# 5

# Controllability and Observability: Fundamental Results

## 5.1 Introduction

The principal goals of this chapter are to introduce the system properties of controllability and observability (and of reachability and constructibility), which play a central role in the study of state feedback controllers and state observers, and in establishing the relations between internal and external system representations, topics that will be studied in Chapters 7, 8, and 9. State controllability refers to the ability to manipulate the state by applying appropriate inputs (in particular, by steering the state vector from one vector value to any other vector value in finite time). Such is the case, for example, in satellite attitude control, where the satellite must change its orientation. State observability refers to the ability to determine the state vector of the system from knowledge of the input and the corresponding output over some finite time interval. Since it is frequently difficult or impossible to measure the state of a system directly (for example, internal temperatures and pressures in an internal combustion engine), it is very desirable to determine such states by observing the inputs and outputs of the system over some finite time interval.

In Section 5.2, the concepts of reachability and controllability and observability and constructibility are introduced, using discrete-time time-invariant systems. Discrete-time systems are selected for this exposition because the mathematical development is much simpler in this case. In subsection 5.2.3 the concept of duality is also introduced. Reachability and controllability are treated in detail in Section 5.3 and observability and constructibility in Section 5.4 for both continuous-time and discrete-time time-invariant systems.

## 5.2 A Brief Introduction to Reachability and Observability

Reachability and controllability are introduced first, followed by observability and constructibility. These important system concepts are more easily explained in the discrete-time case, and this is the approach taken in this section. Duality is also discussed at the end of the section.

### 5.2.1 Reachability and Controllability

The concepts of *state reachability* (or *controllability-from-the-origin*) and *controllability* (or *controllability-to-the-origin*) are introduced here and are discussed at length in Section 5.3.

In the case of time-invariant systems, a state $x_1$ is called *reachable* if there exists an input that transfers the state of the system $x(t)$ from the zero state to $x_1$ in some finite time $T$. The definition of reachability for the discrete-time case is completely analogous. Figure 5.1 shows that different control inputs $u_1(t)$ and $u_2(t)$ may force the state of a continuous-time system to reach the value $x_1$ from the origin at different finite times $T_1$ and $T_2$, following different paths. Note that reachability refers to the ability of the system to reach $x_1$ from the origin in some finite time; it specifies neither the exact time it takes to achieve this nor the trajectory to be followed.

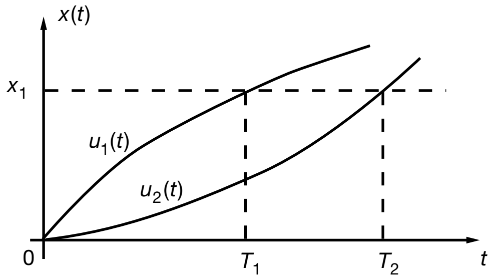


**Figure 5.1.** A reachable state $x_1$

A state $x_0$ is called *controllable* if there exists an input that transfers the state from $x_0$ to the zero state in some finite time $T$. See Figure 5.2. The definition of controllability for the discrete-time case is completely analogous. Similar to reachability, controllability specifies neither the time it takes to achieve the transfer nor the trajectory to be followed.

We note that when particular types of trajectories to be followed are of interest, then one seeks particular control inputs that will achieve such transfers. This leads to various control problem formulations, including the Linear Quadratic (Optimal) Regulator (LQR). The LQR problem is discussed in Chapter 9.

Section 5.3 shows that reachability always implies controllability, but controllability implies reachability only when the state transition matrix $\Phi$ of the system is nonsingular. This is always true for continuous-time systems, but is

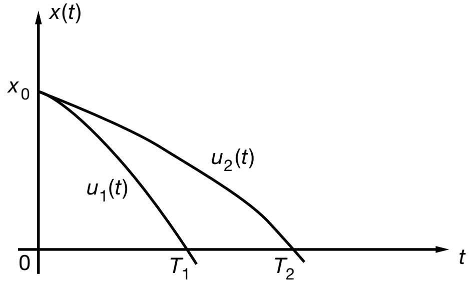


**Figure 5.2.** A controllable state $x_0$

true for discrete-time systems only when the matrix $A$ of the system is nonsingular. If the system is state reachable, then there always exists an input that transfers any state $x_0$ to any other state $x_1$ in finite time.

In the time-invariant case, a system is *reachable* (or *controllable-from-the-origin*) if and only if its *controllability matrix* $\mathcal{C}$,

$$\mathcal{C} \triangleq [B, AB, \ldots, A^{n-1}B] \in R^{n\times mn}, \tag{5.1}$$

has full row rank $n$; that is, rank $\mathcal{C} = n$. The matrices $A \in R^{n\times n}$ and $B \in R^{n\times m}$ come from either the continuous-time state equations

$$\dot{x} = Ax + Bu \tag{5.2}$$

or the discrete-time state equations

$$x(k+1) = Ax(k) + Bu(k), \tag{5.3}$$

$k \geq k_0 = 0$. Alternatively, we say that the pair $(A, B)$ is reachable. The matrix $\mathcal{C}$ should perhaps more appropriately be called the "reachability matrix" or the "controllability-from-the-origin matrix." The term "controllability matrix," however, has been in use for some time and is expected to stay in use. Therefore, we shall call $\mathcal{C}$ the "controllability matrix," having in mind the "controllability-from-the-origin matrix."

We shall now discuss reachability and controllability for discrete-time time-invariant systems (5.3).

If the state $x(k)$ in (5.3) is expressed in terms of the initial vector $x(0)$, then (see Subsection 3.5.1)

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-(i+1)} Bu(i) \tag{5.4}$$

for $k > 0$. Rewriting the summation in terms of matrix-vector multiplication, it follows that it is possible to transfer the state from some value $x(0) = x_0$ to some $x_1$ in $n$ steps, that is, $x(n) = x_1$, if there exists an $n$-step input sequence $\{u(0), u(1), \ldots, u(n-1)\}$ that satisfies the equation

$$x_1 - A^n x_0 = \mathcal{C}_n U_n, \tag{5.5}$$

where $\mathcal{C}_n \triangleq [B, AB, \ldots, A^{n-1}B] = \mathcal{C}$ [see (5.1)] and

$$U_n \triangleq [u^T(n-1), u^T(n-2), \ldots, u^T(0)]^T. \tag{5.6}$$

From the theory of linear algebraic equations, (5.5) has a solution $U_n$ if and only if

$$x_1 - A^n x_0 \in \mathcal{R}(\mathcal{C}), \tag{5.7}$$

where $\mathcal{R}(\mathcal{C}) = \text{range}(\mathcal{C})$. Note that it is not necessary to take more than $n$ steps in the control sequence, since if this transfer cannot be accomplished in $n$ steps, it cannot be accomplished at all. This follows from the Cayley–Hamilton Theorem, in view of which it can be shown that $\mathcal{R}(\mathcal{C}_n) = \mathcal{R}(\mathcal{C}_k)$ for $k \geq n$. Also note that $\mathcal{R}(\mathcal{C}_n)$ includes $\mathcal{R}(\mathcal{C}_k)$ for $k < n$ [i.e., $\mathcal{R}(\mathcal{C}_n) \supset \mathcal{R}(\mathcal{C}_k), k < n$]. (See Exercise 5.1.)

It is now easy to see that the system (5.3) or the pair $(A, B)$ is *reachable* (*controllable-from-the-origin*), if and only if $\text{rank}\,\mathcal{C} = n$, since in this case $\mathcal{R}(\mathcal{C}) = R^n$, the entire state space. Note that $x_1 \in \mathcal{R}(\mathcal{C})$ is the condition for a particular state $x_1$ to be reachable from the zero state. Since $\mathcal{R}(\mathcal{C})$ contains all such states, it is called the *reachable subspace* of the system. It is also clear from (5.5) that if the system is reachable, any state $x_0$ can be transferred to any other state $x_1$ in $n$ steps. In addition, the input that accomplishes this transfer is any solution $U_n$ of (5.5). Note that, depending on $x_1$ and $x_0$, this transfer may be accomplished in fewer than $n$ steps (see Section 5.3).

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***Example 5.1.*** Consider $x(k+1) = Ax(k) + Bu(k)$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Here the controllability (-from-the-origin) matrix $\mathcal{C}$ is $\mathcal{C} = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ with $\text{rank}\,\mathcal{C} = 2$. Therefore, the system [or the pair $(A, B)$] is reachable, meaning that any state $x_1$ can be reached from the zero state in a finite number of steps by applying at most $n$ inputs $\{u(0), u(1), \ldots, u(n-1)\}$ (presently, $n = 2$). To see this, let $x_1 = \begin{bmatrix} a \\ b \end{bmatrix}$. Then (5.5) implies that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$ or $\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b-a \\ a \end{bmatrix}$. Thus, the control $u(0) = a, u(1) = b - a$ will transfer the state from the origin at $k = 0$ to the state $\begin{bmatrix} a \\ b \end{bmatrix}$ at $k = 2$. To verify this, we observe that $x(1) = Ax(0) + Bu(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} a = \begin{bmatrix} 0 \\ a \end{bmatrix}$ and $x(2) = Ax(1) + Bu(1) = \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (b-a) = \begin{bmatrix} a \\ b \end{bmatrix}$.

Reachability of the system also implies that a state $x_1$ can be reached from any other state $x_0$ in at most $n = 2$ steps. To illustrate this, let

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then (5.5) implies that $x_1 - A^2 x_0 = \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-2 \\ b-3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$. Solving, $\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} b-a-1 \\ a-2 \end{bmatrix}$, which will drive the state from $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ at $k = 0$ to $\begin{bmatrix} a \\ b \end{bmatrix}$ at $k = 2$.

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Notice that in general the solution $U_n$ of (5.5) is not unique; i.e., many inputs can accomplish the transfer from $x(0) = x_0$ to $x(n) = x_1$, each corresponding to a particular state trajectory. In control problems, particular inputs are frequently selected that, in addition to transferring the state, satisfy additional criteria, such as, e.g., minimization of an appropriate performance index (optimal control).

A system [or the pair $(A, B)$] is *controllable, or controllable-to-the-origin*, when any state $x_0$ can be driven to the zero state in a finite number of steps. From (5.5) we see that a system is controllable when $A^n x_0 \in \mathcal{R}(\mathcal{C})$ for any $x_0$. If $\operatorname{rank} A = n$, a system is controllable when $\operatorname{rank} \mathcal{C} = n$, i.e., when the reachability condition is satisfied. In this case the $n \times mn$ matrix

$$A^{-n}\mathcal{C} = [A^{-n}B, \ldots, A^{-1}B] \tag{5.8}$$

is of interest and the system is controllable if and only if $\operatorname{rank}(A^{-n}\mathcal{C}) = \operatorname{rank} \mathcal{C} = n$. If, however, $\operatorname{rank} A < n$, then controllability does not imply reachability (see Section 5.3).

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***Example 5.2.*** The system in Example 5.1 is controllable (-to-the-origin). To see this, we let, $x_1 = 0$ in (5.5) and write $-A^2 x_0 = -\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = [B, AB] \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix}$, where $x_0 = \begin{bmatrix} a \\ b \end{bmatrix}$. From this we obtain $\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = -\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -a-b \end{bmatrix}$, which is the input that will drive the state from $\begin{bmatrix} a \\ b \end{bmatrix}$ at $k = 0$ to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $k = 2$.

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***Example 5.3.*** The system $x(k+1) = 0$ is controllable since any state, say, $x(0) = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, can be transferred to the zero state in one step. In this system, however, the input $u$ does not affect the state at all! This example shows that reachability is a more useful concept than controllability for discrete-time systems.

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It should be pointed out that nothing has been said up to now about maintaining the desired system state after reaching it [refer to (5.5)]. Zeroing the input for $k \geq n$, i.e., letting $u(k) = 0$ for $k \geq n$, will not typically work, unless $Ax_1 = x_1$. In general a state starting at $x_1$, will remain at $x_1$ for all $k \geq n$ if and only if there exists an input $u(k), k \geq n$, such that

$$x_1 = Ax_1 + Bu(k), \tag{5.9}$$

that is, if and only if $(I - A)x_1 \in \mathcal{R}(B)$. Clearly, there are states for which this condition may not be satisfied.

### 5.2.2 Observability and Constructibility

In Section 5.4, definitions for state *observability* and *constructibility* are given, and appropriate tests for these concepts are derived. It is shown that observability always implies constructibility, whereas constructibility implies observability only when the state transition matrix $\Phi$ of the system is nonsingular. Whereas this is always true for continuous-time systems, it is true for discrete-time systems only when the matrix $A$ of the system is nonsingular. If a system is state observable, then its present state can be determined from knowledge of the present and future outputs and inputs. Constructibility refers to the ability of determining the present state from present and past outputs and inputs, and as such, it is of greater interest in applications.

In the time-invariant case a system [or a pair $(A, C)$] is observable if and only if its *observability matrix* $\mathcal{O}$, where

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in R^{pn \times n}, \tag{5.10}$$

has full column rank; i.e., $\operatorname{rank} \mathcal{O} = n$. The matrices $A \in R^{n \times n}$ and $C \in R^{p \times n}$ are given by the system description

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{5.11}$$

in the continuous-time case, and by the system description

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k), \tag{5.12}$$

with $k \geq k_0 = 0$, in the discrete-time case.

We shall now briefly discuss observability and constructibility for the discrete-time time-invariant case. As in the case of reachability and controllability, this discussion will provide insight into the underlying concepts and will clarify what these imply for a system.

