stable (refer to Theorem 4.12). ■

Example 4.23. For the system given in Example 4.16 we choose $P = I$, and we compute

$$C = A^T P + PA = A^T + A = 0.$$

According to Theorem 4.22, the equilibrium $x = 0$ of this system is stable (as expected from Example 4.16).

Theorem 4.24. *The equilibrium $x = 0$ of (4.22) is* exponentially stable in the large *if there exists a real, symmetric, and positive definite $n \times n$ matrix P such that the matrix C given in (4.25) is negative definite.*

Proof. We let $\phi(t, x_0) \triangleq \phi(t)$ denote an arbitrary solution of (4.22) with $\phi(0) = x_0$. In view of the hypotheses of the theorem, there exist constants $c_2 \geq c_1 > 0$ and $c_3 \geq c_4 > 0$ such that

$$c_1 \parallel \phi(t) \parallel^2 \leq v(\phi(t)) = \phi(t)^T P \phi(t) \leq c_2 \parallel \phi(t) \parallel^2$$

and

$$-c_3 \parallel \phi(t) \parallel^2 \leq \dot{v}(\phi(t)) = \phi(t)^T C \phi(t) \leq -c_4 \parallel \phi(t) \parallel^2$$

for all $t \geq 0$ and for any $x_0 \in R^n$. Then

$$\begin{aligned} \dot{v}(\phi(t)) &= \frac{d}{dt}[\phi(t)^T P \phi(t)] \leq (-c_4/c_2)\phi(t)^T P \phi(t) \\ &= (-c_4/c_2) v(\phi(t)) \end{aligned}$$

for all $t \geq t_0$. This implies, after multiplication by the appropriate integrating factor, and integrating from 0 to t, that

$$v(\phi(t)) = \phi(t)^T P \phi(t) \leq x_0^T P x_0 e^{-(c_4/c_2)t}$$

or

$$c_1 \parallel \phi(t) \parallel^2 \leq \phi(t)^T P \phi(t) \leq c_2 \parallel x_0 \parallel^2 e^{-(c_4/c_2)t}$$

or

$$\parallel \phi(t) \parallel \leq (c_2/c_1)^{1/2} \parallel x_0 \parallel e^{-\frac{1}{2}(c_4/c_2)t}, \quad t \geq 0.$$

This inequality holds for all $x_0 \in R^n$. Therefore, the equilibrium $x = 0$ of (4.22) is exponentially stable in the large (refer to Sections 4.3 and 4.4). ■

In Figure 4.3 we provide an interpretation of Theorem 4.24 for the two-dimensional case ($n = 2$). The curves C_i, called *level curves*, depict loci where $v(x)$ is constant; i.e., $C_i = \{x \in R^2 : v(x) = x^T P x = c_i\}, i = 0, 1, 2, 3, \ldots$. When the hypotheses of Theorem 4.24 are satisfied, trajectories determined by (4.22) penetrate level curves corresponding to decreasing values of c_i as t increases, tending to the origin as t becomes arbitrarily large.

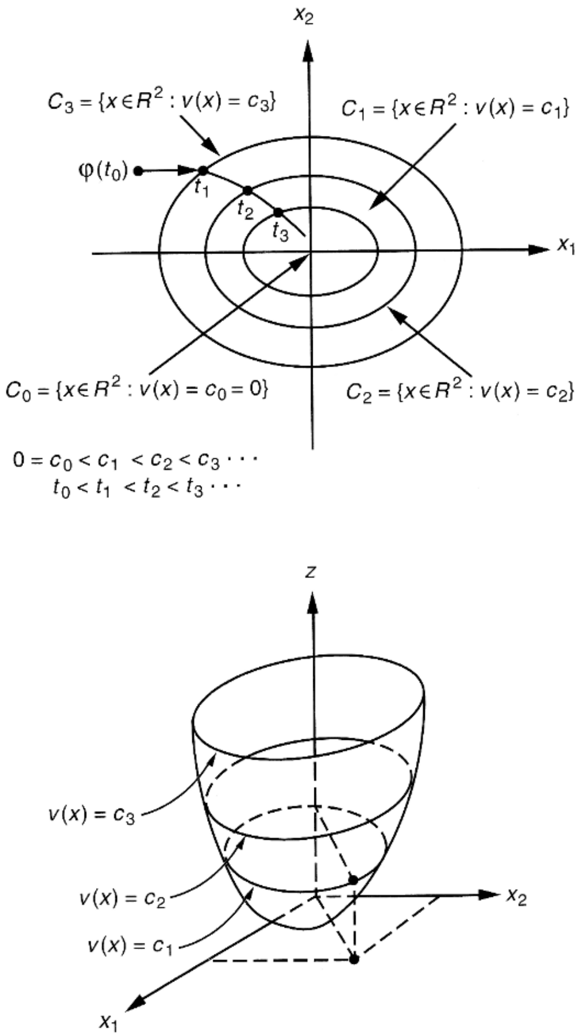

Figure 4.3. Asymptotic stability

Example 4.25. For the system given in Example 4.19, we choose

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix},$$

and we compute the matrix

$$C = A^T P + PA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

According to Theorem 4.24, the equilibrium $x = 0$ of this system is exponentially stable in the large (as expected from Example 4.19).

Theorem 4.26. *The equilibrium $x = 0$ of (4.22) is* unstable *if there exists a real, symmetric $n \times n$ matrix P that is either negative definite or indefinite such that the matrix C given in (4.25) is negative definite.*

Proof. We first assume that P is indefinite. Then P possesses eigenvalues of either sign, and every neighborhood of the origin contains points where the function

$$v(x) = x^T P x$$

is positive and negative. Consider the neighborhood

$$B(\epsilon) = \{x \in R^n : \| x \| < \epsilon\},$$

where $\| \cdot \|$ denotes the Euclidean norm, and let

$$G = \{x \in B(\epsilon) : v(x) < 0\}.$$

On the boundary of G we have either $\| x \| = \epsilon$ or $v(x) = 0$. In particular, note that the origin $x = 0$ is on the boundary of G. Now, since the matrix C is negative definite, there exist constants $c_3 > c_4 > 0$ such that

$$-c_3 \| x \|^2 \leq x^T C x = \dot{v}(x) \leq -c_4 \| x \|^2$$

for all $x \in R^n$. Let $\phi(t, x_0) \triangleq \phi(t)$ and let $x_0 = \phi(0) \in G$. Then $v(x_0) = -a < 0$. The solution $\phi(t)$ starting at x_0 must leave the set G. To see this, note that as long as $\phi(t) \in G, v(\phi(t)) \leq -a$ since $\dot{v}(x) < 0$ in G. Let $-c = \sup\{\dot{v}(x) : x \in G \text{ and } v(x) \leq -a\}$.

Then $c > 0$ and

$$\begin{aligned} v(\phi(t)) &= v(x_0) + \int_0^t \dot{v}(\phi(s)) ds \leq -a - \int_0^t c ds \\ &= -a - tc, t \geq t_0. \end{aligned}$$

This inequality shows that $\phi(t)$ must escape the set G (in finite time) because $v(x)$ is bounded from below on G. But $\phi(t)$ cannot leave G through the surface determined by $v(x) = 0$ since $v(\phi(t)) \leq -a$. Hence, it must leave G through the sphere determined by $\| x \| = \epsilon$. Since the above argument holds for arbitrarily small $\epsilon > 0$, it follows that the origin $x = 0$ of (4.22) is unstable.

Next, we assume that P is negative definite. Then G as defined is all of $B(\epsilon)$. The proof proceeds as above. ■

The proof of Theorem 4.26 shows that for $\epsilon > 0$ sufficiently small when P is negative definite, *all* solutions $\phi(t)$ of (4.22) with initial conditions $x_0 \in B(\epsilon)$ will tend away from the origin. This constitutes a severe case of instability, called *complete instability.*

Example 4.27. For the system given in Example 4.18, we choose

$$P = \begin{bmatrix} -0.28 & -0.96 \\ -0.96 & 0.28 \end{bmatrix},$$

and we compute the matrix

$$C = A^T P + PA = \begin{bmatrix} -20 & 0 \\ 0 & -20 \end{bmatrix}.$$

The eigenvalues of P are ± 1. According to Theorem 4.26, the equilibrium $x = 0$ of this system is unstable (as expected from Example 4.18).

In applying the results derived thus far in this section, we start by choosing (guessing) a matrix P having certain desired properties. Next, we solve for the matrix C, using (4.25). If C possesses certain desired properties (i.e., it is negative definite), we draw appropriate conclusions by applying one of the preceding theorems of this section; if not, we need to choose another matrix P. This points to the principal shortcoming of Lyapunov's Direct Method, when applied to general systems. However, in the *special case* of linear systems described by (4.22), it is possible to *construct* Lyapunov functions of the form $v(x) = x^T Px$ in a *systematic* manner. In doing so, one first chooses the matrix C in (4.25) (having desired properties), and then one solves (4.25) for P. Conclusions are then drawn by applying the appropriate results of this section. In applying this construction procedure, we need to know conditions under which (4.25) possesses a (unique) solution P for a given C. We will address this topic next.

We consider the quadratic form

$$v(x) = x^T Px, \quad P = P^T, \tag{4.27}$$

and the time derivative of $v(x)$ along the solutions of (4.22), given by

$$\dot{v}(x) = x^T Cx, \quad C = C^T, \tag{4.28}$$

where

$$C = A^T P + PA \tag{4.29}$$

and where all symbols are as defined in (4.23) to (4.25). Our objective is to determine the as yet unknown matrix P in such a way that $\dot{v}(x)$ becomes a preassigned negative definite quadratic form, i.e., in such a way that C is a preassigned negative definite matrix.

