9

State Feedback and State Observers

9.1 Introduction

Feedback is a fundamental mechanism arising in nature and is present in many natural processes. Feedback is also common in manufactured systems and is essential in automatic control of dynamic processes with uncertainties in their model descriptions and their interactions with the environment. When feedback is used, the actual values of system variables are sensed, fed back, and used to control the system. Hence, a control law decision process is based not only on predictions about the system behavior derived from a process model, but also on information about the actual behavior. A common example of an automatic feedback control system is the cruise control system in an automobile, which maintains the speed of the automobile at a certain desired value within acceptable tolerances. In this chapter, feedback is introduced and the problem of pole or eigenvalue assignment by means of state feedback is discussed at length in Section 9.2. It is possible to arbitrarily assign all closed-loop eigenvalues by linear static state feedback if and only if the system is completely controllable.

In the study of state feedback, it is assumed that it is possible to measure the values of the states using appropriate sensors. Frequently, however, it may be either impossible or impractical to obtain measurements for all states. It is therefore desirable to be able to estimate the states from measurements of input and output variables that are typically available. In addition to feedback control problems, there are many other problems where knowledge of the state vector is desirable, since such knowledge contains useful information about the system. This is the case, for example, in navigation systems. State estimation is related to observability in an analogous way that state feedback control is related to controllability. The duality between controllability and observability makes it possible to easily solve the estimation problem once the control problem has been solved, and vice versa. In this chapter, asymptotic state estimators, also called state observers, are discussed at length in Section 9.3.

Finally, state feedback static controllers and state dynamic observers are combined to form dynamic output feedback controllers. Such controllers are studied in Section 9.4, using both state-space and transfer function matrix descriptions. In the following discussion, state feedback and state estimation are introduced for continuous- and discrete-time time-invariant systems.

9.2 Linear State Feedback

9.2.1 Continuous-Time Systems

We consider linear, time-invariant, continuous-time systems described by equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{9.1}$$

where $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{p\times n}$, and $D \in R^{p\times m}$.

Definition 9.1. *The* linear, time-invariant, state feedback control law *is defined by*

$$u = Fx + r, \tag{9.2}$$

where $F \in R^{m\times n}$ is a gain matrix and $r(t) \in R^m$ is an external input vector. ■

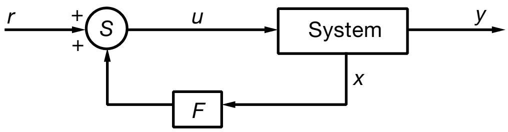

Figure 9.1. Linear state feedback configuration

Note that $r(t)$ is an *external input*, also called a *command or reference input* (see Figure 9.1). It is used to provide an input to the compensated closed-loop system and is omitted when such input is not necessary in a given discussion $[r(t) = 0]$. This is the case, e.g., when the Lyapunov stability of a system is studied. Note that the vector $r(t)$ in (9.2) has the same dimension as $u(t)$. If a different number of inputs is desired, then an input transformation map may be used to accomplish this.

The *compensated closed-loop system* of Figure 9.1 is described by the equations

$$\begin{aligned} \dot{x} &= (A + BF)x + Br, \\ y &= (C + DF)x + Dr, \end{aligned} \tag{9.3}$$

which were determined by substituting $u = Fx + r$ into the description of the *uncompensated open-loop system* (9.1).

The *state feedback gain matrix* F affects the closed-loop system behavior. This is accomplished by altering the matrices A and C of (9.1). In fact, the main influence of F is exercised through the matrix A, resulting in the matrix $A + BF$ of the closed-loop system. The matrix F affects the eigenvalues of $A + BF$ and, therefore, the modes of the closed-loop system. The effects of F can also be thought of as restricting the choices for u ($= Fx$ for $r = 0$) so that for appropriate F, certain properties, such as asymptotic Lyapunov stability, of the equilibrium $x = 0$ are obtained.

Open- Versus Closed-Loop Control

The linear state feedback control law (9.2) can be expressed in terms of the initial state $x(0) = x_0$. In particular, working with Laplace transforms, we obtain $\hat{u} = F\hat{x} + \hat{r} = F[(sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}] + \hat{r}$, in view of $s\hat{x} - x_0 = A\hat{x} + B\hat{u}$, derived from $\dot{x} = Ax + Bu$. Collecting terms, we have $[I - F(sI - A)^{-1}B]\hat{u} = F(sI - A)^{-1}x_0 + \hat{r}$. This yields

$$\hat{u} = F[sI - (A + BF)]^{-1}x_0 + [I - F(sI - A)^{-1}B]^{-1}\hat{r}, \tag{9.4}$$

where the matrix identities $[I - F(sI - A)^{-1}B]^{-1}F(sI - A)^{-1} \equiv F(sI - A)^{-1}[I - BF(sI - A)^{-1}]^{-1} \equiv F[sI - (A + BF)]^{-1}$ have been used.