If the output in (5.12) is expressed in terms of the initial vector $x(0)$, then

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k) \tag{5.13}$$

for $k > 0$ (see Section 3.5). This implies that

$$\tilde{y}(k) = CA^k x_0 \tag{5.14}$$

for $k \geq 0$, where

$$\tilde{y}(k) \triangleq y(k) - \left[ \sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k) \right]$$

for $k > 0, \tilde{y}(0) \triangleq y(0) - Du(0)$, and $x_0 = x(0)$. In (5.14) $x_0$ is to be determined assuming that the system parameters are given and the inputs and outputs are measured. Note that if $u(k) = 0$ for $k \geq 0$, then the problem is simplified, since $\tilde{y}(k) = y(k)$ and since the output is generated only by the initial condition $x_0$. It is clear that the ability of determining $x_0$ from output and input measurements depends only on the matrices $A$ and $C$, since the left-hand side of (5.14) is a known quantity. Now if $x(0) = x_0$ is known, then all $x(k), k \geq 0$, can be determined by means of (5.12). To determine $x_0$, we apply (5.14) for $k = 0, \ldots, n-1$. Then

$$\widetilde{Y}_{0,n-1} = \mathcal{O}_n x_0, \tag{5.15}$$

where $\mathcal{O}_n \triangleq [C^T, (CA)^T, \ldots, (CA^{n-1})^T]^T = \mathcal{O}$ [as in (5.10)] and

$$\widetilde{Y}_{0,n-1} \triangleq [\tilde{y}^T(0), \ldots, \tilde{y}^T(n-1)]^T.$$

Now (5.15) always has a solution $x_0$, by construction. A system is observable if the solution $x_0$ is unique, i.e., if it is the only initial condition that, together with the given input sequence, can generate the observed output sequence. From the theory of linear systems of equations, (5.15) has a unique solution $x_0$ if and only if the null space of $\mathcal{O}$ consists of only the zero vector, i.e., $\mathrm{Null}(\mathcal{O}) = \mathcal{N}(\mathcal{O}) = \{0\}$, or equivalently, if and only if the only $x \in R^n$ that satisfies

$$\mathcal{O}x = 0 \tag{5.16}$$

is the zero vector. This is true if and only if $\mathrm{rank}\, \mathcal{O} = n$. Thus, a system is observable if and only if $\mathrm{rank}\, \mathcal{O} = n$. Any nonzero state vector $x \in R^n$ that satisfies (5.16) is said to be an unobservable state, and $\mathcal{N}(\mathcal{O})$ is said to be the *unobservable subspace*. Note that any such $x$ satisfies $CA^k x = 0$ for $k = 0, 1, \ldots, n-1$. If $\mathrm{rank}\, \mathcal{O} < n$, then all vectors $x_0$ that satisfy (5.15) are given by $x_0 = x_{0p} + x_{0h}$, where $x_{0p}$ is a particular solution and $x_{0h}$ is any vector in $\mathcal{N}(\mathcal{O})$. Any of these state vectors, together with the given inputs, could have generated the measured outputs.

To determine $x_0$ from (5.15) it is not necessary to use more than $n$ values for $\tilde{y}(k), k = 0, \ldots, n-1$, or to observe $y(k)$ for more than $n$ steps in the future. This is true because, in view of the Cayley–Hamilton Theorem, it can be shown that $\mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_k)$ for $k \geq n$. Note also that $\mathcal{N}(\mathcal{O}_n)$ is included in $\mathcal{N}(\mathcal{O}_k)$ $(\mathcal{N}(\mathcal{O}_n) \subset \mathcal{N}(\mathcal{O}_k))$ for $k < n$. Therefore, in general, one has to observe the output for $n$ steps (see Exercise 5.1).

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***Example 5.4.*** Consider the system $x(k+1) = Ax(k), y(k) = Cx(k)$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $C = [0\ 1]$. Here, $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ with $\operatorname{rank} \mathcal{O} = 2$. Therefore, the system [or the pair $(A, C)$] is observable. This means that $x(0)$ can uniquely be determined from $n = 2$ output measurements (in the present cases, the input is zero). In fact, in view of (5.15), $\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ or $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} y(1) - y(0) \\ y(0) \end{bmatrix}$.

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***Example 5.5.*** Consider the system $x(k+1) = Ax(k), y(k) = Cx(k)$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $C = [1\ 0]$. Here, $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ with $\operatorname{rank} \mathcal{O} = 1$. Therefore, the system is not observable. Note that a basis for $\mathcal{N}(\mathcal{O})$ is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, which in view of (5.16) implies that all state vectors of the form $\begin{bmatrix} 0 \\ c \end{bmatrix}, c \in R$, are unobservable. Relation (5.15) implies that $\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$. For a solution $x(0)$ to exist, as it must, we have that $y(0) = y(1) = a$. Thus, this system will generate an identical output for $k \geq 0$. Accordingly, all $x(0)$ that satisfy (5.15) and can generate this output are given by $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$, where $c \in R$.

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In general, a system (5.12) [or a pair $(A, C)$] is *constructible* if the only vector $x$ that satisfies $x = A^k \hat{x}$ with $C\hat{x} = 0$ for every $k \geq 0$ is the zero vector. When $A$ is nonsingular, this condition can be stated more simply, namely, that the system is constructible if the only vector $x$ that satisfies $CA^{-k}x = 0$ for every $k \geq 0$ is the zero vector. Compare this with the condition $CA^k x = 0, k \geq 0$, for $x$ to be an unobservable state; or with the condition that a system is observable if the only vector $x$ that satisfies $CA^k x = 0$ for every $k \geq 0$ is the zero vector. In view of (5.14), the above condition for a system to be constructible is the condition for the existence of a unique solution $x_0$ when past outputs and inputs are used. This, of course, makes sense since constructibility refers to determining the present state from knowledge of past outputs and inputs. Therefore, when $A$ is nonsingular, the system is constructible if and only if the $pn \times n$ matrix

$$\mathcal{O}A^{-n} = \begin{bmatrix} CA^{-n} \\ \vdots \\ CA^{-1} \end{bmatrix} \tag{5.17}$$

has full rank, since in this case the only $x$ that satisfies $CA^{-k}x = 0$ for every $k \geq 0$ is $x = 0$. Note that if the system is observable, then it is also constructible; however, if it is constructible, then it is also observable only when $A$ is nonsingular (see Section 5.3).

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***Example 5.6.*** Consider the (unobservable) system in Example 5.5. Since $A$ is nonsingular, $\mathcal{O}A^{-2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Since $\operatorname{rank} \mathcal{O}A^{-2} = 1 < 2$, the system [or the pair $(A, C)$] is not constructible. This can also be seen from the relation $CA^{-k}x = 0$, $k \geq 0$, that has nonzero solutions $x$, since $C = [1,\ 0] = CA^{-1} = CA^{-2} = \cdots = CA^{-k}$ for $k \geq 0$, which implies that any $x = \begin{bmatrix} 0 \\ c \end{bmatrix}$, $c \in R$, is a solution.

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### 5.2.3 Dual Systems

Consider the system described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{5.18}$$

where $A \in R^{n \times n}, B \in R^{n \times m}, C \in R^{p \times n}$, and $D \in R^{p \times m}$. The *dual system* of (5.18) is defined as the system

$$\dot{x}_D = A_D x_D + B_D u_D, \quad y_D = C_D x_D + D_D u_D, \tag{5.19}$$

where $A_D = A^T, B_D = C^T, C_D = B^T$, and $D_D = D^T$.

**Lemma 5.7.** *System (5.18), denoted by $\{A, B, C, D\}$, is reachable (controllable) if and only if its dual $\{A_D, B_D, C_D, D_D\}$ in (5.19) is observable (constructible), and vice versa.*

*Proof.* System $\{A, B, C, D\}$ is reachable if and only if $\mathcal{C} \triangleq [B, AB, \ldots, A^{n-1}B]$ has full rank $n$, and its dual is observable if and only if

$$\mathcal{O}_D \triangleq \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix}$$

has full rank $n$. Since $\mathcal{O}_D^T = \mathcal{C}, \{A, B, C, D\}$ is reachable if and only if $\{A_D, B_D, C_D, D_D\}$ is observable. Similarly, $\{A, B, C, D\}$ is observable if and only if $\{A_D, B_D, C_D, D_D\}$ is reachable. Now $\{A, B, C, D\}$ is controllable if and only if its dual is constructible, and vice versa, since it is shown in Sections 5.3 and 5.4, that a continuous-time system is controllable if and only if it is reachable; it is constructible if and only if it is observable. ■

For the discrete-time time-invariant case, the dual system is again defined as $A_D = A^T, B_D = C^T, C_D = B^T$, and $D_D = D^T$. That such a system is reachable if and only if its dual is observable can be shown in exactly the same way as in the proof of Lemma 5.7. That such a system is controllable if and only if its dual is constructible in the case when $A$ is nonsingular is because in this case the system is reachable if and only if it is controllable; and the same holds for observability and constructibility. The proof for the case when $A$ is singular involves the controllable and unconstructible subspaces of a system and its dual. We omit the details. The reader is encouraged to complete this proof after studying Sections 5.3 and 5.4.

Figure 5.3 summarizes the relationships between reachability (observability) and controllability (constructibility) for continuous- and discrete-time systems.

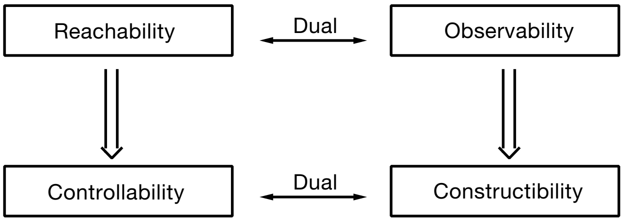


**Figure 5.3.** In continuous-time systems, reachability (observability) always implies and is implied by controllability (constructibility). In discrete-time systems, reachability (observability) always implies but in general is not implied by controllability (constructibility).

## 5.3 Reachability and Controllability

The objective here is to study the important properties of state controllability and reachability when a system is described by a state-space representation. In the previous section, a brief introduction to these concepts was given for discrete-time systems, and it was shown that a system is completely reachable if and only if the controllability (-from-the-origin) matrix $\mathcal{C}$ in (5.1) has full rank $n$ ($\text{rank}\,\mathcal{C} = n$). Furthermore, it was shown that the input sequence necessary to accomplish the transfer can be determined directly from $\mathcal{C}$ by solving a system of linear algebraic equations (5.5). In a similar manner, we would like to derive tests for reachability and controllability and determine the necessary system inputs to accomplish the state transfer for the continuous-time case. We note, however, that whereas the test for reachability can be derived by a number of methods, the appropriate sequence of system inputs to use cannot easily be determined directly from $\mathcal{C}$, as was the case for discrete-time systems. For this reason, we use an approach that utilizes ranges of maps, in particular, the range of an important $n \times n$ matrix—the reachability Gramian. The inputs that accomplish the desired state transfer can be determined directly from this matrix.

### 5.3.1 Continuous-Time Time-Invariant Systems

We consider the state equation

$$\dot{x} = Ax + Bu, \tag{5.20}$$

where $A \in R^{n\times n}, B \in R^{n\times m}$, and $u(t) \in R^m$ is (piecewise) continuous. The state at time $t$ is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)d\tau, \tag{5.21}$$

where $\Phi(t, \tau)$ is the state transition matrix of the system, and $x(t_0) = x_0$ denotes the state at initial time.

Here
$$\Phi(t, \tau) = \Phi(t - \tau, 0) = \exp[(t - \tau)A] = e^{A(t-\tau)}. \tag{5.22}$$

We are interested in using the input to transfer the state from $x_0$ to some other value $x_1$ at some finite time $t_1 > t_0$, [i.e., $x(t_1) = x_1$ in (5.21)]. Because of time invariance, the difference $t_1 - t_0 = T$, rather than the individual times $t_0$ and $t_1$, is important. Accordingly, we can always take $t_0 = 0$ and $t_1 = T$. Equation (5.21) assumes the form

$$x_1 - e^{AT}x_0 = \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau, \tag{5.23}$$

and clearly, there exists $u(t)$, $t \in [0, T]$, that satisfies (5.23) if and only if such transfer of the state is possible. Letting $\hat{x}_1 \triangleq x_1 - e^{AT}x_0$, we note that the $u(t)$ that transfers the state from $x_0$ at time 0 to $x_1$ at time $T$ will also cause the state to reach $\hat{x}_1$ at $T$, starting from the origin at 0 (i.e., $x(0) = 0$).

For the time-invariant system (5.20), we introduce the following concepts.

**Definition 5.8.** *(i) A state $x_1$ is* reachable *if there exists an input $u(t)$, $t \in [0, T]$, that transfers the state $x(t)$ from the origin at $t = 0$ to $x_1$ in some finite time $T$.*

*(ii) The set of all reachable states* $R_r$ *is the* reachable subspace *of the system* $\dot{x} = Ax + Bu$*, or of the pair* $(A, B)$*.*

*(iii) The system* $\dot{x} = Ax + Bu$*, or the pair* $(A, B)$ *is* (completely state) reachable *if every state is reachable, i.e., if* $R_r = R^n$*.* ■

Regarding (ii), note that the set of all reachable states $x_1$ contains the origin and constitutes a linear subspace of the state space $(R^n, R)$.

A reachable state is sometimes also called *controllable-from-the-origin.* Additionally, there are also states defined to be *controllable-to-the-origin* or simply *controllable*; see the definition later in this section.