Equation (4.29) constitutes a system of $n(n+1)/2$ linear equations. We need to determine under what conditions we can solve for the $n(n+1)/2$ elements, p_{ik}, given C and A. To this end, we choose a similarity transformation Q such that

$$QAQ^{-1} = \bar{A}, \tag{4.30}$$

or equivalently,

$$A = Q^{-1}\bar{A}Q, \tag{4.31}$$

where $\bar{A}$ is similar to A and Q is a real $n \times n$ nonsingular matrix. From (4.31) and (4.29) we obtain

$$(\bar{A})^T(Q^{-1})^T PQ^{-1} + (Q^{-1})^T PQ^{-1}\bar{A} = (Q^{-1})^T CQ^{-1} \tag{4.32}$$

or

$$(\bar{A})^T\bar{P} + \bar{P}\bar{A} = \bar{C}, \quad \bar{P} = (Q^{-1})^T PQ^{-1}, \quad \bar{C} = (Q^{-1})^T CQ^{-1}. \tag{4.33}$$

In (4.33), P and C are subjected to a congruence transformation and $\bar{P}$ and $\bar{C}$ have the same definiteness properties as P and C, respectively. Since every real $n \times n$ matrix can be triangularized (refer to [1]), we can choose Q in such a fashion that $\bar{A} = [\bar{a}_{ij}]$ is *triangular*; i.e., $\bar{a}_{ij} = 0$ for $i > j$. Note that in this case the eigenvalues of $A, \lambda_1, \dots, \lambda_n$, appear in the main diagonal of $\bar{A}$. To simplify our notation, we rewrite (4.33) in the form (4.29) by dropping the bars, i.e.,

$$A^T P + PA = C, \quad C = C^T, \tag{4.34}$$

and *we assume that* $A = [a_{ij}]$ *has been triangularized*; i.e., $a_{ij} = 0$ for $i > j$. Since the eigenvalues $\lambda_1, \dots, \lambda_n$ appear in the diagonal of A, we can rewrite (4.34) as

$$\begin{aligned} 2\lambda_1 p_{11} &= c_{11} \\ a_{12}p_{11} + (\lambda_1 + \lambda_2)p_{12} &= c_{12} \\ \dots\dots\dots\dots\dots&\dots\dots \end{aligned} \tag{4.35}$$

Note that λ_1 may be a complex number; in which case, c_{11} will also be complex. Since this system of equations is triangular, and since its determinant is equal to

$$2^n \lambda_1 \dots \lambda_n \prod_{i<j}(\lambda_i + \lambda_j), \tag{4.36}$$

the matrix P can be determined (uniquely) if and only if this determinant is not zero. This is true when all eigenvalues of A are nonzero and no two of them are such that $\lambda_i + \lambda_j = 0$. This condition is not affected by a similarity transformation and is therefore also valid for the original system of equations (4.29).

We summarize the above discussion in the following lemma.

Lemma 4.28. *Let $A \in R^{n\times n}$ and let $\lambda_1, \ldots, \lambda_n$ denote the (not necessarily distinct) eigenvalues of A. Then (4.34) has a unique solution for P corresponding to each $C \in R^{n\times n}$ if and only if*

$$\lambda_i \neq 0, \lambda_i + \lambda_j \neq 0 \text{ for all } i, j. \tag{4.37}$$

■

To construct $v(x)$, we must still check the definiteness of P. This can be done in a purely algebraic way; however, in the present case, it is much easier to apply the results of this section and argue as follows:

(a) If all the eigenvalues λ_i of A have negative real parts, then the equilibrium $x = 0$ of (4.22) is exponentially stable in the large, and if C in (4.29) is negative definite, then P must be positive definite. To prove this, we note that if P is not positive definite, then for $\delta > 0$ and sufficiently small, $(P - \delta I)$ has at least one negative eigenvalue, whereas the function $v(x) = x^T(P - \delta I)x$ has a negative definite derivative; i.e.,

$$v'_{(L)}(x) = x^T[C - \delta(A + A^T)]x < 0$$

for all $x \neq 0$. By Theorem 4.26, the equilibrium $x = 0$ of (4.22) is unstable. We have arrived at a contradiction. Therefore, P must be positive definite.

(b) If A has eigenvalues with positive real parts and no eigenvalues with zero real parts, we can use a similarity transformation $x = Qy$ in such a way that $Q^{-1}AQ$ is a block diagonal matrix of the form diag$[A_1,\ A_2]$, where all the eigenvalues of A_1 have positive real parts, whereas all eigenvalues of A_2 have negative real parts (refer to [1]). (If A does not have any eigenvalues with negative real parts, then we take $A = A_1$). By the result established in (a), noting that all eigenvalues of $-A_1$ have negative real parts, given any negative definite matrices C_1 and C_2, there exist positive definite matrices P_1 and P_2 such that

$$(-A_1^T)P_1 + P_1(-A_1) = C_1, \quad A_2^T P_2 + P_2 A_2 = C_2.$$

Then $w(y) = y^T P y$, with $P = \text{diag}[-P_1,\ P_2]$ is a Lyapunov function for the system $\dot{y} = Q^{-1}AQy$ (and, hence, for the system $\dot{x} = Ax$), which satisfies the hypotheses of Theorem 4.26. Therefore, the equilibrium $x = 0$ of system (4.22) is unstable. If A does not have any eigenvalues with negative real parts, then the equilibrium $x = 0$ of system (4.22) is *completely unstable.*]

In the above proof, we did not invoke Lemma 4.28. We note, however, that if additionally, (4.37) is true, then we can construct the Lyapunov function for (4.22) in a systematic manner.

Summarizing the above discussion, we can now state the main result of this subsection.

Theorem 4.29. *Assume that the matrix A [for system (4.22)] has no eigenvalues with real part equal to zero. If all the eigenvalues of A have negative real parts, or if at least one of the eigenvalues of A has a positive real part, then there exists a quadratic Lyapunov function*

$$v(x) = x^T P x, \quad P = P^T,$$

whose derivative along the solutions of (4.22) is definite (i.e., it is either negative definite or positive definite). ■

This result shows that when A is a stable matrix (i.e., all the eigenvalues of A have negative real parts), then for system (4.22) the conditions of Theorem 4.24 are also necessary conditions for exponential stability in the large. Moreover, in the case when the matrix A has at least one eigenvalue with positive real part and no eigenvalues on the imaginary axis, then the conditions of Theorem 4.26 are also necessary conditions for instability.

Example 4.30. We consider the system (4.22) with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm j$, and therefore condition (4.37) is violated. According to Lemma 4.28, the Lyapunov matrix equation

$$A^T P + PA = C$$

does not possess a unique solution for a given C. We now verify this for two specific cases.

(i) When $C = 0$, we obtain

$$\begin{aligned} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} -2p_{12} & p_{11} - p_{22} \\ p_{11} - p_{22} & 2p_{12} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

or $p_{12} = 0$ and $p_{11} = p_{22}$. Therefore, for any $a \in R$, the matrix $P = aI$ is a solution of the Lyapunov matrix equation. In other words, for $C = 0$, the Lyapunov matrix equation has in this example denumerably many solutions.

(ii) When $C = -2I$, we obtain

$$\begin{bmatrix} -2p_{12} & p_{11} - p_{22} \\ p_{11} - p_{22} & 2p_{12} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

or $p_{11} = p_{22}$ and $p_{12} = 1$ and $p_{12} = -1$, which is impossible. Therefore, for $C = -2I$, the Lyapunov matrix equation has in this example no solutions at all.

It turns out that if all the eigenvalues of matrix A have negative real parts, then we can compute P in (4.29) explicitly.

Theorem 4.31. *If all eigenvalues of a real $n \times n$ matrix A have negative real parts, then for each matrix $C \in R^{n\times n}$, the unique solution of (4.29) is given by*

$$P = \int_0^\infty e^{A^T t}(-C)e^{At}dt. \tag{4.38}$$

Proof. If all eigenvalues of A have negative real parts, then (4.37) is satisfied and therefore (4.29) has a unique solution for every $C \in R^{n\times n}$. To verify that (4.38) is indeed this solution, we first note that the right-hand side of (4.38) is well defined, since all eigenvalues of A have negative real parts. Substituting the right-hand side of (4.38) for P into (4.29), we obtain

$$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{A^T t}(-C)e^{At}dt + \int_0^\infty e^{A^T t}(-C)e^{At}Adt \\ &= \int_0^\infty \frac{d}{dt}[e^{A^T t}(-C)e^{At}]dt \\ &= e^{A^T t}(-C)e^{At}\Big|_0^\infty = C, \end{aligned}$$

which proves the theorem. ■

4.6 Linearization

In this section we consider *nonlinear, finite-dimensional, continuous-time* dynamical systems described by equations of the form

$$\dot{w} = f(w), \tag{4.39}$$

where $f \in C^1(R^n, R^n)$. We assume that $w = 0$ is an equilibrium of (4.39). In accordance with Subsection 1.6.1, we linearize system (4.39) about the origin to obtain

$$\dot{x} = Ax + F(x), \tag{4.40}$$

$x \in R^n$, where $F \in C(R^n, R^n)$ and where A denotes the Jacobian of $f(w)$ evaluated at $w = 0$, given by

$$A = \frac{\partial f}{\partial w}(0), \tag{4.41}$$

and where

$$F(x) = o(\| x \|) \quad \text{as} \quad \| x \| \to 0. \tag{4.42}$$

Associated with (4.40) is the *linearization* of (4.39), given by

$$\dot{y} = Ay. \tag{4.43}$$

In the following discussion, we use the results of Section 4.5 to establish criteria that allow us to deduce the stability properties of the equilibrium $w = 0$ of the nonlinear system (4.39) from the stability properties of the equilibrium $y = 0$ of the linear system (4.43).

Theorem 4.32. *Let $A \in R^{n\times n}$ be a Hurwitz matrix (i.e., all of its eigevnalues have negative real parts), let $F \in C(R^n, R^n)$, and assume that (4.42) holds. Then the equilibrium $x = 0$ of (4.40) [and, hence, of (4.39)] is* exponentially stable*.*

Proof. Theorem 4.29 applies to (4.43) since all the eigenvalues of A have negative real parts. In view of that theorem (and the comments following Lemma 4.28), there exists a symmetric, real, positive definite $n \times n$ matrix P such that

$$PA + A^T P = C, \tag{4.44}$$

where C is negative definite. Consider the Lyapunov function

$$v(x) = x^T P x. \tag{4.45}$$

The derivative of v with respect to t along the solutions of (4.40) is given by

$$\begin{aligned} \dot v(x) &= \dot x^T P x + x^T P \dot x \\ &= (Ax + F(x))^T P x + x^T P(Ax + F(x)) \\ &= x^T C x + 2x^T P F(x). \end{aligned} \tag{4.46}$$

Now choose $\gamma < 0$ such that $x^T C x \le 3\gamma \parallel x \parallel^2$ for all $x \in R^n$. Since it is assumed that (4.42) holds, there is a $\delta > 0$ such that if $\parallel x \parallel \le \delta$, then $\parallel PF(x) \parallel \le -\gamma \parallel x \parallel$ for all $x \in \overline{B(\delta)} = \{x \in R^n : \parallel x \parallel \le \delta\}$. Therefore, for all $x \in \overline{B(\delta)}$, we obtain, in view of (4.46), the estimate

$$\dot v(x) \le 3\gamma \parallel x \parallel^2 - 2\gamma \parallel x \parallel^2 = \gamma \parallel x \parallel^2 . \tag{4.47}$$

Now let $\alpha = \min_{\parallel x \parallel = \delta} v(x)$. Then $\alpha > 0$ (since P is positive definite). Take $\lambda \in (0, \alpha)$, and let

$$C_\lambda = \{x \in B(\delta) = \{x \in R^n : \parallel x \parallel < \delta\} : v(x) \le \lambda\}. \tag{4.48}$$

Then $C_\lambda \subset B(\delta)$. [This can be shown by contradiction. Suppose that C_λ is not entirely inside $B(\delta)$. Then there is a point $\bar x \in C_\lambda$ that lies on the boundary of $B(\delta)$. At this point, $v(\bar x) \ge \alpha > \lambda$. We have thus arrived at a contradiction.] The set C_λ has the property that any solution of (4.40) starting in C_λ at $t = 0$ will stay in C_λ for all $t \ge 0$. To see this, we let $\phi(t, x_0) \triangleq \phi(t)$ and we recall that $\dot v(x) \le \gamma \parallel x \parallel^2, \gamma < 0, x \in B(\delta) \supset C_\lambda$. Then $\dot v(\phi(t)) \le 0$ implies that $v(\phi(t)) \le v(x_0) \le \lambda$ for all $t \ge t_0 \ge 0$. Therefore, $\phi(t) \in C_\lambda$ for all $t \ge t_0 \ge 0$.