Expression (9.4) is an *open-loop (feedforward) control law*, expressed in the Laplace transform domain. It is phrased in terms of the initial conditions $x(0) = x_0$, and if it is applied to the open-loop system (9.1), it generates exactly the same control action $u(t)$ for $t \geq 0$ as the state feedback $u = Fx + r$ in (9.2). It can readily be verified that the descriptions of the compensated system are exactly the same when either control expressions, (9.2) or (9.4), are used. In practice, however, these two control laws hardly behave the same, as explained in the following.

First, notice that in the open-loop scheme (9.4), the initial conditions x_0 are assumed to be known exactly. It is also assumed that the plant parameters in A and B are known exactly. If there are *uncertainties* in the data, this control law may fail miserably, even when the differences are small, since it is based on incorrect information without any way of knowing that these data are not valid. In contrast to the above, the feedback law (9.2) does not require knowledge of x_0. Moreover, it receives feedback information from $x(t)$ and adjusts $u(t)$ to reflect the current system parameters, and consequently, it is more robust to parameter variations. Of course the feedback control law (9.2) will also fail when the parameter variations are too large. In fact, the area of *robust control* relates feedback control law designs to bounds on the uncertainties (due to possible changes) and aims to derive the best design possible under the circumstances.

The point we wish to emphasize here is that although open- and closed-loop control laws may appear to produce identical effects, typically they do not, the reason being that the mathematical system models used are not sufficiently accurate, by necessity or design. Feedback control and closed-loop control are preferred to accommodate ever-present modeling uncertainties in the plant and the environment.

At this point, a few observations are in order. First, we note that feeding back the state in synthesizing a control law is a very powerful mechanism, since the state contains all the information about the history of a system that is needed to uniquely determine the future system behavior, given the input. We observe that the state feedback control law considered presently is linear, resulting in a closed-loop system that is also linear. Nonlinear state feedback control laws are of course also possible. Notice that when a time-invariant system is considered, the state feedback is typically static, unless there is no choice (as in certain optimal control problems), resulting in a closed-loop system that is also time-invariant. These comments justify to a certain extent the choice of linear, time-invariant, state feedback control to compensate linear time-invariant systems.

The problem of stabilizing a system by using state feedback is considered next.

Stabilization

The problem we wish to consider now is to determine a state feedback control law (9.2) having the property that the resulting compensated closed-loop system has an equilibrium $x = 0$ that is asymptotically stable (in the sense of Lyapunov) when $r = 0$. (For a discussion of asymptotic stability, refer to Subsection 3.3.3 and to Chapter 4.) In particular, we wish to determine a matrix $F \in R^{m \times n}$ so that the system

$$\dot{x} = (A + BF)x, \tag{9.5}$$

where $A \in R^{n \times n}$ and $B \in R^{n \times m}$ has equilibrium $x = 0$ that is asymptotically stable. Note that (9.5) was obtained from (9.3) by letting $r = 0$.

One method of deriving such stabilizing F is by formulating the problem as an optimal control problem, e.g., as the Linear Quadratic Regulator (LQR) problem. This is discussed at the end of this section.

Alternatively, in view of Subsection 3.3.3, the equilibrium $x = 0$ of (9.5) is asymptotically stable if and only if the eigenvalues λ_i of $A + BF$ satisfy $Re\lambda_i < 0$, $i = 1, \cdots, n$. Therefore, the stabilization problem for the time-invariant case reduces to the problem of selecting F in such a manner that the eigenvalues of $A + BF$ are shifted into desired locations. This will be studied in the following subsection. Note that stabilization is only one of the control objectives, although a most important one, that can be achieved by shifting eigenvalues. Control system design via eigenvalue (pole) assignment is a topic that is addressed in detail in a number of control books.

9.2.2 Eigenvalue Assignment

Consider again the closed-loop system $\dot{x} = (A+BF)x$ given in (9.5). We shall show that if (A, B) is fully controllable (-from-the-origin, or reachable), all eigenvalues of $A + BF$ can be arbitrarily assigned by appropriately selecting F. In other words, "the eigenvalues of the original system can arbitrarily be changed in this case." This last statement, commonly used in the literature, is rather confusing: The eigenvalues of a given system $\dot{x} = Ax + Bu$ are *not* physically changed by the use of feedback. They are the same as they used to be before the introduction of feedback. Instead, the feedback law $u = Fx + r, r = 0$, generates an input $u(t)$ that, when fed back to the system, makes it behave *as if* the eigenvalues of the system were at different locations [i.e., the input $u(t)$ makes it behave as a different system, the behavior of which is, we hope, more desirable than the behavior of the original system].