**Definition 5.9.** *The* $n \times n$ reachability Gramian *of the time-invariant system* $\dot{x} = Ax + Bu$ *is*

$$W_r(0, T) \triangleq \int_0^T e^{(T-\tau)A} BB^T e^{(T-\tau)A^T} d\tau. \tag{5.24}$$

■

Note that $W_r$ is symmetric and positive semidefinite for every $T > 0$; i.e., $W_r = W_r^T$ and $W_r \geq 0$ (show this).

It can now be shown in [1, p. 230, Lemma 3.2.1] that the reachable subspace of the system (5.20) is exactly the range of the reachability Gramian $W_r$ in (5.24). Let the $n \times mn$ *controllability (-from-the-origin) matrix* be

$$\mathcal{C} \triangleq [B, AB, \ldots, A^{n-1}B]. \tag{5.25}$$

The range of $W_r(0, T)$, denoted by $\mathcal{R}(W_r(0, T))$, is independent of $T$; i.e., it is the same for any finite $T(> 0)$, and in particular, it is equal to the range of the controllability matrix $\mathcal{C}$. Thus, the reachable subspace $R_r$ of system (5.20), which is the set of all states that can be reached from the origin in finite time, is given by the range of $\mathcal{C}, \mathcal{R}(\mathcal{C})$, or the range of $W_r(0, T), \mathcal{R}(W_r(0, T))$, for some finite (and therefore for any) $T > 0$. This is stated as Lemma 5.10 below; for the proof, see [1, p. 236, Lemma 3.2.10].

**Lemma 5.10.** $\mathcal{R}(W_r(0, T)) = \mathcal{R}(\mathcal{C})$ *for every* $T > 0$. ■

---

***Example 5.11.*** For the system $\dot{x} = Ax + Bu$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $e^{At}B = \begin{bmatrix} t \\ 1 \end{bmatrix}$. The reachability Gramian is $W_r(0, T) = \int_0^T \begin{bmatrix} T - \tau \\ 1 \end{bmatrix} [T - \tau, 1] d\tau = \int_0^T \begin{bmatrix} (T-\tau)^2 & T - \tau \\ T - \tau & 1 \end{bmatrix} d\tau = \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix}$. Since $\det W_r(0, T) = \frac{1}{12}T^4 \neq 0$ for any $T > 0$, $\operatorname{rank} W_r(0, T) = n$ and $(A, B)$ is reachable. Note that $\mathcal{C} = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and that $\mathcal{R}(W_r(0, T)) = \mathcal{R}(\mathcal{C}) = R^2$, as expected (Lemma 5.10).

If $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, instead of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $(A, B)$ is not reachable. In this case $e^{At}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the reachability matrix is $W_r(0,T) = \int_0^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} d\tau = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$. Notice again that $\mathcal{R}(\mathcal{C}) = \mathcal{R}(W_r(0,T))$ for every $T > 0$.

---

The following theorems and corollaries 5.12 to 5.15 contain the main reachability results. Their proofs may be found in [1, p. 237, Chapter 3], starting with Theorem 2.11.

**Theorem 5.12.** *Consider the system $\dot{x} = Ax + Bu$, and let $x(0) = 0$. There exists an input $u$ that transfers the state to $x_1$ in finite time if and only if $x_1 \in \mathcal{R}(\mathcal{C})$, or equivalently, if and only if $x_1 \in \mathcal{R}(W_r(0,T))$ for some finite (and therefore for any) $T$. Thus, the reachable subspace $R_r = \mathcal{R}(\mathcal{C}) = \mathcal{R}(W_r(0,T))$. Furthermore, an appropriate $u$ that will accomplish this transfer in time $T$ is given by*

$$u(t) = B^T e^{A^T(T-t)} \eta_1 \tag{5.26}$$

*with $\eta_1$ such that $W_r(0,T)\eta_1 = x_1$ and $t \in [0,T]$.* ■

Note that in (5.26) no restrictions are imposed on time $T$, other than $T$ be finite. $T$ can be as small as we wish; i.e., the transfer can be accomplished in a very short time indeed.

**Corollary 5.13.** *The system $\dot{x} = Ax + Bu$, or the pair $(A, B)$, is (completely state) reachable, if and only if*

$$\operatorname{rank} \mathcal{C} = n, \tag{5.27}$$

*or equivalently, if and only if*

$$\operatorname{rank} W_r(0,T) = n \tag{5.28}$$

*for some finite (and therefore for any) $T$.* ■

**Theorem 5.14.** *There exists an input $u$ that transfers the state of the system $\dot{x} = Ax + Bu$ from $x_0$ to $x_1$ in some finite time $T$ if and only if*

$$x_1 - e^{AT}x_0 \in \mathcal{R}(\mathcal{C}), \tag{5.29}$$

*or equivalently, if and only if*

$$x_1 - e^{AT}x_0 \in \mathcal{R}(W_r(0,T)). \tag{5.30}$$

*Such an input is given by*

$$u(t) = B^T e^{A^T(T-t)}\eta_1 \tag{5.31}$$

*with* $t \in [0, T]$, *where* $\eta_1$ *is a solution of*

$$W_r(0, T)\eta_1 = x_1 - e^{AT}x_0. \tag{5.32}$$

■

The above theorem leads to the next result, which establishes the importance of reachability in determining an input $u$ to transfer the state from any $x_0$ to any $x_1$ in finite time.

**Corollary 5.15.** *Let the system* $\dot{x} = Ax + Bu$ *be (completely state) reachable, or the pair* $(A, B)$ *be reachable. Then there exists an input that will transfer any state* $x_0$ *to any other state* $x_1$ *in some finite time* $T$*. Such input is given by*

$$u(t) = B^T e^{A^T(T-t)} W_r^{-1}(0, T)[x_1 - e^{AT}x_0] \tag{5.33}$$

*for* $t \in [0, T]$. ■

There are many different control inputs $u$ that can accomplish the transfer from $x_0$ to $x_1$ in time $T$. It can be shown that the input $u$ given by (5.33) accomplishes this transfer while expending a minimum amount of energy; in fact, $u$ minimizes the cost functional $\int_0^T \| u(\tau) \|^2 d\tau$, where $\| u(t) \| \triangleq [u^T(t)u(t)]^{1/2}$ denotes the Euclidean norm of $u(t)$.

---

***Example 5.16.*** The system $\dot{x} = Ax + Bu$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is reachable (see Example 5.11). A control input $u(t)$ that will transfer any state $x_0$ to any other state $x_1$ in some finite time $T$ is given by (see Corollary 5.15 and Example 5.11)

$$\begin{aligned} u(t) &= B^T e^{A^T(T-t)} W_r^{-1}(0, T)[x_1 - e^{AT}x_0] \\ &= [T - t, 1] \begin{bmatrix} 12/T^3 & -6/T^2 \\ -6/T^2 & 4/T \end{bmatrix} \left[ x_1 - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_0 \right]. \end{aligned}$$

---

***Example 5.17.*** For the (scalar) system $\dot{x} = -ax + bu$, determine $u(t)$ that will transfer the state from $x(0) = x_0$ to the origin in $T$ sec; i.e., $x(T) = 0$.

We shall apply Corollary 5.15. The reachability Gramian is $W_r(0, T) = \int_0^T e^{-(T-\tau)a} b b e^{-(T-\tau)a} d\tau = e^{-2aT} b^2 \int_0^T e^{2a\tau} d\tau = e^{-2aT} b^2 \frac{1}{2a}[e^{2aT} - 1] = \frac{b^2}{2a}[1 - e^{-2aT}]$. (Note [see (5.36) below] that the controllability Gramian is $W_c(0, T) = \frac{b^2}{2a}[e^{2aT} - 1]$.) Now in view of (5.33), we have

$$\begin{aligned} u(t) &= b e^{-(T-t)a} \frac{2a}{b^2} \frac{1}{1 - e^{-2aT}} [-e^{-aT} x_0] \\ &= -\frac{2a}{b} \frac{e^{-2aT}}{1 - e^{-2aT}} e^{aT} x_0 = -\frac{2a}{b} \frac{1}{e^{2aT} - 1} e^{at} x_0. \end{aligned}$$

To verify that this $u(t)$ accomplishes the desired transfer, we compute $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{-at}x_0 + \int_0^t e^{-at}e^{a\tau}bu(\tau)d\tau = e^{-at}[x_0 + \int_0^t e^{a\tau}b \times \left(-\frac{2a}{b}\frac{1}{e^{2aT}-1} \times e^{a\tau}\right) d\tau = e^{-at}\left[1 - \frac{e^{2at}-1}{e^{2aT}-1}\right] x_0$. Note that $x(T) = 0$, as desired, and also that $x(0) = x_0$. The above expression shows also that for $t > T$, the state does not remain at the origin. An important point to notice here is that as $T \to 0$, the control magnitude $|u| \to \infty$. Thus, although it is (theoretically) possible to accomplish the desired transfer instantaneously, this will require infinite control magnitude. In general the faster the transfer, the larger the control magnitude required.

---

We now introduce the concept of a controllable state.

**Definition 5.18.** *(i) A state $x_0$ is* controllable *if there exists an input $u(t)$, $t \in [0, T]$, which transfers the state $x(t)$ from $x_0$ at $t = 0$ to the origin in some finite time $T$.*
*(ii) The set of all controllable states $R_c$, is the* controllable subspace *of the system $\dot{x} = Ax + Bu$, or of the pair $(A, B)$.*
*(iii) The system $\dot{x} = Ax + Bu$, or the pair $(A, B)$, is* (completely state) controllable *if every state is controllable, i.e., if $R_c = R^n$.* ■

We shall now establish the relationship between reachability and controllability for the continuous-time time-invariant systems (5.20).

In view of (5.23), $x_0$ is controllable when there exists $u(t), t \in [0, T]$, so that

$$-e^{AT}x_0 = \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau$$

or when $e^{AT}x_0 \in \mathcal{R}(W_r(0, T))$ [1, p. 230, Lemma 3.2.1], or equivalently, in view of Lemma 5.10, when

$$e^{AT}x_0 \in \mathcal{R}(\mathcal{C}) \tag{5.34}$$

for some finite $T$. Recall that $x_1$ is reachable when

$$x_1 \in \mathcal{R}(\mathcal{C}). \tag{5.35}$$

We require the following result.

**Lemma 5.19.** *If $x \in \mathcal{R}(\mathcal{C})$, then $Ax \in \mathcal{R}(\mathcal{C})$; i.e., the reachable subspace $R_r = \mathcal{R}(\mathcal{C})$ is an $A$-*invariant *subspace.*

*Proof.* If $x \in \mathcal{R}(\mathcal{C})$, this means that there exists a vector $\alpha$ such that $[B, AB, \ldots, A^{n-1}B]\alpha = x$. Then $Ax = [AB, A^2B, \ldots, A^nB]\alpha$. In view of the Cayley–Hamilton Theorem, $A^n$ can be expressed as a linear combination of $A^{n-1}, \ldots, A, I$, which implies that $Ax = \mathcal{C}\beta$ for some appropriate vector $\beta$. Therefore, $Ax \in \mathcal{R}(\mathcal{C})$. ■

**Theorem 5.20.** *Consider the system $\dot{x} = Ax + Bu$.*

*(i) A state $x$ is reachable if and only if it is controllable.*
*(ii) $R_c = R_r$.*
*(iii) The system (2.3), or the pair $(A, B)$, is (completely state) reachable if and only if it is (completely state) controllable.*

*Proof.* (i) Let $x$ be reachable, that is, $x \in \mathcal{R}(\mathcal{C})$. Premultiply $x$ by $e^{AT} = \sum_{k=0}^{\infty}(T^k/k!)A^k$ and notice that, in view of Lemma 5.19, $Ax, A^2x, \ldots, A^kx \in \mathcal{R}(\mathcal{C})$. Therefore, $e^{AT}x \in \mathcal{R}(\mathcal{C})$ for any $T$ that, in view of (5.34), implies that $x$ is also controllable. If now $x$ is controllable, i.e., $e^{AT}x \in \mathcal{R}(\mathcal{C})$, then premultiplying by $e^{-AT}$, the vector $e^{-AT}\left(e^{AT}x\right) = x$ will also be in $\mathcal{R}(\mathcal{C})$. Therefore, $x$ is also reachable. Note that the second part of (i), that controllability implies reachability, is true because the inverse $(e^{AT})^{-1} = e^{-AT}$ does exist. This is in contrast to the discrete-time case where the state transition matrix $\Phi(k, 0)$ is nonsingular if and only if $A$ is nonsingular [nonreversibility of time in discrete-time systems].

Parts (ii) and (iii) of the theorem follow directly from (i). ■

The reachability Gramian for the time-invariant case, $W_r(0, T)$, was defined in (5.24). For completeness the controllability Gramian is defined below.

**Definition 5.21.** *The* controllability Gramian *in the time-invariant case is the $n \times n$ matrix*

$$W_c(0, T) \triangleq \int_0^T e^{-A\tau}BB^Te^{-A^T\tau}d\tau. \tag{5.36}$$

■

We note that

$$W_r(0, T) = e^{AT}W_c(0, T)e^{A^TT},$$

which can be verified directly.

## Additional Criteria for Reachability and Controllability

We first recall the definition of a set of linearly independent functions of time and consider in particular $n$ complex-valued functions $f_i(t)$, $i = 1, \ldots, n$, where $f_i^T(t) \in C^m$. Recall that the set of functions $f_i$, $i = 1, \ldots, n$, is *linearly dependent* on a time interval $[t_1, t_2]$ over the field of complex numbers $C$ if there exist complex numbers $a_i$, $i = 1, \ldots, n$, not all zero, such that

$$a_1f_1(t) + \cdots + a_nf_n(t) = 0 \quad \text{for all } t \text{ in } [t_1, t_2];$$

otherwise, the set of functions is said to be *linearly independent* on $[t_1, t_2]$ over the field of complex numbers.