We now proceed in a similar manner as in the proof of Theorem 4.24 to complete this proof. In doing so, we first obtain the estimate

$$\dot{v}(\phi(t)) \leq (\gamma/c_2)v(\phi(t)), \tag{4.49}$$

where γ is given in (4.47) and c_2 is determined by the relation

$$c_1 \parallel x \parallel^2 \leq v(x) = x^T P x \leq c_2 \parallel x \parallel^2 . \tag{4.50}$$

Following now in an identical manner as was done in the proof of Theorem 4.22, we have

$$\parallel \phi(t) \parallel \leq (c_2/c_1)^{\frac{1}{2}} \parallel x_0 \parallel e^{\frac{1}{2}(\gamma/c_2)t}, \quad t \geq 0, \tag{4.51}$$

whenever $x_0 \in B(r')$, where r' has been chosen sufficiently small so that $B(r') \subset C_\lambda$. This proves that the equilibrium $x = 0$ of (4.40) is exponentially stable. ■

It is important to recognize that Theorem 4.32 is a *local result* that yields sufficient conditions for the exponential stability of the equilibrium $x = 0$ of (4.40); it does not yield conditions for exponential stability in the large. The proof of Theorem 4.32, however, enables us to determine an estimate of the domain of attraction of the equilibrium $x = 0$ of (4.39), involving the following steps:

1. Determine an equilibrium, x_e, of (4.39) and transform (4.39) to a new system that translates x_e to the origin $x = 0$ (refer to Section 4.2).
2. Linearize (4.39) about the origin and determine $F(x), A$, and the eigenvalues of A.
3. If all eigenvalues of A have negative real parts, choose a negative definite matrix C and solve the Lyapunov matrix equation
$$C = A^T P + PA.$$
4. Determine the Lyapunov function
$$v(x) = x^T P x.$$
5. Compute the derivative of v along the solutions of (4.40), given by
$$\dot{v}(x) = x^T C x + 2x^T P F(x).$$
6. Determine $\delta > 0$ such that $\dot{v}(x) < 0$ for all $x \in B(\delta) - \{0\}$.
7. Determine the largest $\lambda = \lambda_M$ such that $C_{\lambda_M} \subset B(\delta)$, where
$$C_\lambda = \{x \in R^n : v(x) < \lambda\}.$$
8. C_{λ_M} is a subset of the domain of attraction of the equilibrium $x = 0$ of (4.40) and, hence, of (4.39).

The above procedure may be repeated for different choices of matrix C given in step (3), resulting in different matrices P_i, which in turn may result in different estimates for the domain of attraction, $C^i_{\lambda_M}$, $i \in \Lambda$, where Λ is an index set. The union of the sets $C^i_{\lambda_M} \triangleq D_i, D = \cup_i D_i$, is also a subset of the domain of attraction of the equilibrium $x = 0$ of (4.39).

Theorem 4.33. *Assume that A is a real $n \times n$ matrix that has at least one eigenvalue with positive real part and no eigenvalue with zero real part. Let $F \in C(R^n, R^n)$, and assume that (4.42) holds. Then the equilibrium $x = 0$ of (4.40) [and, hence, of (4.39)] is* unstable.

Proof. We use Theorem 4.29 to choose a real, symmetric $n \times n$ matrix P such that the matrix $PA + A^T P = C$ is negative definite. The matrix P is not positive definite, or even positive semidefinite (refer to the comments following Lemma 4.28). Hence, the function $v(x) = x^T P x$ is negative at some points arbitrarily close to the origin. The derivative of $v(x)$ with respect to t along the solutions of (4.40) is given by (4.46). As in the proof of Theorem 4.32, we can choose a $\gamma < 0$ such that $x^T C x \leq 3\gamma \parallel x \parallel^2$ for all $x \in R^n$, and in view of (4.42) we can choose a $\delta > 0$ such that $\parallel PF(x) \parallel \leq -\gamma \parallel x \parallel$ for all $x \in B(\delta)$. Therefore, for all $x \in B(\delta)$, we obtain that

$$\dot{v}(x) \leq 3\gamma \parallel x \parallel^2 - 2\gamma \parallel x \parallel^2 = \gamma \parallel x \parallel^2 .$$

Now let

$$G = \{x \in B(\delta) : v(x) < 0\}.$$

The boundary of G is made up of points where $v(x) = 0$ and where $\parallel x \parallel = \delta$. Note in particular that the equilibrium $x = 0$ of (4.40) is in the boundary of G. Now following an identical procedure as in the proof of Theorem 4.26, we show that any solution $\phi(t)$ of (4.40) with $\phi(0) = x_0 \in G$ must escape G in finite time through the surface determined by $\parallel x \parallel = \delta$. Since the above argument holds for arbitrarily small $\delta > 0$, it follows that the origin $x = 0$ of (4.40) is unstable. ■

Before concluding this section, we consider a few specific cases.

Example 4.34. The *Lienard Equation* is given by

$$\ddot{w} + f(w)\dot{w} + w = 0, \tag{4.52}$$

where $f \in C^1(R, R)$ with $f(0) > 0$. Letting $x_1 = w$ and $x_2 = \dot{w}$, we obtain

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - f(x_1)x_2. \end{aligned} \tag{4.53}$$

Let $x^T = (x_1, x_2)$, $f(x)^T = (f_1(x), f_2(x))$, and let

$$J(0) = A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(0) & \frac{\partial f_1}{\partial x_2}(0) \\ \frac{\partial f_2}{\partial x_1}(0) & \frac{\partial f_2}{\partial x_2}(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -f(0) \end{bmatrix}.$$

Then

$$\dot{x} = Ax + [f(x) - Ax] = Ax + F(x),$$

where

$$F(x) = \begin{bmatrix} 0 \\ [f(0) - f(x_1)]\, x_2 \end{bmatrix}.$$

The origin $x = 0$ is clearly an equilibrium of (4.52) and hence of (4.53). The eigenvalues of A are given by

$$\lambda_1, \lambda_2 = \frac{-f(0) \pm \sqrt{f(0)^2 - 4}}{2},$$

and therefore, A is a Hurwitz matrix. Also, (4.42) holds. Therefore, all the conditions of Theorem 4.32 are satisfied. We conclude that the equilibrium $x = 0$ of (4.53) is *exponentially stable.*

Example 4.35. We consider the system given by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= -x_2 + x_2(x_1^2 + x_2^2). \end{aligned} \tag{4.54}$$

The origin is clearly an equilibrium of (4.54). Also, the system is already in the form (4.40) with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F(x) = \begin{bmatrix} x_1(x_1^2 + x_2^2) \\ x_2(x_1^2 + x_2^2) \end{bmatrix},$$

and condition (4.42) is clearly satisfied. The eigenvalues of A are $\lambda_1 = -1, \lambda_2 = -1$. Therefore, all conditions of Theorem 4.32 are satisfied and we conclude that the equilibrium $x^T = (x_1, x_2) = 0$ is *exponentially stable*; however, we cannot conclude that this equilibrium is exponentially stable in the large. Accordingly, we seek to determine an estimate for the domain of attraction of this equilibrium.

We choose $C = -I$ (where $I \in R^{2\times 2}$ denotes the identity matrix), and we solve the matrix equation $A^T P + PA = C$ to obtain $P = (1/2)I$, and therefore,

$$v(x_1, x_2) = x^T P x = \frac{1}{2}(x_1^2 + x_2^2).$$

Along the solutions of (4.54) we obtain

$$\begin{aligned} \dot{v}(x_1, x_2) &= x^T C x + 2x^T P F(x) \\ &= -(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2. \end{aligned}$$

Clearly, $\dot{v}(x_1, x_2) < 0$ when $(x_1, x_2) \neq (0, 0)$ and $x_1^2 + x_2^2 < 1$. In the language of the proof of Theorem 4.32, we can therefore choose $\delta = 1$.

Now let

$$C_{1/2} = \{x \in R^2 : v(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) < \frac{1}{2}\}.$$

Then clearly, $C_{1/2} \subset B(\delta), \delta = 1$, in fact $C_{1/2} = B(\delta)$. Therefore, the set $\{x \in R^2 : x_1^2 + x_2^2 < 1\}$ is a subset of the domain of attraction of the equilibrium $(x_1, x_2)^T = 0$ of system (4.54).

Example 4.36. The differential equation governing the motion of a pendulum is given by

$$\ddot{\theta} + a \sin\theta = 0, \tag{4.55}$$

where $a > 0$ is a constant (refer to Chapter 1). Letting $\theta = x_1$ and $\dot{\theta} = x_2$, we obtain the system description

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a \sin x_1. \end{aligned} \tag{4.56}$$

The points $x_e^{(1)} = (0, 0)^T$ and $x_e^{(2)} = (\pi, 0)^T$ are equilibria of (4.56).

(i) Linearizing (4.56) about the equilibrium $x_e^{(1)}$, we put (4.56) into the form (4.40) with

$$A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm j\sqrt{a}$. Therefore, the results of this section (Theorem 4.32 and 4.33) are not applicable in the present case.

(ii) In (4.56), we let $y_1 = x_1 - \pi$ and $y_2 = x_2$. Then (4.56) assumes the form

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -a \sin(y_1 + \pi). \end{aligned} \tag{4.57}$$

The point $(y_1, y_2)^T = (0, 0)^T$ is clearly an equilibrium of system (4.57). Linearizing about this equilibrium, we put (4.57) into the form (4.40), where

$$A = \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \quad F(y_1, y_2) = \begin{bmatrix} 0 \\ -a(\sin(y_1 + \pi) + y_1) \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = a, -a$. All conditions of Theorem 4.33 are satisfied, and we conclude that the equilibrium $x_e^{(2)} = (\pi, 0)^T$ of system (4.56) is *unstable*.