It is possible to test linear independence using the *Gram matrix of the functions $f_i$*.

**Lemma 5.22.** *Let $F(t) \in C^{n\times m}$ be a matrix with $f_i(t) \in C^{1\times m}$ in its ith row. Define the* Gram matrix *of $f_i(t)$, $i = 1, \dots, n$, by*

$$W(t_1, t_2) \triangleq \int_{t_1}^{t_2} F(t)F^*(t)dt, \tag{5.37}$$

*where $(\cdot)^*$ denotes the complex conjugate transpose. The set $f_i(t)$, $i = 1, \dots, n$, is linearly independent on $[t_1, t_2]$ over the field of complex numbers if and only if the Gram matrix $W(t_1, t_2)$ is nonsingular, or equivalently, if and only if the* Gram determinant $\det W(t_1, t_2) \neq 0$.

*Proof.* (*Necessity*) Assume the set $f_i, i = 1, \dots, n$, is linearly independent but $W(t_1, t_2)$ is singular. Then there exists some nonzero $\alpha \in C^{1\times n}$ so that $\alpha W(t_1, t_2) = 0$, from which $\alpha W(t_1, t_2)\alpha^* = \int_{t_1}^{t_2} (\alpha F(t))(\alpha F(t))^* dt = 0$. Since $(\alpha F(t))(\alpha F(t))^* \geq 0$ for all $t$, this implies that $\alpha F(t) = 0$ for all $t$ in $[t_1, t_2]$, which is a contradiction. Therefore, $W(t_1, t_2)$ is nonsingular.

(*Sufficiency*) Assume that $W(t_1, t_2)$ is nonsingular but the set $f_i, i = 1, \dots, n$, is linearly dependent. Then there exists some nonzero $\alpha \in C^{1\times n}$ so that $\alpha F(t) = 0$. Then $\alpha W(t_1, t_2) = \int_{t_1}^{t_2} \alpha F(t)F^*(t)dt = 0$, which is a contradiction. Therefore, the set $f_i$, $i = 1, \dots, n$, is linearly independent. ∎

We now introduce a number of additional tests for reachability and controllability of time-invariant systems. Some earlier results are also repeated here for convenience.

**Theorem 5.23.** *The system $\dot{x} = Ax + Bu$ is reachable (controllable-from-the-origin)*

*(i)* *if and only if*

$$\text{rank}\, W_r(0, T) = n \quad \textit{for some finite } T > 0,$$

*where*

$$W_r(0, T) \triangleq \int_0^T e^{(T-\tau)A}BB^T e^{(T-\tau)A^T} d\tau, \tag{5.38}$$

*the reachability Gramian; or*

*(ii)* *if and only if the $n$ rows of*

$$e^{At}B \tag{5.39}$$

*are linearly independent on $[0, \infty)$ over the field of complex numbers; or alternatively, if and only if the $n$ rows of*

$$(sI - A)^{-1}B \tag{5.40}$$

*are linearly independent over the field of complex numbers; or*

*(iii)* *if and only if*

$$\text{rank}\, \mathcal{C} = n, \tag{5.41}$$

*where $\mathcal{C} \triangleq [B, A, B, \dots, A^{n-1}B]$, the controllability matrix; or* *(iv) if and only if*

$$\operatorname{rank}[s_i I - A, B] = n \tag{5.42}$$

*for all complex numbers $s_i$; or alternatively, for $s_i$, $i = 1, \ldots, n$, the eigenvalues of $A$.*

*Proof.* Parts (i) and (ii) were proved in Corollary 5.13.

In part (ii), $\operatorname{rank} W_r(0,T) = n$ implies and is implied by the linear independence of the $n$ rows of $e^{(T-t)A}B$ on $[0,T]$ over the field of complex numbers, in view of Lemma 5.22, or by the linear independence of the $n$ rows of $e^{\hat{t}A}B$, where $\hat{t} \triangleq T - t$, on $[0,T]$. Therefore, the system is reachable if and only if the $n$ rows of $e^{At}B$ are linearly independent on $[0,\infty)$ over the field of complex numbers. Note that the time interval can be taken to be $[0,\infty)$ since in $[0,T]$, $T$ can be taken to be any finite positive real number. To prove the second part of (ii), recall that $\mathcal{L}(e^{At}B) = (sI - A)^{-1}B$ and that the Laplace transform is a one-to-one linear operator.

Part (iv) will be proved later in Section 6.3. ■

Since reachability implies and is implied by controllability, the criteria developed in the theorem for reachability are typically used to test the controllability of a system as well.

---

***Example 5.24.*** For the system $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (as in Example 5.11), we shall verify Theorem 5.23. The system is reachable since

(i) the reachability Gramian $W_r(0,T) = \begin{bmatrix} \frac{1}{3}T^3 & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & T \end{bmatrix}$ has $\operatorname{rank} W_r(0,T) = 2 = n$ for any $T > 0$, or since

(ii) $e^{At}B = \begin{bmatrix} t \\ 1 \end{bmatrix}$ has rows that are linearly independent on $[0,\infty)$ over the field of complex numbers (since $a_1 \times t + a_2 \times 1 = 0$, where $a_1$ and $a_2$ are complex numbers implies that $a_1 = a_2 = 0$). Similarly, the rows of $(sI - A)^{-1}B = \begin{bmatrix} 1/s^2 \\ 1/s \end{bmatrix}$ are linearly independent over the field of complex numbers. Also, since

(iii) $\operatorname{rank} \mathcal{C} = \operatorname{rank}[B, AB] = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 = n$, or

(iv) $\operatorname{rank}[s_i I - A, B] = \operatorname{rank} \begin{bmatrix} s_i & -1 & 0 \\ 0 & s_i & 1 \end{bmatrix} = 2 = n$ for $s_i = 0$, $i = 1, 2$, the eigenvalues of $A$.

If $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in place of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

(i) $W_r(0,T) = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ (see Example 5.11) with $\operatorname{rank} W_r(0,T) = 1 < 2 = n$, and

(ii) $e^{At}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $(sI - A)^{-1}B = \begin{bmatrix} 1/s \\ 0 \end{bmatrix}$, neither of which has rows that are linearly independent over the complex numbers. Also,

(iii) $\operatorname{rank} \mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < 2 = n$, and

(iv) $\operatorname{rank}[s_i I - A, B] = \operatorname{rank} \begin{bmatrix} s_i & -1 & 1 \\ 0 & s_i & 0 \end{bmatrix} = 1 < 2 = n$ for $s_i = 0$.

Based on any of the above tests, it is concluded that the system is not reachable.

---

### 5.3.2 Discrete-Time Systems

The response of discrete-time systems was studied in Section 3.5. We consider systems described by equations of the form

$$x(k+1) = Ax(k) + Bu(k), \quad k \geq k_0, \tag{5.43}$$

where $A \in R^{n \times n}$ and $B \in R^{n \times m}$. The state $x(k)$ is given by

$$x(k) = \Phi(k, k_0)x(k_0) + \sum_{i=k_0}^{k-1} \Phi(k, i+1)Bu(i), \tag{5.44}$$

where the state transition matrix is

$$\Phi(k, k_0) = A^{k-k_0}, \quad k \geq k_0. \tag{5.45}$$

Let the state at time $k_0$ be $x_0$. For the state at some time $k_1 > k_0$ to assume the value $x_1$, an input $u$ must exist that satisfies $x(k_1) = x_1$ in (5.44).

For the time-invariant system the elapsed time $k_1 - k_0$ is of interest, and we therefore take $k_0 = 0$ and $k_1 = K$. Recalling that $\Phi(k, 0) = A^k$, for the state $x_1$ to be reached from $x(0) = x_0$ in $K$ steps, i.e., $x(K) = x_1$, an input $u$ must exist that satisfies

$$x_1 = A^K x_0 + \sum_{i=0}^{K-1} A^{K-(i+1)} Bu(i), \tag{5.46}$$

when $K > 0$, or

$$x_1 = A^K x_0 + \mathcal{C}_K U_K, \tag{5.47}$$

where

$$\mathcal{C}_K \triangleq [B, AB, \ldots, A^{K-1}B] \tag{5.48}$$

and

$$U_K \triangleq [u^T(K-1), u^T(K-2), \ldots, u^T(0)]^T. \tag{5.49}$$

The definitions of *reachable state* $x_1$, *reachable subspace* $R_r$, and a *system* being *(completely state) reachable*, or *the pair (A,B) being reachable*, are the same as in the continuous-time case (see Definition 5.8, and use integer $K$ in place of real time $T$).

To determine the finite input sequence for discrete-time systems that will accomplish a desired state transfer, if such a sequence exists, one does not have to define matrices comparable with the reachability Gramian $W_r$, as in the case for continuous-time systems, but we can work directly with the controllability matrix $\mathcal{C}_n = \mathcal{C}$; see also the introductory discussion in Section 5.2.1. In particular, we have the following result.

**Theorem 5.25.** *Consider the system $x(k+1) = Ax(k) + Bu(k)$ given in (5.43), and let $x(0) = 0$. There exists an input $u$ that transfers the state to $x_1$ in finite time if and only if*

$$x_1 \in \mathcal{R}(\mathcal{C}).$$

*In this case, $x_1$ is reachable and $R_r = \mathcal{R}(\mathcal{C})$. An appropriate input sequence $\{u(k)\}$, $k = 0, \ldots, n-1$, that accomplishes this transfer in $n$ steps is determined by $U_n \triangleq [u^T(n-1), u^T(n-2), \ldots, u^T(0)]^T$, which is a solution to the equation*

$$\mathcal{C} U_n = x_1. \tag{5.50}$$

*Henceforth, with an abuse of language, we will refer to $U_n$ as a control sequence, when in fact we actually have in mind $\{u(k)\}$.*

*Proof.* In view of (5.47), $x_1$ can be reached from the origin in $K$ steps if and only if $x_1 = \mathcal{C}_K U_K$ has a solution $U_K$, or if and only if $x_1 \in \mathcal{R}(\mathcal{C}_K)$. Furthermore, all input sequences that accomplish this are solutions to the equation $x_1 = \mathcal{C}_K U_K$. For $x_1$ to be reachable we must have $x_1 \in \mathcal{R}(\mathcal{C}_K)$ for some finite $K$. This range, however, cannot increase beyond the range of $\mathcal{C}_n = \mathcal{C}$; i.e., $\mathcal{R}(\mathcal{C}_K) = \mathcal{R}(\mathcal{C}_n)$ for $K \geq n$ [see Exercise 5.1]. This follows from the Cayley–Hamilton Theorem, which implies that any vector $x$ in $\mathcal{R}(\mathcal{C}_K)$, $K \geq n$, can be expressed as a linear combination of $B, AB, \ldots, A^{n-1}B$. Therefore, $x \in \mathcal{R}(\mathcal{C}_n)$. It is of course possible to have $x_1 \in \mathcal{R}(\mathcal{C}_K)$ with $K < n$, for a particular $x_1$; however, in this case $x_1 \in \mathcal{R}(\mathcal{C}_n)$, since $\mathcal{C}_K$ is a subset of $\mathcal{C}_n$. Thus, $x_1$ is reachable if and only if it is in the range of $\mathcal{C}_n = \mathcal{C}$. Clearly, any $U_n$ that accomplishes the transfer satisfies (5.50). ∎

As pointed out in the above proof, for given $x_1$ we may have $x_1 \in \mathcal{R}(\mathcal{C}_K)$ for some $K < n$. In this case the transfer can be accomplished in fewer than $n$ steps, and appropriate inputs are obtained by solving the equation $\mathcal{C}_K U_K = x_1$.

**Corollary 5.26.** *The system* $x(k+1) = Ax(k) + Bu(k)$ *in (5.43) is* (completely state) reachable*, or the pair* $(A, B)$ *is reachable, if and only if*

$$\operatorname{rank} \mathcal{C} = n. \tag{5.51}$$

*Proof.* Apply Theorem 5.25, noting that $\mathcal{R}(\mathcal{C}) = R_r = R^n$ if and only if $\operatorname{rank} \mathcal{C} = n$. ■

**Theorem 5.27.** *There exists an input* $u$ *that transfers the state of the system* $x(k+1) = Ax(k) + Bu(k)$ *in (5.43) from* $x_0$ *to* $x_1$ *in some finite number of steps* $K$*, if and only if*

$$x_1 - A^K x_0 \in \mathcal{R}(\mathcal{C}_K). \tag{5.52}$$

*Such an input sequence* $U_K \triangleq [u^T(K-1), u^T(K-2), \ldots, u^T(0)]^T$ *is determined by solving the equation*

$$\mathcal{C}_K U_K = x_1 - A^K x_0. \tag{5.53}$$

*Proof.* The proof follows directly from (5.47). ■

The above theorem leads to the following result that establishes the importance of reachability in determining $u$ to transfer the state from any $x_0$ to any $x_1$ in a finite number of steps.

**Corollary 5.28.** *Let the system* $x(k+1) = Ax(k) + Bu(k)$ *given in (5.43) be (completely state) reachable or the pair* $(A, B)$ *be reachable. Then there exists an input sequence that transfers the state from any* $x_0$ *to any* $x_1$ *in a finite number of steps. Such input is determined by solving Eq. (5.54).*

*Proof.* Consider (5.47). Since $(A, B)$ is reachable, $\operatorname{rank} \mathcal{C}_n = \operatorname{rank} \mathcal{C} = n$ and $\mathcal{R}(\mathcal{C}) = R^n$. Then

$$\mathcal{C} U_n = x_1 - A^n x_0 \tag{5.54}$$

always has a solution $U_n = [u^T(n-1), \ldots, u^T(0)]^T$ for any $x_0$ and $x_1$. This input sequence transfers the state from $x_0$ to $x_1$ in $n$ steps. ■

Note that, in view of Theorem 5.27, for particular $x_0$ and $x_1$, the state transfer may be accomplished in $K < n$ steps, using (5.53).

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***Example 5.29.*** Consider the system in Example 5.1, namely, $x(k+1) = Ax(k) + Bu(k)$, where $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Since $\operatorname{rank} \mathcal{C} = \operatorname{rank}[B, AB] = \operatorname{rank} \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] = 2 = n$, the system is reachable and any state $x_0$ can be transferred to any other state $x_1$ in two steps. Let $x_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, $x_0 = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$. Then (5.54) implies that $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}$ or $\begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} b - 1 - b_0 \\ a - a_0 - b_0 \end{bmatrix}$. This agrees with the results obtained in Example 5.1. In view of (5.53), if $x_1$ and $x_0$ are chosen so that $x_1 - Ax_0 = \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a - b_0 \\ b - a_0 - b_0 \end{bmatrix}$ is in the $\mathcal{R}(\mathcal{C}_1) = \mathcal{R}(B) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, then the state transfer can be achieved in one step. For example, if $x_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $Bu(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(0) = x_1 - Ax_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ implies that the transfer from $x_0$ to $x_1$ can be accomplished in this case in $1 < 2 = n$ steps with $u(0) = 2$.

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***Example 5.30.*** Consider the system $x(k+1) = Ax(k) + Bu(k)$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since $\mathcal{C} = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has full rank, there exists an input sequence that will transfer the state from any $x(0) = x_0$ to any $x(n) = x_1$ (in $n$ steps), given by (5.54), $U_2 = \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \mathcal{C}^{-1}(x_1 - A^2 x_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (x_1 - x_0)$. Compare this with Example 5.16, where the continuous-time system had the same system parameters $A$ and $B$.

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*Additional Criteria for Reachability.* Note that completely analogous results to Theorem 5.23(ii)–(iv) exist for the discrete-time case.

We now turn to the concept of controllability. The definitions of *controllable state* $x_0$, *controllable subspace* $R_c$, and a *system* being *(completely state) controllable*, or *the pair (A,B) being controllable* are similar to the corresponding concepts given in Definition 5.18 for the case of continuous-time systems.

We shall now establish the relationship between reachability and controllability for the discrete-time time-invariant systems $x(k+1) = Ax(k) + Bu(k)$ in (5.43).

Consider (5.46). The state $x_0$ is controllable if it can be steered to the origin $x_1 = 0$ in a finite number of steps $K$. That is, $x_0$ is controllable if and only if

$$-A^K x_0 = \mathcal{C}_K U_K \tag{5.55}$$

for some finite positive integer $K$, or when

$$A^K x_0 \in \mathcal{R}(\mathcal{C}_K) \tag{5.56}$$

for some $K$. Recall that $x_1$ is reachable when

$$x_1 \in \mathcal{R}(\mathcal{C}). \tag{5.57}$$

**Theorem 5.31.** *Consider the system $x(k+1) = Ax(k) + Bu(k)$ in (5.43).*

*(i) If state $x$ is reachable, then it is controllable.*

*(ii)* $R_r \subset R_c$.
*(iii)* *If the system is (completely state) reachable, or the pair* $(A, B)$ *is reachable, then the system is also (completely state) controllable, or the pair* $(A, B)$ *is controllable.*

*Furthermore, if A is nonsingular, then relations (i) and (iii) become if and only if statements, since controllability also implies reachability, and relation (ii) becomes an equality; i.e.,* $R_c = R_r$.

*Proof.* (i) If $x$ is reachable, then $x \in \mathcal{R}(\mathcal{C})$. In view of Lemma 5.19, $\mathcal{R}(\mathcal{C})$ is an $A$-invariant subspace and so $A^n x \in \mathcal{R}(\mathcal{C})$, which in view of (5.56), implies that $x$ is also controllable. Since $x$ is an arbitrary vector in $R_r$, this implies (ii). If $\mathcal{R}(\mathcal{C}) = R^n$, the whole state space, then $A^n x$ for any $x$ is in $\mathcal{R}(\mathcal{C})$ and so any vector $x$ is also controllable. Thus, reachability implies controllability. Now, if $A$ is nonsingular, then $A^{-n}$ exists. If $x$ is controllable, i.e., $A^n x \in \mathcal{R}(\mathcal{C})$, then $x \in \mathcal{R}(\mathcal{C})$, i.e., $x$ is also reachable. This can be seen by noting that $A^{-n}$ can be written as a power series in terms of $A$, which in view of Lemma 5.19, implies that $A^{-n}(A^n x) = x$ is also in $\mathcal{R}(\mathcal{C})$. ■

Matrix $A$ being nonsingular is the necessary and sufficient condition for the state transition matrix $\Phi(k, k_0)$ to be nonsingular, which in turn is the condition for *time reversibility* in discrete-time systems. Recall that reversibility in time may not be present in such systems since $\Phi(k, k_0)$ may be singular. In contrast to this, in continuous-time systems, $\Phi(t, t_0)$ is always nonsingular. This causes differences in behavior between continuous- and discrete-time systems and implies that in discrete-time systems controllability may not imply reachability (see Theorem 5.31). Note that, in view of Theorem 5.20, in the case of continuous-time systems, it is not only reachability that always implies controllability, but also vice versa, controllability always implies reachability.

When $A$ is nonsingular, the input that will transfer the state from $x_0$ at $k = 0$ to $x_1 = 0$ in $n$ steps can be determined using (5.54). In particular, one needs to solve

$$[A^{-n}\mathcal{C}]U_n = [A^{-n}B, \ldots, A^{-1}B]U_n = -x_0 \tag{5.58}$$

for $U_n = [u^T(n-1), \ldots, u^T(0)]^T$. Note that $x_0$ is controllable if and only if $-A^n x_0 \in \mathcal{R}(\mathcal{C})$, or if and only if $x_0 \in \mathcal{R}(A^{-n}\mathcal{C})$ for $A$ nonsingular.

Clearly, in the case of controllability (and under the assumption that $A$ is nonsingular), the matrix $A^{-n}\mathcal{C}$ is of interest, instead of $\mathcal{C}$ [see also (5.8)]. In particular, a system is controllable if and only if $\operatorname{rank}(A^{-n}\mathcal{C}) = \operatorname{rank}\mathcal{C} = n$.

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***Example 5.32.*** Consider the system $x(k+1) = Ax(k) + Bu(k)$, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\operatorname{rank}\mathcal{C} = \operatorname{rank}[B, AB] = \operatorname{rank}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 1 < 2 = n$, this system is not (completely) reachable (controllable-from-the-origin). All reachable states are of the form $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where $\alpha \in R$ since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the $\mathcal{R}(\mathcal{C}) = R_r$, the reachability subspace.

In view of (5.56) and the Cayley–Hamilton Theorem, all controllable states $x_0$ satisfy $A^2 x_0 \in \mathcal{R}(\mathcal{C})$; i.e., all controllable states are of the form $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where $\alpha \in R$. This verifies Theorem 5.31 for the case when $A$ is nonsingular. Note that presently $R_r = R_c$.

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***Example 5.33.*** Consider the system $x(k+1) = Ax(k) + Bu(k)$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since $\operatorname{rank} \mathcal{C} = \operatorname{rank}[B, AB] = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < 2 = n$, the system is not (completely) reachable. All reachable states are of the form $\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, where $\alpha \in R$ since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathcal{R}(\mathcal{C}) = R_r$, the reachability subspace.

To determine the controllable subspace $R_c$, consider (5.56) for $K = n$, in view of the Cayley–Hamilton Theorem. Note that $A^{-1}\mathcal{C}$ cannot be used in the present case, since $A$ is singular. Since $A^2 x_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathcal{R}(\mathcal{C})$, any state $x_0$ will be a controllable state; i.e., the system is (completely) controllable and $R_c = R^n$. This verifies Theorem 5.31 and illustrates that controllability does not in general imply reachability.

Note that (5.54) can be used to determine the control sequence that will drive any state $x_0$ to the origin ($x_1 = 0$). In particular,

$$\mathcal{C} U_n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -A^2 x_0.$$

Therefore, $u(0) = \alpha$ and $u(1) = 0$, where $\alpha \in R$ will drive any state to the origin. To verify this, we consider $x(1) = Ax(0) + Bu(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \alpha = \begin{bmatrix} x_{02} + \alpha \\ 0 \end{bmatrix}$ and $x(2) = Ax(1) + Bu(1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{02} + \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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## 5.4 Observability and Constructibility

In applications, the state of a system is frequently required but not accessible. Under such conditions, the question arises whether it is possible to determine the state by observing the response of the system to some input over some period of time. It turns out that the answer to this question is affirmative if the system is observable. *Observability* refers to the ability of determining the present state $x(t_0)$ from knowledge of current and future system outputs, $y(t)$, and system inputs, $u(t), t \geq t_0$. *Constructibility* refers to the ability of determining the present state $x(t_0)$ from knowledge of current and past system outputs, $y(t)$, and system inputs, $u(t), t \leq t_0$. Observability was briefly addressed in Section 5.2. In this section this concept is formally defined and the (present) state is explicitly determined from input and output measurements.

### 5.4.1 Continuous-Time Time-Invariant Systems

We shall now study observability and constructibility for time-invariant systems described by equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{5.59}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}, D \in R^{p\times m}$, and $u(t) \in R^m$ is (piecewise) continuous. As was shown in Section 3.3, the output of this system is given by

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{5.60}$$

We recall that the initial time can always be taken to be $t_0 = 0$. We will find it convenient to rewrite (5.60) as

$$\tilde{y}(t) = Ce^{At}x_0, \tag{5.61}$$

where $\tilde{y}(t) \triangleq y(t) - \left[\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\right]$ and $x_0 = x(0)$.

**Definition 5.34.** *A* state $x$ *is* unobservable *if the zero-input response of the system (5.59) is zero for every $t \geq 0$, i.e., if*

$$Ce^{At}x = 0 \quad \textit{for every } t \geq 0. \tag{5.62}$$

*The set of all unobservable states $x$, $R_{\bar{o}}$, is called the* unobservable subspace *of (5.59).* System *(5.59) is* (completely state) observable*, or the pair $(A, C)$ is observable, if the only state $x \in R^n$ that is unobservable is $x = 0$, i.e., if $R_{\bar{o}} = \{0\}$.* ■

Definition 5.34 states that a state is unobservable precisely when it cannot be distinguished as an initial condition at time 0 from the initial condition $x(0) = 0$. This is because in this case the output is the same as if the initial condition were the zero vector. Note that the set of all unobservable states contains the zero vector and it can be shown to be a linear subspace. We now define the observability Gramian.

**Definition 5.35.** *The* observability Gramian *of system (5.59) is the* $n \times n$ *matrix*

$$W_o(0,T) \triangleq \int_0^T e^{A^T\tau} C^T C e^{A\tau} d\tau. \tag{5.63}$$

■

We note that $W_o$ is symmetric and positive semidefinite for every $T > 0$; i.e., $W_o = W_o^T$ and $W_o \geq 0$ (show this). Recall that the *$pn \times n$ observability matrix*

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{5.64}$$

was defined in Section 5.2.

We now show that the null space of $W_o(0,T)$, denoted by $\mathcal{N}(W_o(0,T))$, is independent of $T$; i.e., it is the same for any $T > 0$, and in particular, it is equal to the null space of the observability matrix $\mathcal{O}$. Thus, the unobservable subspace $R_{\bar{o}}$ of the system is given by the null space of $\mathcal{O}, \mathcal{N}(\mathcal{O})$, or the null space of $W_o(0,T), \mathcal{N}(W_o(0,T))$ for some finite (and therefore for all) $T > 0$.

**Lemma 5.36.** $\mathcal{N}(\mathcal{O}) = \mathcal{N}(W_o(0,T))$ *for every* $T > 0$.

*Proof.* If $x \in \mathcal{N}(\mathcal{O})$, then $\mathcal{O}x = 0$. Thus, $CA^k x = 0$ for all $0 \leq k \leq n-1$, which is also true for every $k > n-1$, in view of the Cayley–Hamilton Theorem. Then $Ce^{At}x = C[\Sigma_{k=0}^{\infty}(t^k/k!)A^i]x = 0$ for every finite $t$. Therefore, in view of (5.63), $W_o(0,T)x = 0$ for every $T > 0$; i.e., $x \in \mathcal{N}(W_o(0,T))$ for every $T > 0$. Now let $x \in \mathcal{N}(W_o(0,T))$ for some $T > 0$, so that $x^T W(0,T)x = \int_0^T \| Ce^{A\tau}x \|^2 d\tau = 0$, or $Ce^{At}x = 0$ for every $t \in [0,T]$. Taking derivatives of the last equation with respect to $t$ and evaluating at $t = 0$, we obtain $Cx = CAx = \cdots = CA^k x = 0$ for every $k > 0$. Therefore, $CA^k x = 0$ for every $k \geq 0$, i.e., $\mathcal{O}x = 0$ or $x \in \mathcal{N}(\mathcal{O})$. ■

**Theorem 5.37.** *A state $x$ is unobservable if and only if*

$$x \in \mathcal{N}(\mathcal{O}), \tag{5.65}$$

*or equivalently, if and only if*

$$x \in \mathcal{N}(W_o(0,T)) \tag{5.66}$$

*for some finite (and therefore for all) $T > 0$. Thus, the unobservable subspace $R_{\bar{o}} = \mathcal{N}(\mathcal{O}) = \mathcal{N}(W_o(0,T))$ for some $T > 0$.*

*Proof.* If $x$ is unobservable, (5.62) is satisfied. Taking derivatives with respect to $t$ and evaluating at $t = 0$, we obtain $Cx = CAx = \cdots = CA^k x = 0$ for $k > 0$ or $CA^k x = 0$ for every $k \geq 0$. Therefore, $\mathcal{O}x = 0$ and (5.65) is satisfied.