4.7 Input–Output Stability

We now turn our attention to systems described by the state equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \tag{4.58}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$, and $D \in R^{p\times m}$. In the preceding sections of this chapter we investigated the *internal stability properties* of system (4.58) by studying the Lyapunov stability of the trivial solution of the associated system

$$\dot{w} = Aw. \tag{4.59}$$

In this approach, system inputs and system outputs played no role. To account for these, we now consider the *external stability properties* of system (4.58), called *input–output stability*: Every bounded input of a system should produce a bounded output. More specifically, in the present context, we say that system (4.58) is *bounded-input/bounded-output (BIBO) stable*, if for zero initial conditions at $t = 0$, every bounded input defined on $[0, \infty)$ gives rise to a bounded response on $[0, \infty)$.

Matrix D does not affect the BIBO stability of (4.58). Accordingly, we will consider without any loss of generality the case where $D \equiv 0$; i.e., throughout this section we will concern ourselves with systems described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx. \end{aligned} \tag{4.60}$$

We will say that the system (4.60) is *BIBO stable* if there exists a constant $c > 0$ such that the conditions

$$\begin{aligned} x(0) &= 0, \\ ||u(t)|| &\le 1, \quad t \ge 0, \end{aligned}$$

imply that $\| y(t) \| \le c$ for all $t \ge 0$. (The symbol $\| \cdot \|$ denotes the Euclidean norm.)

Recall that for system (4.60) the impulse response matrix is given by

$$\begin{aligned} H(t) &= Ce^{At}B, \quad t \ge 0, \\ &= 0, \qquad\quad\ \ t < 0, \end{aligned} \tag{4.61}$$

and the transfer function matrix is given by

$$\widehat{H}(s) = C(sI - A)^{-1}B. \tag{4.62}$$

Theorem 4.37. *The system (4.60) is* BIBO stable *if and only if there exists a finite constant $L > 0$ such that for all t,*

$$\int_0^t \| H(t-\tau) \| \, d\tau \le L. \tag{4.63}$$

Proof. The first part of the proof of Theorem 4.37 (sufficiency) is straightforward. Indeed, if $\| u(t) \| \leq 1$ for all $t \geq 0$ and if (4.63) is true, then we have for all $t \geq 0$ that

$$\begin{aligned}
\| y(t) \| &= \| \int_0^t H(t-\tau)u(\tau)d\tau \| \\
&\leq \int_0^t \| H(t-\tau)u(\tau) \| d\tau \\
&\leq \int_0^t \| H(t-\tau) \| \| u(\tau) \| d\tau \\
&\leq \int_0^t \| H(t-\tau) \| d\tau \leq L.
\end{aligned}$$

Therefore, system (4.60) is BIBO stable.

In proving the second part of Theorem 4.37 (necessity), we simplify matters by first considering in (4.60) the single-variable case ($n = 1$) with the input–output description given by

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau. \tag{4.64}$$

For purposes of contradiction, we assume that the system is BIBO stable, but no finite L exists such that (4.63) is satisfied. Another way of stating this is that for *every finite* L, there exists $t_1 = t_1(L), t_1 > 0$, such that

$$\int_0^{t_1} |h(t_1, \tau)| d\tau > L.$$

We now choose in particular the input given by

$$u(t) = \begin{cases} +1 & \text{if } h(t-\tau) > 0, \\ 0 & \text{if } h(t-\tau) = 0, \\ -1 & \text{if } h(t-\tau) < 0, \end{cases} \tag{4.65}$$

$0 \leq t \leq t_1$. Clearly, $|u(t)| \leq 1$ for all $t \geq 0$. The output of the system at $t = t_1$ due to the above input, however, is

$$y(t_1) = \int_0^{t_1} h(t_1 - \tau)u(\tau)d\tau = \int_0^{t_1} |h(t_1 - \tau)| d\tau > L,$$

which contradicts the assumption that the system is BIBO stable.

The above can now be generalized to the multivariable case. In doing so, we apply the single-variable result to every possible pair of input and output vector components, we make use of the fact that the sum of a finite number of bounded sums will be bounded, and we recall that a vector is bounded if and only if each of its components is bounded. We leave the details to the reader. ■

In the preceding argument we made the tacit assumption that u is continuous, or piecewise continuous. However, our particular choice of u may involve nondenumerably many switchings (discontinuities) over a given finite-time interval. In such cases, u is no longer piecewise continuous; however, it is measurable (in the Lebesgue sense). This generalization can be handled, although in a broader mathematical setting that we do not wish to pursue here. The interested reader may want to refer, e.g., to the books by Desoer and Vidyasagar [5], Michel and Miller [13], and Vidyasagar [20] and the papers by Sandberg [17] to [19] and Zames [21], [22] for further details.

From Theorem 4.37 and from (4.61) it follows readily that a necessary and sufficient condition for the BIBO stability of system (4.60) is the condition

$$\int_0^\infty \| H(t) \| \, dt < \infty. \tag{4.66}$$

Corollary 4.38. *Assume that the equilibrium $w = 0$ of (4.59) is exponentially stable. Then system (4.60) is* BIBO stable.

Proof. Under the hypotheses of the corollary, we have

$$\begin{aligned}\| \int_0^t H(t-\tau) d\tau \| &\leq \int_0^t \| H(t-\tau) \| \, d\tau \\ &= \int_0^t \| C\Phi(t-\tau)B \| \, d\tau \leq \| C \| \| B \| \int_0^t \| \Phi(t-\tau) \| \, d\tau.\end{aligned}$$

Since the equilibrium $w = 0$ of (4.59) is exponentially stable, there exist $\delta > 0$, $\lambda > 0$ such that $\| \Phi(t,\tau) \| \leq \delta e^{-\lambda(t-\tau)}, t \geq \tau$. Therefore,

$$\begin{aligned}\int_0^t \| H(t-\tau) \| \, d\tau &\leq \int_0^t \| C \| \| B \| \, \delta e^{-\lambda(t-\tau)} d\tau \\ &\leq (\| C \| \| B \| \, \delta)/\lambda \triangleq L\end{aligned}$$

for all τ, t with $t \geq \tau$. It now follows from Theorem 4.37 that system (4.60) is BIBO stable. ■

In Section 7.3 we will establish a connection between the BIBO stability of (4.60) and the exponential stability of the trivial solution of (4.59).

Next, we recall that a complex number s_p is a *pole* of $\widehat{H}(s) = [\hat{h}_{ij}(s)]$ if for some pair (i,j), we have $|\hat{h}_{ij}(s_p)| = \infty$. If each entry of $\widehat{H}(s)$ has only poles with negative real values, then, as shown in Chapter 3, each entry of $H(t) = [h_{ij}(t)]$ has a sum of exponentials with exponents with real part negative. It follows that the integral

$$\int_0^\infty \| H(t) \| \, dt$$

is finite, and any realization of $\widehat{H}(s)$ will result in a system that is BIBO stable.

Now conversely, if

$$\int_0^\infty \| H(t) \| \, dt$$

is finite, then the exponential terms in any entry of $H(t)$ must have negative real parts. But then every entry of $\widehat{H}(s)$ has poles whose real parts are negative.

We have proved the following result.

Theorem 4.39. *The system (4.60) is BIBO stable if and only if all poles of the transfer function $\widehat{H}(s)$ given in (4.62) have only poles with negative real parts.* ■

Example 4.40. A system with $H(s) = 1/s$ is not BIBO stable. To see this consider a step input. The response is then given by $y(t) = t$, $t \geq 0$, which is not bounded.

4.8 Discrete-Time Systems

In this section we address the Lyapunov stability of an equilibrium of discrete-time systems (internal stability) and the input–output stability of discrete-time systems (external stability). We establish results for discrete-time systems that are analogous to practically all the stability results that we presented for continuous-time systems.

This section is organized into five subsections. In the first subsection we provide essential preliminary material. In the second and third subsections we establish results for the stability, instability, asymptotic stability, and exponential stability of an equilibrium and boundedness of solutions of systems described by linear autonomous ordinary difference equations. These results are used to develop Lyapunov stability results for linearizations of nonlinear systems described by ordinary difference equations in the fourth subsection. In the last subsection we present results for the input–output stability of linear time-invariant discrete-time systems.

4.8.1 Preliminaries

We concern ourselves here with finite-dimensional discrete-time systems described by difference equations of the form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{4.67}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, k \geq k_0$, and $k, k_0 \in Z^+$. Since (4.67) is time-invariant, we will assume without loss of generality that $k_0 = 0$, and thus, $x : Z^+ \to R^n, y : Z^+ \to R^p$, and $u : Z^+ \to R^m$.

The internal dynamics of (4.67) under conditions of no input are described by equations of the form

$$x(k+1) = Ax(k). \tag{4.68}$$

Such equations may arise in the modeling process, or they may be the consequence of the linearization of nonlinear systems described by equations of the form

$$x(k+1) = g(x(k)), \tag{4.69}$$

where $g : R^n \to R^n$. For example, if $g \in C^1(R^n, R^n)$, then in linearizing (4.69) about, e.g., $x = 0$, we obtain

$$x(k+1) = Ax(k) + f(x(k)), \tag{4.70}$$

where $A = \frac{\partial f}{\partial x}(x)\Big|_{x=0}$ and where $f : R^n \to R^n$ is $o(||x||)$ as a norm of x (e.g., the Euclidean norm) approaches zero. Recall that this means that given $\epsilon > 0$, there is a $\delta > 0$ such that $\| f(x) \| < \epsilon \| x \|$ for all $\| x \| < \delta$.

As in Section 4.7, we will study the *external qualitative properties* of system (4.67) by means of the BIBO stability of such systems. Consistent with the definition of input–output stability of continuous-time systems, we will say that the system (4.67) is *BIBO stable* if there exists a constant $L > 0$ such that the conditions

$$x(0) = 0,$$
$$\| u(k) \| \leq 1, \quad k \geq 0,$$

imply that $\| y(k) \| \leq L$ for all $k \geq 0$.

We will study the *internal qualitative properties* of system (4.67) by studying the *Lyapunov stability* properties of an equilibrium of (4.68).

Since system (4.69) is time-invariant, we will assume without loss of generality that $k_0 = 0$. As in Chapters 1 and 2, we will denote for a given set of initial data $x(0) = x_0$ the solution of (4.69) by $\phi(k, x_0)$. When x_0 is understood or of no importance, we will frequently write $\phi(k)$ in place of $\phi(k, x_0)$. Recall that for system (4.69) [as well as systems (4.67), (4.68), and (4.70)], there are no particular difficulties concerning the existence and uniqueness of solutions, and furthermore, as long as g in (4.69) is continuous, the solutions will be continuous with respect to initial data. Recall also that in contrast to systems described by ordinary differential equations, the solutions of systems described by ordinary difference equations [such as (4.69)] exist only in the forward direction of time ($k \geq 0$).