Assume now that $\mathcal{O}x = 0$; i.e., $CA^k x = 0$ for $0 \leq k \leq n-1$, which is also true for every $k > n-1$, in view of the Cayley–Hamilton Theorem. Then $Ce^{At}x = C[\Sigma_{k=0}^{\infty}(t^k/k!)A^i]x = 0$ for every finite $t$; i.e., (5.62) is satisfied and $x$ is unobservable. Therefore, $x$ is unobservable if and only if (5.65) is satisfied. In view of Lemma 5.36, (5.66) follows. ■

Clearly, $x$ is observable if and only if $\mathcal{O}x \neq 0$ or $W_o(0,T)x \neq 0$ for some $T > 0$.

**Corollary 5.38.** *The system (5.59) is (completely state) observable, or the pair $(A, C)$ is observable, if and only if*

$$\operatorname{rank} \mathcal{O} = n, \tag{5.67}$$

*or equivalently, if and only if*

$$\operatorname{rank} W_o(0,T) = n \tag{5.68}$$

*for some finite (and therefore for all) $T > 0$. If the system is observable, the state $x_0$ at $t = 0$ is given by*

$$x_0 = W_o^{-1}(0,T)\left[\int_0^T e^{A^T\tau}C^T\tilde{y}(\tau)d\tau\right]. \tag{5.69}$$

*Proof.* The system is observable if and only if the only vector that satisfies (5.62) or (5.65) is the zero vector. This is true if and only if the null space is empty, i.e., if and only if (5.67) or (5.68) are true. To determine the state $x_0$ at $t = 0$, given the output and input values over some interval $[0, T]$, we premultiply (5.61) by $e^{A^T\tau}C^T$ and integrate over $[0, T]$ to obtain

$$W_o(0,T)x_0 = \int_0^T e^{A^T\tau}C^T\tilde{y}(\tau)d\tau, \tag{5.70}$$

in view of (5.63). When the system is observable, (5.70) has the unique solution (5.69). ■

*Note that $T > 0$, the time span over which the input and output are observed, is arbitrary. Intuitively, one would expect in practice to have difficulties in evaluating $x_0$ accurately when $T$ is small, using any numerical method. Note that for very small $T$, $||W_o(0,T)||$ can be very small, which can lead to numerical difficulties in solving (5.70). Compare this with the analogous case for reachability, where small $T$ leads in general to large values in control action.*

It is clear that if the state at some time $t_0$ is determined, then the state $x(t)$ at any subsequent time is easily determined, given $u(t), t \geq t_0$.

Alternative methods to (5.69) to determine the state of the system when the system is observable are provided in Section 9.3 on state observers.

***Example 5.39.*** (i) Consider the system $\dot{x} = Ax, y = Cx$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = [1,\ 0]$. Here $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $Ce^{At} = [1,\ t]$. The observability Gramian is then $W_o(0,T) = \int_0^T \begin{bmatrix} 1 \\ \tau \end{bmatrix} [1\ \tau] d\tau = \int_0^T \begin{bmatrix} 1 & \tau \\ \tau & \tau^2 \end{bmatrix} d\tau = \begin{bmatrix} T & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & \frac{1}{3}T^3 \end{bmatrix}$. Notice that $\det W_o(0,T) = \frac{1}{12}T^4 \neq 0$ for any $T > 0$, i.e., $\operatorname{rank} W_o(0,T) = 2 = n$ for any $T > 0$, and therefore (Corollary 5.38), the system is observable. Alternatively, note that the observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\operatorname{rank} \mathcal{O} = 2 = n$. Clearly, in this case $\mathcal{N}(\mathcal{O}) = \mathcal{N}(W_o(0,T)) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, which verifies Lemma 5.36.

(ii) If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, as before, but $C = [0,\ 1]$, in place of $[1,\ 0]$, then $Ce^{At} = [0,\ 1]$ and the observability Gramian is $W_o(0,T) = \int_0^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0,\ 1] d\tau = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$. We have $\operatorname{rank} W_o(0,T) = 1 < 2 = n$, and the system is not completely observable. In view of Theorem 5.37, all unobservable states $x \in \mathcal{N}(W_o(0,T))$ and are therefore of the form $\begin{bmatrix} \alpha \\ 0 \end{bmatrix}, \alpha \in R$. Alternatively, the observability matrix $\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $\mathcal{N}(\mathcal{O}) = \mathcal{N}(W_0(0,T)) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

Observability utilizes future output measurements to determine the present state. In (re)constructibility, past output measurements are used. Constructibility is defined in the following, and its relation to observability is determined.

**Definition 5.40.** *A* state $x$ *is* unconstructible *if the zero-input response of the system (5.59) is zero for all* $t \leq 0$*; i.e.,*

$$Ce^{At}x = 0 \quad \text{for every } t \leq 0. \tag{5.71}$$

*The set of all unconstructible states* $x, R_{\overline{cn}}$*, is called the* unconstructible subspace *of (5.59). The system (5.59) is (completely state)* (re)constructible*, or the pair* $(A, C)$ *is* (re)constructible*, if the only state* $x \in R^n$ *that is unconstructible is* $x = 0$*; i.e.,* $R_{\overline{cn}} = \{0\}$.

We shall now establish a relationship between observability and constructibility for the continuous-time time-invariant systems (5.59). Recall that $x$ is unobservable if and only if

$$Ce^{At}x = 0 \quad \text{for every } t \geq 0. \tag{5.72}$$

**Theorem 5.41.** *Consider the system $\dot{x} = Ax + Bu, y = Cx + Du$ given in (5.59).*

*(i)* *A state $x$ is unobservable if and only if it is unconstructible.*
*(ii)* $R_{\bar{o}} = R_{\overline{cn}}$.
*(iii)* *The system, or the pair $(A, C)$, is (completely state) observable if and only if it is (completely state) (re)constructible.*

*Proof.* (i) If $x$ is unobservable, then $Ce^{At}x = 0$ for every $t \geq 0$. Taking derivatives with respect to $t$ and evaluating at $t = 0$, we obtain $Cx = CAx = \cdots = CA^k x = 0$ for $k > 0$ or $CA^k x = 0$ for every $k \geq 0$. This, in view of $Ce^{At}x = \sum_{k=0}^{\infty}(t^k/k!)CA^k x$, implies that $Ce^{At}x = 0$ for every $t \leq 0$; i.e., $x$ is unconstructible. The converse is proved in a similar manner. Parts (ii) and (iii) of the theorem follow directly from (i). ∎

The observability Gramian for the time-invariant case, $W_o(0, T)$, was defined in (5.63). The constructibility Gramian is now defined.

**Definition 5.42.** *The* constructibility Gramian *of system (5.59) is the $n \times n$ matrix*

$$W_{cn}(0, T) \triangleq \int_0^T e^{A^T(\tau - T)} C^T C e^{A(\tau - T)} d\tau. \tag{5.73}$$

∎

Note that

$$W_o(0, T) = e^{A^T T} W_{cn}(0, T) e^{AT}, \tag{5.74}$$

as can be verified directly.

### Additional Criteria for Observability and Constructibility

We shall now use Lemma 5.22 to develop additional tests for observability and constructibility. These are analogous to the corresponding results established for reachability and controllability in Theorem 5.23.

**Theorem 5.43.** *The system $\dot{x} = Ax + Bu, y = Cx + Du$ is observable*

*(i)* *if and only if*

$$\operatorname{rank} W_o(0, T) = n \tag{5.75}$$

*for some finite $T > 0$, where $W_0(0, T) \triangleq \int_0^T e^{A^T\tau} C^T C e^{A\tau} d\tau$, the observability Gramian, or* *(ii) if and only if the $n$ columns of*

$$Ce^{At} \tag{5.76}$$

*are linearly independent on $[0, \infty)$ over the field of complex numbers, or alternatively, if and only if the $n$ columns of*

$$C(sI - A)^{-1} \tag{5.77}$$

*are linearly independent over the field of complex numbers, or*

*(iii) if and only if*

$$\operatorname{rank} \mathcal{O} = n, \tag{5.78}$$

*where* $\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$, *the observability matrix, or*

*(iv) if and only if*

$$\operatorname{rank} \begin{bmatrix} s_i I - A \\ C \end{bmatrix} = n \tag{5.79}$$

*for all complex numbers $s_i$, or alternatively, for all eigenvalues of $A$.*

*Proof.* The proof of this theorem is completely analogous to the (dual) results on reachability (Theorem 5.23) and is omitted. ■

Since it was shown (in Theorem 5.41) that observability implies and is implied by constructibility, the tests developed in the theorem for observability are typically also used to test for constructibility.

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***Example 5.44.*** Consider the system $\dot{x} = Ax, y = Cx$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = [1,\ 0]$, as in Example 5.39(i). We shall verify (i) to (iv) of Theorem 5.43 for this case.

(i) For the observability Gramian, $W_o(0, T) = \begin{bmatrix} T & \frac{1}{2}T^2 \\ \frac{1}{2}T^2 & \frac{1}{3}T^3 \end{bmatrix}$, we have $\operatorname{rank} W_o(0, T) = 2 = n$ for any $T > 0$.

(ii) The columns of $Ce^{At} = [1,\ t]$ are linearly independent on $[0, \infty)$ over the field of complex numbers, since $a_1 \times 1 + a_2 \times t = 0$ implies that the complex numbers $a_1$ and $a_2$ must both be zero. Similarly, the columns of $C(sI - A)^{-1} = \left[\frac{1}{s}, \frac{1}{s^2}\right]$ are linearly independent over the field of complex numbers.

(iii) $\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2 = n.$

(iv) $\operatorname{rank} \begin{bmatrix} s_i I - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} s_i & -1 \\ 0 & s_i \\ 1 & 0 \end{bmatrix} = 2 = n$ for $s_i = 0$, $i = 1, 2$, the eigenvalues of $A$.

Consider again $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but $C = [0,\ 1]$ [in place of $[1,\ 0]$, as in Example 5.39(ii)].

The system is not observable for the reasons given below.

(i) $W_o(0,T) = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix}$ with $\operatorname{rank} W_o(0,T) = 1 < 2 = n$.

(ii) $Ce^{At} = [0,\ 1]$ and its columns are not linearly independent. Similarly, the columns of $C(sI - A)^{-1} = [0,\ \frac{1}{s}]$ are not linearly independent.

(iii) $\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1 < 2 = n$.

(iv) $\operatorname{rank} \begin{bmatrix} s_i I - A \\ C \end{bmatrix} = \operatorname{rank} \begin{bmatrix} s_i & -1 \\ 0 & s_i \\ 0 & 1 \end{bmatrix} = 1 < 2 = n$ for $s_i = 0$ an eigenvalue of $A$.

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### 5.4.2 Discrete-Time Time-Invariant Systems

We consider systems described by equations of the form

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) + Du(k), \quad k \geq k_0, \tag{5.80}$$

where $A \in R^{n \times n}, C \in R^{n \times m}, C \in R^{p \times n}, D \in R^{p \times m}$. The output $y(k)$ for $k > k_0$ is given by

$$y(k) = C(k)\Phi(k, k_0)x(k_0) + \sum_{i=k_0}^{k-1} C(k)\Phi(k, i+1)B(i)u(i) + D(k)u(k), \tag{5.81}$$

where the state transition matrix $\Phi(k, k_0)$ is given by

$$\Phi(k, k_0) = A^{k-k_0}, \quad k \geq k_0. \tag{5.82}$$

Observability and (re)constructibility for discrete-time systems are defined as in the continuous-time case. Observability refers to the ability to uniquely determine the state from knowledge of current and future outputs and inputs, whereas constructibility refers to the ability to determine the state from knowledge of current and past outputs and inputs. Without loss of generality, we take $k_0 = 0$. Then

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k) \tag{5.83}$$

for $k > 0$ and $y(0) = Cx(0) + Du(0)$. Rewrite as

$$\tilde{y}(k) = CA^k x_0 \tag{5.84}$$

for $k \geq 0$, where $\tilde{y}(k) \triangleq y(k) - \left[\sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k)\right]$ for $k > 0$ and $\tilde{y}(0) \triangleq y(0)$, and $x_0 = x(0)$.