We say that $x_e \in R^n$ is an *equilibrium* of system (4.69) if $\phi(k, x_e) \equiv x_e$ for all $k \geq 0$, or equivalently,

$$g(x_e) = x_e. \tag{4.71}$$

As in the continuous-time case, we will assume without loss of generality that the equilibrium of interest will be the origin; i.e., $x_e = 0$. If this is not the case, then we can always transform (similarly as in the continuous-time case) system (4.69) into a system of equations that has an equilibrium at the origin.

Example 4.41. The system described by the equation

$$x(k+1) = x(k)[x(k) - 1]$$

has two equilibria, one at $x_{e1} = 0$ and another at $x_{e2} = 1$.

Example 4.42. The system described by the equations

$$x_1(k+1) = x_2(k),$$
$$x_2(k+1) = -x_1(k)$$

has an equilibrium at $x_e^T = (0, 0)$.

Throughout this section we will assume that the function g in (4.69) is continuous, or if required, continuously differentiable. The various definitions of Lyapunov stability of the equilibrium $x = 0$ of system (4.69) are essentially identical to the corresponding definitions of Lyapunov stability of an equilibrium of continuous-time systems described by ordinary differential equations, replacing $t \in R^+$ by $k \in Z^+$. We will concern ourselves with stability, instability, asymptotic stability, and exponential stability of the equilibrium $x = 0$ of (4.69).

We say that the equilibrium $x = 0$ of (4.69) is *stable* if for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\| \phi(k, x_0) \| < \epsilon$ for all $k \geq 0$ whenever $\| x_0 \| < \delta$. If the equilibrium $x = 0$ of (4.69) is not stable, it is said to be *unstable*. We say that the equilibrium $x = 0$ of (4.69) is *asymptotically stable* if (i) it is stable and (ii) there exists an $\eta > 0$ such that if $\| x_0 \| < \eta$, then $\lim_{k\to\infty} \| \phi(k, x_0) \| = 0$. If the equilibrium $x = 0$ satisfies property (ii), it is said to be *attractive*, and we call the set of all $x_0 \in R^n$ for which $x = 0$ is attractive the *domain of attraction* of this equilibrium. If $x = 0$ is asymptotically stable and if its domain of attraction is all of R^n, then it is said to be *asymptotically stable in the large* or *globally asymptotically stable*. We say that the equililbrium $x = 0$ of (4.69) is *exponentially stable* if there exists an $\alpha > 0$ and for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that $\| \phi(k, x_0) \| \leq \epsilon e^{-\alpha k}$ for all $k \geq 0$ whenever $\| x_0 \| < \delta(\epsilon)$. The equilibrium $x = 0$ of (4.69) is *exponentially stable in the large* if there exists $\alpha > 0$ and for any $\beta > 0$, there exists $k(\beta) > 0$ such that $\| \phi(t, x_0) \| \leq k(\beta) \| x_0 \| e^{-\alpha k}$ for all $k > 0$ whenever $\| x_0 \| < \beta$. Finally, we say that a solution of (4.69) through x_0 is *bounded* if there is a constant M such that $\| \phi(k, x_0) \| \leq M$ for all $k \geq 0$.

4.8.2 Linear Systems

In proving some of the results of this section, we require a result for system (4.68) that is analogous to Theorem 3.1. As in the proof of that theorem, we note that the linear combination of solutions of system (4.68) is also a solution of system (4.68), and hence, the set of solutions $\{\phi : Z^+ \times R^n \to R^n\}$ constitutes a vector space (over $F = R$ or $F = C$). The dimension of this vector space is n. To show this, we choose a set of linearly independent vectors $x_0^1, \dots, x_0^n$ in the n-dimensional x-space (R^n or C^n) and we show, in an identical manner as in the proof of Theorem 3.1, that the set of solutions $\phi(k, x_0^i), i = 1, \dots, n$, is linearly independent and spans the set of solutions of system (4.68). (We ask the reader in the Exercise section to provide the details of the proof of the above assertions.) This yields the following result.

Theorem 4.43. *The set of solutions of system (4.68) over the time interval Z^+ forms an n-dimensional vector space.* ■

Incidentally, if in particular we choose $\phi(k, e^i), i = 1, \dots, n$, where $e^i, i = 1, \dots, n$, denotes the natural basis for R^n, and if we let $\Phi(k, k_0 = 0) \triangleq \Phi(k) = [\phi(k, e^1), \dots, \phi(k, e^n)]$, then it is easily verified that the $n \times n$ matrix $\Phi(k)$ satisfies the matrix equation

$$\Phi(k+1) = A\Phi(k), \quad \Phi(0) = I,$$

and that $\Phi(k) = A^k, k \geq 0$ [i.e., $\Phi(k)$ is the state transition matrix for system (4.68)].

Theorem 4.44. *The equilibrium $x = 0$ of system (4.68) is stable if and only if the solutions of (4.68) are bounded.*

Proof. Assume that the equilibrium $x = 0$ of (4.68) is stable. Then for $\epsilon = 1$ there is a $\delta > 0$ such that $\| \phi(k, x_0) \| < 1$ for all $k \geq 0$ and all $\| x_0 \| \leq \delta$. In this case

$$\| \phi(k, x_0) \| = \| A^k x_0 \| = \| A^k x_0 \delta / \| x_0 \| \| (\| x_0 \| / \delta) < \| x_0 \| / \delta$$

for all $x_0 \neq 0$ and all $k \geq 0$. Using the definition of matrix norm [refer to Section A.7] it follows that $\| A^k \| \leq \delta^{-1}, k \geq 0$. We have proved that if the equilibrium $x = 0$ of (4.68) is stable, then the solutions of (4.68) are bounded.

Conversely, suppose that all solutions $\phi(k, x_0) = A^k x_0$ are bounded. Let $\{e^1, \dots, e^n\}$ denote the natural basis for n-space and let $\| \phi(k, e^j) \| < \beta_j$ for all $k \geq 0$. Then for any vector $x_0 = \sum_{j=1}^n \alpha_j e^j$ we have that

$$\begin{aligned} \| \phi(k, x_0) \| &= \| \sum_{j=1}^n \alpha_j \phi(k, e^j) \| \leq \sum_{j=1}^n |\alpha_j| \beta_j \leq (\max_j \beta_j) \sum_{j=1}^n |\alpha_j| \\ &\leq c \| x_0 \|, \quad k \geq 0, \end{aligned}$$

for some constant c. For given $\epsilon > 0$, we choose $\delta = \epsilon/c$. Then, if $\| x_0 \| < \delta$, we have $\| \phi(k, x_0) \| < c \| x_0 \| < \epsilon$ for all $k \geq 0$. We have proved that if the solutions of (4.68) are bounded, then the equilibrium $x = 0$ of (4.68) is stable. ■

Theorem 4.45. *The following statements are equivalent:*

(i) The equilibrium $x = 0$ of (4.68) is asymptotically stable,
(ii) The equilibrium $x = 0$ of (4.68) is asymptotically stable in the large,
(iii) $\lim_{k\to\infty} \| A^k \| = 0$.

Proof. Assume that statement (i) is true. Then there is an $\eta > 0$ such that when $\| x_0 \| \leq \eta$, then $\phi(k, x_0) \to 0$ as $k \to \infty$. But then we have for *any* $x_0 \neq 0$ that

$$\phi(k, x_0) = A^k x_0 = [A^k(\eta x_0 / \| x_0 \|)] \| x_0 \| / \eta \to 0 \text{ as } k \to \infty.$$

It follows that statement (ii) is true.

Next, assume that statement (ii) is true. Then for any $\epsilon > 0$ there must exist a $K = K(\epsilon)$ such that for all $k \geq K$ we have that $\| \phi(k, x_0) \| = \| A^k x_0 \| < \epsilon$. To see this, let $\{e^1, \ldots, e^n\}$ be the natural basis for R^n. Thus, for a fixed constant $c > 0$, if $x_0 = (\alpha_1, \ldots, \alpha_n)^T$ and if $\| x_0 \| \leq 1$, then $x_0 = \sum_{j=1}^n \alpha_j e^j$ and $\sum_{j=1}^n |\alpha_j| \leq c$. For each j there is a $K_j = K_j(\epsilon)$ such that $\| A^k e^j \| < \epsilon/c$ for $k \geq K_j$. Define $K = K(\epsilon) = \max\{K_j(\epsilon) : j = 1, \ldots, n\}$. For $\| x_0 \| \leq 1$ and $k \geq K$ we have that

$$\| A^k x_0 \| = \| \sum_{j=1}^n \alpha_j A^k e^j \| \leq \sum_{j=1}^n |\alpha_j| (\epsilon/c) \leq \epsilon.$$

By the definition of matrix norm [see Section A.7], this means that $\| A^k \| \leq \epsilon$ for $k > K$. Therefore, statement (iii) is true.

Finally, assume that statement (iii) is true. Then $\| A^k \|$ is bounded for all $k \geq 0$. By Theorem 4.44, the equilibrium $x = 0$ is stable. To prove asymptotic stability, fix $\epsilon > 0$. If $\| x_0 \| < \eta = 1$, then $\| \phi(k, x_0) \| \leq \| A^k \| \| x_0 \| \to 0$ as $k \to \infty$. Therefore, statement (i) is true. This completes the proof. ■

Theorem 4.46. *The equilibrium $x = 0$ of (4.68) is asymptotically stable if and only if it is exponentially stable.*

Proof. The exponential stability of the equilibrium $x = 0$ implies the asymptotic stability of the equilibrium $x = 0$ of systems (4.69) in general and, hence, for systems (4.68) in particular.

Conversely, assume that the equilibrium $x = 0$ of (4.68) is asymptotically stable. Then there is a $\delta > 0$ and a $K > 0$ such that if $\| x_0 \| \leq \delta$, then

$$\| \Phi(k + K) x_0 \| \leq \frac{\delta}{2}$$

for all $k \geq 0$. This implies that

$$\| \Phi(k+K) \| \leq \frac{1}{2} \quad \text{if } k \geq 0. \tag{4.72}$$

From Section 3.5.1 we have that $\Phi(k-l) = \Phi(k-s)\Phi(s-l)$ for any k, l, s. Therefore,

$$\| \Phi(k+2K) \| = \| \Phi[(k+2K)-(k+K)]\Phi(k+K) \| \leq \frac{1}{4}$$

in view of (4.72). By induction we obtain for $k \geq 0$ that

$$\| \Phi(k+nK) \| \leq 2^{-n}. \tag{4.73}$$

Let $\alpha = \frac{(\ln 2)}{K}$. Then (4.73) implies that for $0 \leq k < K$ we have that

$$\begin{aligned} \| (k+nK, x_0) \| &\leq 2 \| x_0 \| 2^{-(n+1)} \\ &= 2 \| x_0 \| e^{-\alpha(n+1)K} \\ &\leq 2 \| x_0 \| e^{-\alpha(k+nK)}, \end{aligned}$$

which proves the result. ■

To arrive at the next result, we make reference to the results of Subsection 3.5.5. Specifically, by inspecting the expressions for the modes of system (4.68) given in (3.131) and (3.132), or by utilizing the Jordan canonical form of A [refer to (3.135) and (3.136)], the following result is evident.

Theorem 4.47. *(i) The equilibrium $x = 0$ of system (4.68) is* asymptotically stable *if and only if all eigenvalues of A are within the unit circle of the complex plane (i.e., if $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A, then $|\lambda_j| < 1, j = 1, \ldots, n$). In this case we say that* the matrix A is Schur stable*, or simply,* the matrix A is stable.

(ii) The equilibrium $x = 0$ of system (4.68) is stable *if and only if $|\lambda_j| \leq 1, j = 1, \ldots, n$, and for each eigenvalue with $|\lambda_j| = 1$ having multiplicity $n_j > 1$, it is true that*

$$\lim_{z \to \lambda_j} \left\{ \frac{d^{n_j-1-l}}{dz^{n_j-1-l}} [(z-\lambda_j)^{n_j}(zI-A)^{-1}] \right\} = 0, \quad l = 1, \ldots, n_j - 1.$$

(iii) The equilibrium $x = 0$ of system (4.68) is unstable *if and only if the conditions in (ii) above are not true.* ■

Alternatively, it is evident that the equilibrium $x = 0$ of system (4.68) is *stable* if and only if all eigenvalues of A are within or on the unit circle of the complex plane, and every eigenvalue that is on the unit circle has an associated Jordan block of order 1.

Example 4.48. (i) For the system in Example 4.42 we have

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm\sqrt{-1}$. According to Theorem 4.47, the equilibrium $x = 0$ of the system is stable, and according to Theorem 4.44 the matrix A^k is bounded for all $k \geq 0$.

(ii) For system (4.68) let

$$A = \begin{bmatrix} 0 & -1/2 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm 1/\sqrt{2}$. According to Theorem 4.47, the equilibrium $x = 0$ of the system is asymptotically stable, and according to Theorem 4.45, $\lim_{k\to\infty} A^k = 0$.

(iii) For system (4.68) let

$$A = \begin{bmatrix} 0 & -1/2 \\ -3 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1, \lambda_2 = \pm\sqrt{3/2}$. According to Theorem 4.47, the equilibrium $x = 0$ of the system is unstable, and according to Theorem 4.44, the matrix A^k is not bounded with increasing k.

(iv) For system (4.68) let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The matrix A is a Jordan block of order 2 for the eigenvalue $\lambda = 1$. Accordingly, the equilibrium $x = 0$ of the system is unstable (refer to the remark following Theorem 4.47) and the matrix A^k is unbounded with increasing k.

4.8.3 The Lyapunov Matrix Equation

In this subsection we obtain another characterization of stable matrices by means of the *Lyapunov matrix equation*.

Returning to system (4.68) we choose as a Lyapunov function

$$v(x) = x^T B x, B = B^T, \tag{4.74}$$

and we evaluate the first forward difference of v along the solutions of (4.68) as

$$\begin{aligned}
Dv(x(k)) \triangleq v(x(k+1)) - v(x(k)) &= x(k+1)^T Bx(k+1) - x(k)^T Bx(k) \\
&= x(k)^T A^T BAx(k) - x(k)^T Bx(k) \\
&= x(k)^T (A^T BA - B)x(k),
\end{aligned}$$

and therefore,

$$Dv(x) = x^T (A^T BA - B)x \triangleq -x^T Cx,$$

where

$$A^T BA - B = C, \quad C^T = C. \tag{4.75}$$

Theorem 4.49. *(i) The equilibrium $x = 0$ of system (4.68) is* stable *if there exists a real, symmetric, and positive definite matrix B such that the matrix C given in (4.75) is negative semidefinite.*

(ii) The equilibrium $x = 0$ of system (4.68) is asymptotically stable in the large *if there exists a real, symmetric, and positive definite matrix B such that the matrix C given in (4.75) is negative definite.*

(iii) The equilibrium $x = 0$ of system (4.68) is unstable *if there exists a real, symmetric matrix B that is either negative definite or indefinite such that the matrix C given in (4.75) is negative definite.* ■

In proving Theorem 4.49 one can follow a similar approach as in the proofs of Theorems 4.22, 4.24 and 4.26. We leave the details to the reader as an exercise.

In applying Theorem 4.49, we start by choosing (guessing) a matrix B having certain desired properties and we then solve for the matrix C, using equation (4.75). If C possesses certain desired properties (i.e., it is negative definite), we can draw appropriate conclusions by applying one of the results given in Theorem 4.49; if not, we need to choose another matrix B. This approach is not very satisfactory, and in the following we will derive results that will allow us (as in the case of continuous-time systems) to *construct* Lyapunov functions of the form $v(x) = x^T Bx$ in a systematic manner. In doing so, we first choose a matrix C in (4.75) that is either negative definite or positive definite, and then we solve (4.75) for B. Conclusions are then made by applying Theorem 4.49. In applying this construction procedure, we need to know conditions under which (4.75) possesses a (unique) solution B for *any* definite (i.e., positive or negative definite) matrix C. We will address this issue next.

We first show that if A is stable, i.e., if all eigenvalues of matrix A [in system (4.68)] are inside the unit circle of the complex plane, then we can compute B in (4.75) explicitly. To show this, we assume that in (4.75) C is a given matrix and that A is stable. Then

$$(A^T)^{k+1} BA^{k+1} - (A^T)^k BA^k = (A^T)^k CA^k,$$

and summing from $k = 0$ to l yields

$$A^TBA-B+(A^T)^2BA^2-A^TBA+\cdots+(A^T)^{l+1}BA^{l+1}-(A^T)^lBA^l=\sum_{k=0}^{l}(A^T)^kCA^k$$

or

$$(A^T)^{l+1}BA^{l+1}-B=\sum_{k=0}^{l}(A^T)^kCA^k.$$

Letting $l\to\infty$, we obtain

$$B=-\sum_{k=0}^{\infty}(A^T)^kCA^k. \tag{4.76}$$

It is easy to verify that (4.76) is a solution of (4.75). We have

$$-A^T\left[\sum_{k=0}^{\infty}(A^T)^kCA^k\right]A+\sum_{k=0}^{\infty}(A^T)^kCA^k=C$$

or

$$-A^TCA+C-(A^T)^2CA^2+A^TCA-(A^T)^3CA^3+(A^T)^2CA^2-\cdots=C.$$

Therefore (4.76) is a solution of (4.75). Furthermore, if C is negative definite, then B is positive definite.

Combining the above with Theorem 4.49(ii) we have the following result.

Theorem 4.50. *If there is a positive definite and symmetric matrix B and a negative definite and symmetric matrix C satisfying (4.75), then the matrix A is stable. Conversely, if A is stable, then, given* any *symmetric matrix C, (4.75) has a unique solution, and if C is negative definite, then B is positive definite.* ■

Next, we determine conditions under which the system of equations (4.75) has a (unique) solution $B=B^T\in R^{n\times n}$ for a given matrix $C=C^T\in R^{n\times n}$. To accomplish this, we consider the more general equation

$$A_1XA_2-X=C, \tag{4.77}$$

where $A_1\in R^{m\times m}$, $A_2\in R^{n\times n}$, and X and C are $m\times n$ matrices.

Lemma 4.51. *Let $A_1\in R^{m\times m}$ and $A_2\in R^{n\times n}$. Then (4.77) has a unique solution $X\in R^{m\times n}$ for a given $C\in R^{m\times n}$ if and only if no eigenvalue of A_1 is a reciprocal of an eigenvalue of A_2.*

Proof. We need to show that the condition on A_1 and A_2 is equivalent to the condition that $A_1XA_2=X$ implies $X=0$. Once we have proved that $A_1XA_2=X$ has the unique solution $X=0$, then it can be shown that (4.77) has a unique solution for every C, since (4.77) is a linear equation.

Assume first that the condition on A_1 and A_2 is satisfied. Now $A_1XA_2 = X$ implies that $A_1^{k-j}XA_2^{k-j} = X$ and

$$A_1^j X = A_1^k X A_2^{k-j} \quad \text{for } k \geq j \geq 0.$$

Now for a polynomial of degree k,

$$p(\lambda) = \sum_{j=0}^{k} a_j \lambda^j,$$

we define the polynomial of degree k,

$$p^*(\lambda) = \sum_{j=0}^{k} a_j \lambda^{k-j} = \lambda^k p(1/\lambda),$$

from which it follows that

$$p(A_1)X = A_1^k X p^*(A_2).$$

Now let $\phi_i(\lambda)$ be the characteristic polynomial of $A_i, i = 1, 2$. Since $\phi_1(\lambda)$ and $\phi_2^*(\lambda)$ are relatively prime, there are polynomials $p(\lambda)$ and $q(\lambda)$ such that

$$p(\lambda)\phi_1(\lambda) + q(\lambda)\phi_2^*(\lambda) = 1.$$

Now define $\phi(\lambda) = q(\lambda)\phi_2^*(\lambda)$ and note that $\phi^*(\lambda) = q^*(\lambda)\phi_2(\lambda)$. It follows that $\phi^*(A_2) = 0$ and $\phi(A_1) = I$. From this it follows that $A_1XA_2 = X$ implies $X = 0$.

To prove the converse, we assume that λ is an eigenvalue of A_1 and λ^{-1} is an eigenvalue of A_2 (and, hence, is also an eigenvalue of A_2^T). Let $A_1x^1 = \lambda x^1$ and $A_2^T x^2 = \lambda^{-1}x^2, x^1 \neq 0$ and $x^2 \neq 0$. Define $X = (x_1^2 x^1, x_2^2 x^1, \ldots, x_n^2 x^1)$. Then $X \neq 0$ and $A_1XA_2 = X$. ■

To construct $v(x)$ by using Lemma 4.51, we must still check the definiteness of B. To accomplish this, we use Theorem 4.49.

1. If all eigenvalue of A [for system (4.68)] are inside the unit circle of the complex plane, then no reciprocal of an eigenvalue of A is an eigenvalue, and Lemma 4.51 gives another way of showing that (4.75) has a unique solution B for each C if A is stable. If C is negative definite, then B is positive definite. This can be shown as was done for the case of linear ordinary differential equations.
2. Suppose that at least one of the eigenvalues of A is outside the unit circle in the complex plane and that A has no eigenvalues on the unit circle. As in the case of linear differential equations (4.22) (Section 4.5), we use a similarity transformation $x = Qy$ in such a way that $Q^{-1}AQ = \text{diag}[A_1,\ A_2]$, where all eigenvalues of A_1 are outside the unit circle while all eigenvalues

of A_2 are within the unit circle. We then proceed identically as in the case of linear differential equations to show that under the present assumptions there exists for system (4.68) a Lyapunov function that satisfies the hypotheses of Theorem 4.49(iii). Therefore, the equilibrium $x = 0$ of system (4.68) is unstable. If A does not have any eigenvalues within the unit circle, then the equilibrium $x = 0$ of (4.68) is completely unstable. In this proof, Lemma 4.51 has not been invoked. If additionally, the hypotheses of Lemma 4.51 are true (i.e., no reciprocal of an eigenvalue of A is an eigenvalue of A), then we can construct the Lyapunov function for system (4.68) in a systematic manner.