**Definition 5.45.** *A* state $x$ *is* unobservable *if the zero-input response of system (5.80) is zero for all* $k \geq 0$*, i.e., if*

$$CA^k x = 0 \quad \text{for every } k \geq 0. \tag{5.85}$$

*The set of all unobservable states* $x, R_{\bar{o}}$*, is called the* unobservable subspace *of (5.80). The* system *(5.80) is* (completely state) observable*, or the pair* $(A, C)$ *is observable, if the only state* $x \in R^n$ *that is unobservable is* $x = 0$*, i.e., if* $R_{\bar{o}} = \{0\}$*.* ■

The $pn \times n$ *observability matrix* $\mathcal{O}$ was defined in (5.64). Let $\mathcal{N}(\mathcal{O})$ denote the null space of $\mathcal{O}$.

**Theorem 5.46.** *A state* $x$ *is unobservable if and only if*

$$x \in \mathcal{N}(\mathcal{O}); \tag{5.86}$$

*i.e., the unobservable subspace* $R_{\bar{o}} = \mathcal{N}(\mathcal{O})$*.*

*Proof.* If $x \in \mathcal{N}(\mathcal{O})$, then $\mathcal{O}x = 0$ or $CA^k x = 0$ for $0 \leq k \leq n-1$. This statement is also true for $k > n-1$, in view of the Cayley–Hamilton Theorem. Therefore, (5.85) is satisfied and $x$ is unobservable. Conversely, if $x$ is unobservable, then (5.85) is satisfied and $\mathcal{O}x = 0$. ■

Clearly, $x$ is observable if and only if $\mathcal{O}x \neq 0$.

**Corollary 5.47.** *The system (5.80) is (completely state) observable, or the pair* $(A, C)$ *is observable, if and only if*

$$\operatorname{rank} \mathcal{O} = n. \tag{5.87}$$

*If the system is observable, the state* $x_0$ *at* $k = 0$ *can be determined as the unique solution of*

$$[Y_{0,n-1} - M_n U_{0,n-1}] = \mathcal{O}x_0, \tag{5.88}$$

*where*

$$\begin{aligned} Y_{0,n-1} &\triangleq [y^T(0), y^T(1), \ldots, y^T(n-1)]^T \text{ is a } pn \times 1 \text{ matrix},\\ U_{0,n-1} &\triangleq [u^T(0), u^T(1), \ldots, u^T(n-1)]^T \text{ is an } mn \times 1 \text{ matrix}, \end{aligned}$$

*and* $M_n$ *is the* $pn \times mn$ *matrix given by*

$$M_n \triangleq \begin{bmatrix} D & 0 & \cdots & 0 & 0 \\ CB & D & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{n-2}B & CA^{n-3}B & \cdots & D & \\ CA^{n-1}B & CA^{n-2}B & \cdots & CB & D \end{bmatrix}.$$

*Proof.* The system is observable if and only if the only vector that satisfies (5.85) is the zero vector. This is true if and only if $\mathcal{N}(\mathcal{O}) = \{0\}$, or if (5.87) is true. To determine the state $x_0$, apply (5.83) for $k = 0, 1, \ldots, n-1$, and rearrange in a form of a system of linear equations to obtain (5.88). ■

The matrix $M_n$ defined above has the special structure of a *Toeplitz* matrix. Note that a matrix $T$ is Toeplitz if its $(i, j)$th entry depends on the value $i - j$; that is, $T$ is "constant along the diagonals."

*Additional Criteria for Observability.* Note that completely analogous results to Theorem 5.43(ii)–(iv) exist for the discrete-time case.

Constructibility refers to the ability to determine uniquely the state $x(0)$ from knowledge of current and past outputs and inputs. This is in contrast to observability, which utilizes future outputs and inputs. The easiest way to define constructibility is by the use of (5.84), where $x(0) = x_0$ is to be determined from past data $\tilde{y}(k)$, $k \leq 0$. Note, however, that for $k \leq 0$, $A^k$ may not exist; in fact, it exists only when $A$ is nonsingular. To avoid making restrictive assumptions, we shall define unconstructible states in a slightly different way than anticipated. Unfortunately, this definition is not very transparent. It turns out that by using this definition, an unconstructible state can be related to an unobservable state in a manner analogous to the way a controllable state was related to a reachable state in Section 5.3 (see also the discussion of duality in Section 5.2).

**Definition 5.48.** *A state $x$ is* unconstructible *if for every $k \geq 0$, there exists $\hat{x} \in R^n$ such that*

$$x = A^k \hat{x}, \quad C\hat{x} = 0. \tag{5.89}$$

*The set of all unconstructible states, $R_{\overline{cn}}$, is called the* unconstructible subspace*. The system (5.80) is* (completely state) constructible*, or the pair $(A, C)$ is constructible, if the only state $x \in R^n$ that is unconstructible is $x = 0$, i.e., if $R_{\overline{cn}} = \{0\}$.* ■

Note that if $A$ is nonsingular, then (5.89) simply states that $x$ is unconstructible if $CA^{-k}x = 0$ for every $k \geq 0$ (compare this with Definition 5.45 of an unobservable state).

The results that can be derived for constructibility are simply dual to the results on controllability. They are presented briefly below, but first, a technical result must be established.

**Lemma 5.49.** *If $x \in \mathcal{N}(\mathcal{O})$, then $Ax \in \mathcal{N}(\mathcal{O})$; i.e., the unobservable subspace $R_{\bar{o}} = \mathcal{N}(\mathcal{O})$ is an* A-invariant subspace.

*Proof.* Let $x \in \mathcal{N}(\mathcal{O})$, so that $\mathcal{O}x = 0$. Then $CA^k x = 0$ for $0 \leq k \leq n-1$. This statement is also true for $k > n-1$, in view of the Cayley–Hamilton Theorem. Therefore, $\mathcal{O}Ax = 0$; i.e., $Ax \in \mathcal{N}(\mathcal{O})$. ■

**Theorem 5.50.** *Consider the system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$ given in (5.80).*

*(i) If a state $x$ is unconstructible, then it is unobservable.*
*(ii) $R_{\overline{cn}} \subset R_{\bar{o}}$.*
*(iii) If the system is (completely state) observable, or the pair $(A, C)$ is observable, then the system is also (completely state) constructible, or the pair $(A, C)$ is constructible.*

*If $A$ is nonsingular, then relations (i) and (iii) are if and only if statements. In this case, constructibility also implies observability. Furthermore, in this case, (ii) becomes an equality; i.e., $R_{\overline{cn}} = R_{\bar{o}}$.*

*Proof.* This theorem is dual to Theorem 5.31, which relates reachability and controllability in the discrete-time case. To verify (i), assume that $x$ satisfies (5.89) and premultiply by $C$ to obtain $Cx = CA^k\hat{x}$ for every $k \geq 0$. Note that $Cx = 0$ since for $k = 0$, $x = \hat{x}$, and $C\hat{x} = 0$. Therefore, $CA^k\hat{x} = 0$ for every $k \geq 0$; i.e., $\hat{x} \in \mathcal{N}(\mathcal{O})$. In view of Lemma 5.49, $x = A^k\hat{x} \in \mathcal{N}(\mathcal{O})$, and thus, $x$ is unobservable. Since $x$ is arbitrary, we have also verified (ii). When the system is observable, $R_{\bar{o}}$ is empty, which in view of (ii), implies that $R_{\bar{c}n} = \{0\}$ or that the system is constructible. This proves (iii). Alternatively, one could also prove this directly: Assume that the system is observable but not constructible. Then there exist $x, \hat{x} \neq 0$, which satisfy (5.89). As above, this implies that $\hat{x} \in \mathcal{N}(\mathcal{O})$, which is a contradiction since the system is observable.

Consider now the case when $A$ is nonsingular and let $x$ be unobservable. Then, in view of Lemma 5.49, $\hat{x} \triangleq A^{-k}x$ is also in $\mathcal{N}(\mathcal{O})$; i.e., $C\hat{x} = 0$. Therefore, $x = A^k\hat{x}$ is unconstructible, in view of Definition 5.48. This implies also that $R_{\bar{o}} \subset R_{\overline{cn}}$, and therefore, $R_{\bar{o}} = R_{\overline{cn}}$, which proves that in the present case constructibility also implies observability. ∎

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***Example 5.51.*** Consider the system in Example 5.5, $x(k+1) = Ax(k), y(k) = Cx(k)$, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $C = [1,\ 0]$. As shown, $\operatorname{rank} \mathcal{O} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = 1 < 2 = n$; i.e., the system is not observable. All unobservable states are of the form $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $\alpha \in R$ since $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{N}(\mathcal{O}) = R_{\bar{o}}$, the unobservable subspace.

In Example 5.6 it was shown that all the states $x$ that satisfy $CA^{-k}x = 0$ for every $k \geq 0$, i.e., all the unconstructible states, are given by $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha \in R$. This verifies Theorem 5.50(i) and (ii) for the case when $A$ is nonsingular.

---

***Example 5.52.*** Consider the system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$, where $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $C = [1,\ 0]$. The observability matrix $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is of rank 1, and therefore, the system is not observable. In fact, all states of the form $\alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are unobservable states since $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathcal{N}(\mathcal{O})$.

To check constructibility, the defining relations (5.89) must be used since $A$ is singular. $C\hat{x} = [1,\ 0]\hat{x} = 0$ implies $\hat{x} = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$. Substituting into $x = A^k \hat{x}$, we obtain for $k = 0$, $x = \hat{x}$, and $x = 0$ for $k \geq 1$. Therefore, the only unconstructible state is $x = 0$, which implies that the system is constructible (although it is unobservable). This means that the initial state $x(0)$ can be uniquely determined from past measurements. In fact, from $x(k+1) = Ax(k)$ and $y(k) = Cx(k)$, we obtain $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(-1) \\ x_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ x_1(-1) \end{bmatrix}$ and $y(-1) = Cx(-1) = [1,\ 0] \begin{bmatrix} x_1(-1) \\ x_2(-1) \end{bmatrix} = x_1(-1)$. Therefore, $x(0) = \begin{bmatrix} 0 \\ y(-1) \end{bmatrix}$.

---

When $A$ is nonsingular, the state $x_0$ at $k = 0$ can be determined from past outputs and inputs in the following manner. We consider (5.84) and note that in this case

$$\tilde{y}(k) = CA^k x_0$$

is valid for $k \leq 0$ as well. This implies that

$$\widetilde{Y}_{-1,-n} = \mathcal{O}A^{-n} x_0 = \begin{bmatrix} CA^{-n} \\ \vdots \\ CA^{-1} \end{bmatrix} x_0 \tag{5.90}$$

with $\widetilde{Y}_{-1,-n} \triangleq [\tilde{y}^T(-n), \ldots, \tilde{y}^T(-1)]^T$. Equation (5.90) must be solved for $x_0$. Clearly, in the case of constructibility (and under the assumption that $A$ is nonsingular), the matrix $\mathcal{O}A^{-n}$ is of interest instead of $\mathcal{O}$ [compare this with the dual results in (5.58)]. In particular, the system is constructible if and only if $\operatorname{rank}(\mathcal{O}A^{-n}) = \operatorname{rank} \mathcal{O} = n$.

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***Example 5.53.*** Consider the system in Example 5.4, namely, $x(k+1) = Ax(k), y(k) = Cx(k)$, where $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $C = [0,\ 1]$. Since $A$ is nonsingular, to check constructibility we consider $\mathcal{O}A^{-2} = \left[\begin{smallmatrix} CA^{-2} \\ CA^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, which has full rank. Therefore, the system is constructible (as expected), since it is observable. To determine $x(0)$, in view of (5.90), we note that $\begin{bmatrix} y(-1) \\ y(-2) \end{bmatrix} = \mathcal{O}A^{-2}x(0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, from which $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y(-1) \\ y(-2) \end{bmatrix} = \begin{bmatrix} y(-2) \\ y(-1) + y(-2) \end{bmatrix}$.

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## 5.5 Summary and Highlights

*Reachability and Controllability*

- In continuous-time systems, reachability always implies and is implied by controllability. In discrete-time systems, reachability always implies controllability, but controllability implies reachability only when $A$ is nonsingular. See Definitions 5.8 and 5.18 and Theorems 5.20 and 5.31.
- When a discrete-time system $x(k+1) = Ax(k)+Bu(k)$ [denoted by $(A, B)$] is completely reachable (controllable-from-the-origin), the input sequence $\{u(i)\}$, $i = 0, \ldots, K-1$ that transfers the state from any $x_0 (= x(0))$ to any $x_1$ in some finite time $K$ ($x_1 = x(K),\ K > 0$) is determined by solving

$$x_1 = A^K x_0 + \sum_{i=0}^{K-1} A^{K-(i+1)} Bu(i) \quad \text{or}$$

$$x_1 - A^K x_0 = [B, AB, \ldots, A^{K-1}]\ [u^T(K-1), \ldots, u^T(0)]^T.$$

  A solution for this always exists when $K = n$. See Theorem 5.27.

- $$\mathcal{C} = [B, AB, \ldots, A^{n-1}B]\ (n \times mn) \qquad (5.25)$$

  is the controllability matrix for both discrete- and continuous-time time-invariant systems, and it has full (row) rank when the system, denoted by $(A, B)$, is (completely) reachable (controllable-from-the-origin).
- When a continuous-time system $\dot{x} = Ax + Bu$ [denoted by $(A, B)$] is controllable, an input that transfers any state $x_0 (= x(0))$ to any other state $x_1$ in some finite time $T$ ($x_1 = x(T)$ is

$$u(t) = B^T e^{A^T(T-t)} W_r^{-1}(0, T)[x_1 - e^{AT} x_0] \quad t \in [0, T], \qquad (5.33)$$

  where

$$W_r(0, T) = \int_0^T e^{(T-\tau)A} BB^T e^{(T-\tau)A^T} d\tau \qquad (5.24)$$

  is the reachability Gramian of the system.
- $(A, B)$ is reachable if and only if

$$\text{rank}[s_i I - A, B] = n \qquad (5.42)$$

  for $s_i$, $i = 1, \ldots, n$, all the eigenvalues of $A$.