Summarizing the above discussion, we have proved the following result.

Theorem 4.52. *Assume that the matrix A for system (4.68) has no eigenvalues on the unit circle in the complex plane. If all the eigenvalues of the matrix A are within the unit circle of the complex plane, or if at least one eigenvalue is outside the unit circle of the complex plane, then there exists a Lyapunov function of the form $v(x) = x^T Bx, B = B^T$, whose first forward difference along the solutions of system (4.68) is definite (i.e., it is either negative definite or positive definite).* ■

Theorem 4.52 shows that when all the eigenvalues of A are within the unit circle, then for system (4.68), the conditions of Theorem 4.49(ii) are also necessary conditions for exponential stability in the large. Furthermore, when at least one eigenvalue of A is outside the unit circle and no eigenvalues are on the unit circle, then the conditions of Theorem 4.49(iii) are also necessary conditions for instability.

We conclude this subsection with some specific examples.

Example 4.53. (i) For system (4.68), let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $B = I$, which is positive definite. From (4.75) we obtain

$$C = A^T A - I = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows from Theorem 4.49(i) that the equilibrium $x = 0$ of this system is stable. This is the same conclusion that was made in Example 4.48.

(ii) For system (4.68), let

$$A = \begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}.$$

Choose

$$B = \begin{bmatrix} \frac{8}{3} & 0 \\ 0 & \frac{5}{3} \end{bmatrix},$$

which is positive definite. From (4.75) we obtain

$$C = A^T BA - B = \begin{bmatrix} 0 & -1 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{8}{3} & 0 \\ 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \frac{8}{3} & 0 \\ 0 & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is negative definite. It follows from Theorem 4.49(ii) that the equilibrium $x = 0$ of this system is asymptotically stable in the large. This is the same conclusion that was made in Example 4.48(ii).

(iii) For system (4.68), let

$$A = \begin{bmatrix} 0 & -\frac{1}{2} \\ -3 & 0 \end{bmatrix}.$$

Choose

$$C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is negative definite. From (4.75) we obtain

$$C = A^T BA - B = \begin{bmatrix} 0 & -3 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

or

$$\begin{bmatrix} (9b_{22} - b_{11}) & \frac{1}{2}b_{12} \\ \frac{1}{2}b_{12} & (\frac{1}{4}b_{11} - b_{22}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which yields

$$B = \begin{bmatrix} -8 & 0 \\ 0 & -1 \end{bmatrix},$$

which is also negative definite. It follows from Theorem 4.49(iii) that the equilibrium $x = 0$ of this system is unstable. This conclusion is consistent with the conclusion made in Example 4.48(iii).

(iv) For system (4.68), let

$$A = \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & 3 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = \frac{1}{3}$ and $\lambda_2 = 3$. According to Lemma 4.51, for a given C, (4.77) does *not* have a unique solution in this case since $\lambda_1 = 1/\lambda_2$. For purposes of illustration, we choose $C = -I$. Then

$$-I = A^T BA - B = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}$$

or

$$\begin{bmatrix} -\frac{8}{9}b_{11} & \frac{1}{3}b_{11} \\ \frac{1}{3}b_{11} & b_{11} + 6b_{12} + 8b_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which shows that for $C = -I$, (4.77) does not have any solution (for B) at all.

4.8.4 Linearization

In this subsection we determine conditions under which the stability properties of the equilibrium $w = 0$ of the linear system

$$w(k+1) = Aw(k) \tag{4.78}$$

determine the stability properties of the equilibrium $x = 0$ of the nonlinear system

$$x(k+1) = Ax(k) + f(x(k)), \tag{4.79}$$

under the assumption that $f(x) = o(\| x \|)$ as $\| x \| \to 0$ (i.e., given $\epsilon > 0$, there exists $\delta > 0$ such that $\| f(x(k)) \| < \epsilon \| x(k) \|$ for all $k \geq 0$ and all $\| x(k) \| < \delta$). [Refer to the discussion concerning (4.68) to (4.70) in Subsection 4.8.1.]

Theorem 4.54. *Assume that $f \in C(R^n, R^n)$ and that $f(x)$ is $o(\| x \|)$ as $\| x \| \to 0$. (i) If A is stable (i.e., all the eigenvalues of A are within the unit circle of the complex plane), then the equilibrium $x = 0$ of system (4.79) is* asymptotically stable. *(ii) If at least one eigenvalue of A is outside the unit circle of the complex plane and no eigenvalue is on the unit circle, then the equilibrium $x = 0$ of system (4.79) is* unstable. ■

In proving Theorem 4.54 one can follow a similar approach as in the proofs of Theorems 4.32 and 4.33. We leave the details to the reader as an exercise.

Before concluding this subsection, we consider some specific examples.

Example 4.55. (i) Consider the system

$$\begin{aligned} x_1(k+1) &= -\frac{1}{2}x_2(k) + x_1(k)^2 + x_2(k)^2, \\ x_2(k+1) &= -x_1(k) + x_1(k)^2 + x_2(k)^2. \end{aligned} \tag{4.80}$$

Using the notation of (4.79), we have

$$A = \begin{bmatrix} 0 & -\frac{1}{2} \\ -1 & 0 \end{bmatrix}, \quad f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2^2 \\ \\ x_1^2 + x_2^2 \end{bmatrix}.$$

The linearization of (4.80) is given by

$$w(k+1) = Aw(k). \tag{4.81}$$

From Example 4.48(ii) [and Example 4.53(ii)], it follows that the equilibrium $w = 0$ of (4.81) is asymptotically stable. Furthermore, in the present case $f(x) = o(\| x \|)$ as $\| x \| \to 0$. Therefore, in view of Theorem 4.54, the equilibrium $x = 0$ of system (4.80) is asymptotically stable.

(ii) Consider the system

$$
\begin{aligned}
x_1(k+1) &= -\frac{1}{2}x_2(k) + x_1(k)^3 + x_2(k)^2, \\
x_2(k+1) &= -3x_1(k) + x_1^4(k) - x_2(k)^5.
\end{aligned}
\tag{4.82}
$$

Using the notation of (4.78) and (4.79), we have in the present case

$$
A = \begin{bmatrix} 0 & -\frac{1}{2} \\ -3 & 0 \end{bmatrix}, \quad f(x_1, x_2) = \begin{bmatrix} x_1^3 + x_2^2 \\ \\ x_1^4 - x_2^5 \end{bmatrix}.
$$

Since A is unstable [refer to Example 4.53(iii) and Example 4.48(iii)] and since $f(x) = o(\| x \|)$ as $\| x \| \to 0$, it follows from Theorem 4.54 that the equilibrium $x = 0$ of system (4.82) is unstable.

4.8.5 Input–Output Stability

We conclude this chapter by considering the input–output stability of discrete-time systems described by equations of the form

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k),
\end{aligned}
\tag{4.83}
$$

where all matrices and vectors are defined as in (4.67). Throughout this subsection we will assume that $k_0 = 0, x(0) = 0$, and $k \geq 0$.

As in the continuous-time case, we say that system (4.83) is *BIBO stable* if there exists a constant $c > 0$ such that the conditions

$$
\begin{aligned}
x(0) &= 0, \\
\| u(k) \| &\leq 1, \quad k \geq 0,
\end{aligned}
$$

imply that $\| y(k) \| \leq c$ for all $k \geq 0$.

The results that we will present involve the impulse response matrix of (4.83) given by

$$
H(k) = \begin{cases} CA^{k-1}B, & k > 0, \\ 0, & k \leq 0, \end{cases}
\tag{4.84}
$$

and the transfer function matrix given by

$$
\widehat{H}(z) = C(zI - A)^{-1}B.
\tag{4.85}
$$

Recall that

$$
y(n) = \sum_{k=0}^{n} H(n-k)u(k).
\tag{4.86}
$$

Associated with system (4.83) is the free dynamical system described by the equation

$$
p(k+1) = Ap(k).
\tag{4.87}
$$

Theorem 4.56. *The system (4.83) is* BIBO stable *if and only if there exists a constant* $L > 0$ *such that for all* $n \geq 0$*,*

$$\sum_{k=0}^{n} \| H(k) \| \leq L. \tag{4.88}$$

■

As in the continous-time case, the first part of the proof of Theorem 4.56 (sufficiency) is straightforward. Specifically, if $\| u(k) \| \leq 1$ for all $k \geq 0$ and if (4.88) is true, then we have for all $n \geq 0$,

$$\begin{aligned} \| y(n) \| &= \| \sum_{k=0}^{n} H(n-k)u(k) \| \leq \sum_{k=0}^{n} \| H(n-k)u(k) \| \\ &\leq \sum_{k=0}^{n} \| H(n-k) \| \| u(k) \| \leq \sum_{k=0}^{n} \| H(n-k) \| \leq L. \end{aligned}$$

Therefore, system (4.83) is BIBO stable.

In proving the second part of Theorem 4.56 (necessity), we simplify matters by first considering in (4.83) the single-variable case ($n = 1$) with the system description given by

$$y(t) = \sum_{k=0}^{t} h(t-k)u(k), \quad t > 0. \tag{4.89}$$

For purposes of contradiction, we assume that the system is BIBO stable, but no finite L exists such that (4.88) is satisfied. Another way of expressing the last assumption is that for *any finite* L, there exists $t = k_1(L) \triangleq k_1$ such that

$$\sum_{k=0}^{k_1} |h(k_1 - k)| > L.$$

We now choose in particular the input u given by

$$u(k) = \begin{cases} +1 & \text{if } h(t-k) > 0, \\ 0 & \text{if } h(t-k) = 0, \\ -1 & \text{if } h(t-k) < 0, \end{cases}$$

$0 \leq k \leq k_1$. Clearly, $|u(k)| \leq 1$ for all $k \geq 0$. The output of the system at $t = k_1$ due to the above input, however, is

$$y(k_1) = \sum_{k=0}^{k_1} h(k_1 - k)u(k) = \sum_{k=0}^{k_1} |h(k_1 - k)| > L,$$

which contradicts the assumption that the system is BIBO stable.

The above can now be extended to the multivariable case. In doing so we apply the single-variable result to every possible pair of input and output vector components, we make use of the fact that the sum of a finite number of bounded sums will be bounded, and we note that a vector is bounded if and only if each of its components is bounded. We leave the details to the reader.

Next, as in the case of continuous-time systems, we note that the asymptotic stability of the equilibrium $p = 0$ of system (4.87) implies the BIBO stability of system (4.83) since the sum

$$\| \sum_{k=1}^{\infty} CA^{k-1}B \| \leq \sum_{k=1}^{\infty} \| C \| \| A^{k-1} \| \| B \|$$

is finite.

Next, we recall that a complex number z_p is a *pole* of $\widehat{H}(z) = [\hat{h}_{ij}(z)]$ if for some (i, j) we have $|\hat{h}_{ij}(z_p)| = \infty$. If each entry of $\widehat{H}(z)$ has only poles with modulus (magnitude) less than 1, then, as shown in Chapter 3, each entry of $H(k) = [h_{ij}(k)]$ consists of a sum of convergent terms. It follows that under these conditions the sum

$$\sum_{k=0}^{\infty} \| H(k) \|$$

is finite, and any realization of $\widehat{H}(z)$ will result in a system that is BIBO stable.

Conversely, if

$$\sum_{k=0}^{\infty} \| H(k) \|$$

is finite, then the terms in every entry of $H(k)$ must be convergent. But then every entry of $\widehat{H}(z)$ has poles whose modulus is within the unit circle of the complex plane. We have proved the final result of this section.

Theorem 4.57. *The time-invariant system (4.83) is BIBO stable if and only if the poles of the transfer function*

$$\widehat{H}(z) = C(zI - A)^{-1}B$$

are within the unit circle of the complex plane. ■

4.9 Summary and Highlights

In this chapter we first addressed the stability of an equilibrium of continuous-time finite-dimensional systems. In doing so, we first introduced the concept of equilibrium and defined several types of stability in the sense of Lyapunov (Sections 4.2 and 4.3). Next, we established several stability conditions of an equilibrium for linear systems $\dot{x} = Ax$, $t \geq 0$ in terms of the state transition matrix in Theorems 4.12–4.14 and in terms of eigenvalues in Theorem 4.15 (Section 4.4). Next, we established various stability conditions that are phrased in terms of the Lyapunov matrix equation (4.25) for system $\dot{x} = Ax$ (Section 4.5). The existence of Lyapunov functions for $\dot{x} = Ax$ of the form $x^T Px$ is established in Theorem 4.29. In Section 4.6 we established conditions under which the asymptotic stability and the instability of an equilibrium for a nonlinear time-invariant system can be deduced via linearization; see Theorems 4.32 and 4.33.

Next, we addressed the input–output stability of time-invariant linear, continuous-time, finite-dimensional systems (Section 4.7). For such systems we established several conditions for bounded input/bounded output stability (BIBO stability); see Theorems 4.37 and 4.39.

The chapter is concluded with Section 4.8, where we addressed the Lyapunov stability and the input–output stability of linear, time-invariant, discrete-time systems. For such systems, we established results that are analogous to the stability results of continuous-time systems. The stability of an equilibrium is expressed in terms of the state transition matrix in Theorem 4.45, in terms of the eigenvalues in Theorem 4.47, and in terms of the Lyapunov Matrix Equation in Theorems 4.49 and 4.50. The existence of Lyapunov functions of the form $x^T Px$ for $x(k+1) = Ax(k)$ is established in Theorem 4.52. Stability results based on linearization are presented in Theorem 4.54 and for BIBO stability in Theorems 4.56 and 4.57.

4.10 Notes

The initial contributions to stability theory that took place toward the end of the nineteenth century are primarily due to physicists and mathematicians (Lyapunov [11]), whereas input–output stability is the brainchild of electrical engineers (Sandberg [17] to [19], Zames [21], [22]). Sources with extensive coverage of Lyapunov stability theory include, e.g., Hahn [6], Khalil [8], LaSalle [9], LaSalle and Lefschetz [10], Michel and Miller [13], Michel et al. [14], Miller and Michel [15], and Vidyasagar [20]. Input–output stability is addressed in great detail in Desoer and Vidyasagar [5], Vidyasagar [20], and Michel and Miller [13]. For a survey that traces many of the important developments of stability in feedback control, refer to Michel [12].

In the context of *linear systems*, sources on both Lyapunov stability and input–output stability can be found in numerous texts, including Antsaklis and Michel [1], Brockett [2], Chen [3], DeCarlo [4], Kailath [7], and Rugh [16]. In developing our presentation, we found the texts by Antsaklis and Michel [1], Brockett [2], Hahn [6], LaSalle [9], and Miller and Michel [15] especially helpful.

In this chapter, we addressed various types of Lyapunov stability and bounded input/bounded output stability of time-invariant systems. In the various stability concepts for such systems, the initial time t_0 (resp., k_0) plays no significant role, and for this reason, we chose without loss of generality $t_0 = 0$ (resp., $k_0 = 0$). In the case of time-varying systems, this is in general not true, and in defining the various Lyapunov stability concepts and the concept of bounded input/bounded output stability, one has to take into account the effects of initial time. In doing so, we have to distinguish between *uniformity* and *nonuniformity* when defining the various types of Lyapunov stability of an equilibrium and the BIBO stability of a system. For a treatment of the Lyapunov stability and the BIBO stablity of the time-varying counterparts of systems (4.14), (4.15) and (4.67), (4.68), we refer the reader to Chapter 6 in Antsaklis and Michel [1].

We conclude by noting that there are graphical criteria (i.e., frequency domain criteria), such as, the Leonhard–Mikhailov criterion, and algebraic criteria, such as the Routh–Hurwitz criterion and the Schur–Cohn criterion, which yield necessary and sufficient conditions for the asymptotic stability of the equilibrium $x = 0$ for system (4.15) and (4.68). For a presentation of these results, the reader should consult Chapter 6 in Antsaklis and Michel [1] and Michel [12].

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Exercises

4.1. Determine the set of equilibrium points of a system described by the differential equations

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2 + x_3,\\ \dot{x}_2 &= 2x_1 + 3x_2 + x_3,\\ \dot{x}_3 &= 3x_1 + 2x_2 + 2x_3.\end{aligned}$$

4.2. Determine the set of equilibria of a system described by the differential equations

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \begin{cases} x_1 \sin(1/x_1), & \text{when } x_1 \neq 0,\\ 0, & \text{when } x_1 = 0.\end{cases}\end{aligned}$$

4.3. Determine the equilibrium points and their stability properties of a system described by the ordinary differential equation

$$\dot{x} = x(x-1) \tag{4.90}$$

by solving (4.90) and then applying the definitions of stability, asymptotic stability, etc.

4.4. Prove that the equilibrium $x = 0$ of (4.16) is stable (resp., asymptotically stable or unstable) if and only if $y = 0$ of (4.19) is stable (resp., asymptotically stable or unstable).

4.5. Apply Proposition 4.20 to determine the definiteness properties of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 1 & -1 & 10 \end{bmatrix}.$$

4.6. Use Theorem 4.26 to prove that the trivial solution of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is unstable.

4.7. Determine the equilibrium points of a system described by the differential equation

$$\dot{x} = -x + x^2,$$

and determine the stability properties of the equilibrium points, if applicable, by using Theorem 4.32 or 4.33.

4.8. The system described by the differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= -x_1 + x_2(x_1^2 + x_2^2) \end{aligned} \tag{4.91}$$

has an equilibrium at the origin $x^T = (x_1, x_2) = (0, 0)$. Show that the trivial solution of the *linearization* of system (4.91) is stable. Prove that the equilibrium $x = 0$ of system (4.91) is unstable. (This example shows that the assumptions on the matrix A in Theorems 4.32 and 4.33 are absolutely essential.)

4.9. Use Corollary 4.38 to analyze the stability properties of the system given by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ A &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [0, 1]. \end{aligned}$$

4.10. Determine all equilibrium points for the discrete-time systems given by

(a)

$$\begin{aligned} x_1(k+1) &= x_2(k) + |x_1(k)|, \\ x_2(k+1) &= -x_1(k) + |x_2(k)|. \end{aligned}$$

(b)

$$x_1(k+1) = x_1(k)x_2(k) - 1,$$
$$x_2(k+1) = 2x_1(k)x_2(k) + 1.$$

4.11. Prove Theorem 4.43.

4.12. Determine the stability properties of the trivial solution of the discrete-time system given by the equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

with θ fixed.

4.13. Analyze the stability of the equilibrium $x = 0$ of the system described by the scalar-valued difference equation

$$x(k+1) = \sin[x(k)].$$

4.14. Analyze the stability of the equilibrium $x = 0$ of the system described by the difference equations

$$x_1(k+1) = x_1(k) + x_2(k)[x_1(k)^2 + x_2(k)^2],$$
$$x_2(k+1) = x_2(k) - x_1(k)[x_1(k)^2 + x_2(k)^2].$$

4.15. Determine a basis of the solution space of the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}.$$

Use your answer in analyzing the stability of the trivial solution of this system.

4.16. Let $A \in R^{n\times n}$. Prove that part (iii) of Theorem 4.45 is equivalent to the statement that all eigenvalues of A have modulus less than 1; i.e.,

$$\lim_{k\to\infty} \| A^k \| = 0$$

if and only if for any eigenvalue λ of A, it is true that $|\lambda| < 1$.

4.17. Use Theorem 4.44 to show that the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} x(k)$$

is unstable.

4.18. (a) Use Theorem 4.47 to determine the stability of the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 9 & -1 \end{bmatrix} x(k).$$

(b) Use Theorem 4.47 to determine the stability of the equilibrium $x = 0$ of the system

$$x(k+1) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 9 & -1 \end{bmatrix} x(k).$$

4.19. Apply Theorems 4.24 and 4.49 to show that if the equilibrium $x = 0$ ($x \in R^n$) of the system

$$x(k+1) = e^A x(k)$$

is asymptotically stable, then the equilibrium $x = 0$ of the system

$$\dot{x} = Ax$$

is also asymptotically stable.

4.20. Apply Theorem 4.49 to show that the trivial solution of the system given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

is unstable.

4.21. Determine the stability of the equilibrium $x = 0$ of the scalar-valued system given by

$$x(k+1) = \frac{1}{2}x(k) + \frac{2}{3}\sin x(k).$$

4.22. Analyze the stability properties of the discrete-time system given by

$$x(k+1) = x(k) + \frac{1}{2}u(k)$$
$$y(k) = \frac{1}{2}x(k)$$

where x, y, and u are scalar-valued variables. Is this system BIBO stable?

4.23. Prove Theorem 4.47.

4.24. Prove Theorem 4.49 by following a similar approach as was used in the proofs of Theorems 4.22, 4.24, and 4.26.

4.25. Prove Theorem 4.54 by following a similar approach as was used in the proofs of Theorems 4.32 and 4.33.