*Observability and Constructibility*

- In continuous-time systems, observability always implies and is implied by constructibility. In discrete-time systems, observability always implies constructibility, but constructibility implies observability only when $A$ is nonsingular. See Definitions 5.34 and 5.40 and Theorems 5.41 and 5.50.

- When a discrete-time system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$ [denoted by $(A, C)$] is completely observable, any initial state $x(0) = x_0$ can be uniquely determined by observing the input and output over some finite period of time, and using the relation

$$\tilde{y}(k) = CA^k x_0 \quad k = 0, 1, \ldots, n-1, \tag{5.84}$$

where $\tilde{y}(k) = y(k) - \left[\sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + D(k)u(k)\right]$. To determine $x_0$, solve

$$\begin{bmatrix} \tilde{y}(0) \\ \tilde{y}(1) \\ \vdots \\ \tilde{y}(n-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0.$$

See (5.88).

- $$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (pn \times n) \tag{5.64}$$

is the observability matrix for both discrete- and continuous-time, time-invariant systems and it has full (column) rank when the system is completely observable.

- Consider the continuous-time system $\dot{x} = Ax + Bu$, $y = Cx + Du$. When this system [denoted by $(A, C)$] is completely observable, any initial state $x_0 = x(0)$ can be uniquely determined by observing the input and output over some finite period of time $T$ and using the relation

$$\tilde{y}(t) = Ce^{At} x_0,$$

where $\tilde{y}(t) = y(t) - \left[\int_0^t Ce^{A(t-\tau)} Bu(\tau) d\tau + Du(t)\right]$. The initial state $x_0$ may be determined from

$$x_0 = W_o^{-1}(0, T) \left[\int_0^T e^{A^T \tau} C^T \tilde{y}(\tau) d\tau\right], \tag{5.69}$$

where

$$W_o(0, T) = \int_0^T e^{A^T \tau} C^T C e^{A\tau} d\tau \tag{5.63}$$

is the observability Gramian of the system.

- $(A, C)$ is observable if and only if

$$\text{rank} \begin{bmatrix} s_i I - A \\ C \end{bmatrix} = n \tag{5.79}$$

for $s_i$, $i = 1, \ldots, n$, all the eigenvalues of $A$.

*Dual Systems*

- $(A_D = A^T, B_D = C^T, C_D = B^T, D_D = D^T)$ is the dual of $(A, B, C, D)$. Reachability is dual to observability. If a system is reachable (observable), its dual is observable (reachable).

## 5.6 Notes

The concept of controllability was first encountered as a technical condition in certain optimal control problems and also in the so-called finite-settling-time design problem for discrete-time systems (see Kalman [4]). In the latter, an input must be determined that returns the state $x_0$ to the origin as quickly as possible. Manipulating the input to assign particular values to the initial state in (analog-computer) simulations was not an issue since the individual capacitors could initially be charged independently. Also, observability was not an issue in simulations due to the particular system structures that were used (corresponding, e.g., to observer forms). The current definitions for controllability and observability and the recognition of the duality between them were worked out by Kalman in 1959–1960 (see Kalman [7] for historical comments) and were presented by Kalman in [5]. The significance of realizations that were both controllable and observable (see Chapter 5) was established later in Gilbert [2], Kalman [6], and Popov [8]. For further information regarding these historical issues, consult Kailath [3] and the original sources. Note that [3] has extensive references up to the late seventies with emphasis on the time-invariant case and a rather complete set of original references together with historical remarks for the period when the foundations of the state-space system theory were set, in the late fifties and sixties.

## References

1. P.J. Antsaklis and A.N. Michel, *Linear Systems*, Birkhäuser, Boston, MA, 2006.
2. E. Gilbert, "Controllability and observability in multivariable control systems," *SIAM J. Control*, Vol. 1, pp. 128–151, 1963.
3. T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980.
4. R.E. Kalman, "Optimal nonlinear control of saturating systems by intermittent control," IRE WESCON Rec., Sec. IV, pp. 130–135, 1957.
5. R.E. Kalman, "On the general theory of control systems," in *Proc. of the First Intern. Congress on Automatic Control*, pp. 481–493, Butterworth, London, 1960.
6. R.E. Kalman, "Mathematical descriptions of linear systems," *SIAM J. Control*, Vol. 1, pp. 152–192, 1963.
7. R.E. Kalman, *Lectures on Controllability and Observability*, C.I.M.E., Bologna, 1968.
8. V.M. Popov, "On a new problem of stability for control systems," *Autom. Remote Control*, Vol. 24, No. 1, pp. 1–23, 1963.

## Exercises

**5.1.** (a) Let $\mathcal{C}_k \triangleq [B, AB, \ldots, A^{k-1}B]$, where $A \in R^{n\times n}, B \in R^{n\times m}$. Show that

$$\mathcal{R}\,(\mathcal{C}_k) = \mathcal{R}\,(\mathcal{C}_n) \text{ for } k \geq n, \quad \text{and} \quad \mathcal{R}(\mathcal{C}_k) \subset \mathcal{R}(\mathcal{C}_n) \text{ for } k < n.$$

(b) Let $\mathcal{O}_k \triangleq [C^T, (CA)^T, \ldots, (CA^{k-1})^T]^T$, where $A \in R^{n\times n}, C \in R^{p\times n}$. Show that

$$\mathcal{N}(\mathcal{O}_k) = \mathcal{N}(\mathcal{O}_n) \text{ for } k \geq n, \text{ and } \mathcal{N}(\mathcal{O}_k) \supset \mathcal{N}(\mathcal{O}_n) \text{ for } k < n.$$

**5.2.** Consider the state equation $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which was obtained by linearizing the nonlinear equations of motion of an orbiting satellite about a steady-state solution. In the state $x = [x_1, x_2, x_3, x_4]^T$, $x_1$ is the differential radius, whereas $x_3$ is the differential angle. In the input vector $u = [u_1, u_2]^T$, $u_1$ is the radial thrust and $u_2$ is the tangential thrust.

(a) Is this system controllable from $u$? If $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$, is the system observable from $y$?
(b) Can the system be controlled if the radial thruster fails? What if the tangential thruster fails?
(c) Is the system observable from $y_1$ only? From $y_2$ only?

**5.3.** Consider the state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} u.$

(a) If $x(0) = \begin{bmatrix} a \\ b \end{bmatrix}$, derive an input that will drive the state to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $T$ sec.
(b) For $x(0) = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$, plot $u(t), x_1(t), x_2(t)$ for $T = 1, 2,$ and 5 sec. Comment on the magnitude of the input in your results.

**5.4.** Consider the state equation $x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(k), y(k) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k).$

(a) Is $x^1 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ reachable? If yes, what is the minimum number of steps required to transfer the state from the zero state to $x^1$? What inputs do you need?

(b) Determine all states that are reachable.

(c) Determine all states that are unobservable.

(d) If $\dot{x} = Ax + Bu$ is given with $A, B$ as in (a), what is the minimum time required to transfer the state from the zero state to $x^1$? What is an appropriate $u(t)$?

**5.5.** *Output reachability (controllability)* can be defined in a manner analogous to state reachability (controllability). In particular, a system will be called output reachable if there exists an input that transfers the output from some $y_0$ to any $y_1$ in finite time.

Consider now a discrete-time time-invariant system $x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k) + Du(k)$ with $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$ and $D \in R^{p\times m}$. Recall that

$$y(k) = CA^k x(0) + \sum_{i=0}^{k-1} CA^{k-(i+1)} Bu(i) + Du(k).$$

(a) Show that the system $\{A, B, C, D\}$ is output reachable if and only if

$$\text{rank}[D, CB, CAB, \ldots, CA^{n-1}B] = p.$$

Note that this rank condition is also the condition for output reachability for continuous-time time-invariant systems $\dot{x} = Ax + Bu, y = Cx + Du$. It should be noted that, in general, state reachability is neither necessary nor sufficient for output reachability. Notice for example that if $\text{rank}\, D = p$, then the system is output reachable.

(b) Let $D = 0$. Show that if $(A, B)$ is (state) reachable, then $\{A, B, C, D\}$ is output reachable if and only if $\text{rank}\, C = p$.

(c) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $C = [1,\ 1,\ 0]$, and $D = 0$.

(i) Is the system output reachable? Is it state reachable?

(ii) Let $x(0) = 0$. Determine an appropriate input sequence to transfer the output to $y_1 = 3$ in minimum time. Repeat for $x(0) = [1, -1, 2]^T$.

**5.6.** (a) Given $\dot{x} = Ax + Bu, y = Cx + Du$, show that this system is output reachable if and only if the rows of the $p \times m$ transfer matrix $H(s)$ are linearly independent over the field of complex numbers. In view of this result, is the system $H(s) = \begin{bmatrix} \frac{1}{s+2} \\ \frac{s}{s+1} \end{bmatrix}$ output reachable?

(b) Similarly, for discrete-time systems, the system is output reachable if and only if the rows of the transfer function matrix $H(z)$ are linearly independent over the field of complex numbers. Consider now the system of Exercise 5.5 and determine whether it is output reachable.

**5.7.** Show that the circuit depicted in Figure 5.4 with input $u$ and output $y$ is neither state reachable nor observable but is output reachable.

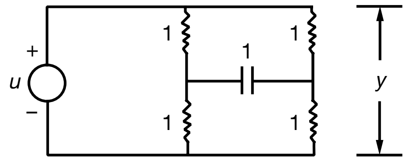


**Figure 5.4.** Circuit for Exercise 5.7

**5.8.** A system $\dot{x} = Ax + Bu, y = Cx + Du$ is called *output function controllable* if there exists an input $u(t)$, $t \in [0, \infty)$, that will cause the output $y(t)$ to follow a prescribed trajectory for $0 \leq t < \infty$, assuming that the system is at rest at $t = 0$. It is easiest to derive a test for output function controllability in terms of the $p \times m$ transfer function matrix $H(s)$, and this is the approach taken in the following. We say that the $m \times p$ rational matrix $H_R(s)$ is a *right inverse of* $H(s)$ if

$$H(s)H_R(s) = I_p.$$

(a) Show that the right inverse $H_R(s)$ exists if and only if $\operatorname{rank} H(s) = p$. *Hint:* In the sufficiency proof, select $H_R = H^T(HH^T)^{-1}$, the (right) pseudoinverse of $H$.

(b) Show that the system is output function controllable if and only if $H(s)$ has a right inverse $H_R(s)$. *Hint:* Consider $\hat{y} = H\hat{u}$. In the necessity proof, show that if $\operatorname{rank} H < p$, then the system may not be output function controllable.

*Input function observability* is the dual to output function controllablity. Here, the *left inverse of* $H(s)$, $H_L(s)$, is of interest and is defined by

$$H_L(s)H(s) = I_m.$$

(c) Show that the left inverse $H_L(s)$ of $H(s)$ exists if and only if $\operatorname{rank} H(s) = m$. *Hint:* This is the dual result to part (a).

(d) Let $H(s) = \left[\frac{s+1}{s}, \frac{1}{s}\right]$ and characterize all inputs $u(t)$ that will cause the system (at rest at $t = 0$) to exactly follow a step, $\hat{y}(s) = 1/s$.

Part (d) points to a variety of questions that may arise when inverses are considered, including: Is $H_R(s)$ proper? Is it unique? Is it stable? What is the minimum degree possible?

**5.9.** Consider the system $\dot{x} = Ax + Bu, y = Cx$. Show that output function controllability implies output controllability (-from-the-origin, or reachability).

**5.10.** Given $x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$, $y(k) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k)$, and assume zero initial conditions.

(a) Is there a sequence of inputs $\{u(0), u(1), \dots\}$ that transfers the output from $y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in finite time? If the answer is yes, determine such a sequence.

(b) Characterize all outputs that can be reached from the zero output ($y(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$), in one step.

**5.11.** Suppose that for system $x(k+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k)$, $y(k) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k)$, it is known that $y(0) = y(1) = y(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Based on this information, what can be said about the initial condition $x(0)$?

**5.12.** (a) Consider the system $\dot{x} = Ax + Bu, y = Cx + Du$, where $(A, C)$ is assumed to be observable. Express $x(t)$ as a function of $y(t), u(t)$ and their derivatives. *Hint:* Write $y(t), y^{(1)}(t), \dots, y^{(n-1)}(t)$ in terms of $x(t)$ and $u(t), u^{(1)}(t), \dots, u^{(n-1)}(t)$ ( $x(t) \in R^n$ ).

(b) Given the system $\dot{x} = Ax + Bu, y = Cx + Du$ with $(A, C)$ observable. Determine $x(0)$ in terms of $y(t), u(t)$ and their derivatives up to order $n - 1$. Note that in general this is not a practical way of determining $x(0)$, since this method requires differentiation of signals, which is very susceptible to measurement noise.

(c) Consider the system $x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k) + Du(k)$, where $(A, C)$ is observable. Express $x(k)$ as a function of $y(k), y(k+1), \dots, y(k+n-1)$ and $u(k), u(k+1), \dots, y(k+n-1)$. *Hint:* Express $y(k), \dots, y(k+n-1)$ in terms of $x(k)$ and $u(k), u(k+1), \dots, u(k+n-1)$ [ $x(k) \in R^n$ ]. Note the relation to expression (5.88) in Section 5.4.