Theorem 9.2. *Given $A \in R^{n\times n}$ and $B \in R^{n\times m}$, there exists $F \in R^{m\times n}$ such that the n eigenvalues of $A+BF$ can be assigned to arbitrary, real, or complex conjugate locations if and only if (A, B) is controllable (-from-the-origin, or reachable).*

Proof. (*Necessity*): Suppose that the eigenvalues of $A+BF$ have been arbitrarily assigned, and assume that (A, B) in (9.1) is not fully controllable. We shall show that this leads to a contradiction. Since (A, B) is not fully controllable, in view of the results in Section 6.2, there exists a similarity transformation that will separate the controllable part from the uncontrollable part in (9.5). In particular, there exists a nonsingular matrix Q such that

$$Q^{-1}(A+BF)Q = Q^{-1}AQ + (Q^{-1}B)(FQ) = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} [F_1, F_2]$$
$$= \begin{bmatrix} A_1 + B_1F_1 & A_{12} + B_1F_2 \\ 0 & A_2 \end{bmatrix}, \tag{9.6}$$

where $[F_1, F_2] \triangleq FQ$ and (A_1, B_1) is controllable. The eigenvalues of $A + BF$ are the same as the eigenvalues of $Q^{-1}(A+BF)Q$, which implies that $A+BF$ has certain fixed eigenvalues, the eigenvalues of A_2, that cannot be shifted via F. These are the uncontrollable eigenvalues of the system. Therefore, the eigenvalues of $A + BF$ have not been arbitrarily assigned, which is a contradiction. Thus, (A, B) is fully controllable.

(*Sufficiency*): Let (A, B) be fully controllable. Then by using any of the eigenvalue assignment algorithms presented later in this section, all the eigenvalues of $A + BF$ can be arbitrarily assigned. ∎

Lemma 9.3. *The uncontrollable eigenvalues of (A, B) cannot be shifted via state feedback.*

Proof. See the necessity part of the proof of Theorem 9.2. Note that the uncontrollable eigenvalues are the eigenvalues of A_2. ∎

Example 9.4. Consider the uncontrollable pair (A, B), where $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This pair can be transformed to a standard form for uncontrollable systems, namely, $\hat{A} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$, $\hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, from which it can easily be seen that -1 is the uncontrollable eigenvalue, whereas -2 is the controllable eigenvalue.

Now if $F = [f_1, f_2]$, then $\det(sI-(A+BF)) = \det \begin{bmatrix} s-f_1, & 2-f_2 \\ -1-f_1 & s+3-f_2 \end{bmatrix} = s^2 + s(-f_1 - f_2 + 3) + (-f_1 - f_2 + 2) = (s+1)(s + (-f_1 - f_2 + 2))$. Clearly, the uncontrollable eigenvalue -1 cannot be shifted via state feedback. The controllable eigenvalue -2 can be shifted arbitrarily to $(f_1 + f_2 - 2)$ by $F = [f_1, f_2]$.

It is now quite clear that a given system (9.1) can be made asymptotically stable via the state feedback control law (9.2) only when all the uncontrollable eigenvalues of (A, B) are already in the open left part of the s-plane. This is so because state feedback can alter only the controllable eigenvalues.

Definition 9.5. *The pair (A, B) is called* stabilizable *if all its uncontrollable eigenvalues are stable.* ∎

Before presenting methods to select F for eigenvalue assignment, it is of interest to examine how the linear feedback control law $u = Fx + r$ given in (9.2) affects controllability and observability. We write

$$\begin{bmatrix} sI-(A+BF) & B \\ -(C+DF) & D \end{bmatrix} = \begin{bmatrix} sI-A & B \\ -C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -F & I \end{bmatrix} \tag{9.7}$$

and note that

$$\operatorname{rank}[\lambda I - (A+BF), B] = \operatorname{rank}[\lambda I - A, B]$$

for all complex λ. Thus, if (A, B) is controllable, then so is $(A+BF, B)$ for any F. Furthermore, notice that in view of

$$\begin{aligned} \mathcal{C}_F &= [B, (A+BF)B, (A+BF)^2B, \ldots, (A+BF)^{n-1}B] \\ &= [B, AB, A^2B, \ldots, A^{n-1}B] \begin{bmatrix} I & FB & F(A+BF)B & \cdot \\ 0 & I & FB & \cdot \\ & & I & \cdot \\ & & & \ddots \\ & & & & I \end{bmatrix}, \end{aligned} \tag{9.8}$$

$\mathcal{R}(\mathcal{C}_F) = \mathcal{R}([B, AB, \ldots, A^{n-1}B]) = \mathcal{R}\,(\mathcal{C})$. This shows that F does not alter the controllability subspace of the system. This in turn proves the following lemma.

Lemma 9.6. *The controllability subspaces of* $\dot{x} = Ax + Bu$ *and* $\dot{x} = (A + BF)x + Br$ *are the same for any* F. ■

Although the controllabiliity of the system is not altered by linear state feedback $u = Fx + r$, this is not true for the observability property. Note that the observability of the closed-loop system (9.3) depends on the matrices $(A + BF)$ and $(C + DF)$, and it is possible to select F to make certain eigenvalues unobservable from the output. In fact this mechanism is quite common and is used in several control design methods. It is also possible to make observable certain eigenvalues of the open-loop system that were unobservable.

Several methods are now presented to select F to arbitrarily assign the closed-loop eigenvalues.

Methods for Eigenvalue Assignment by State Feedback

In view of Theorem 9.2, the *eigenvalue assignment problem* can now be stated as follows. Given a controllable pair (A, B), determine F to assign the n eigenvalues of $A + BF$ to arbitrary real and/or complex conjugate locations. This problem is also known as the *pole assignment problem*, where by the term "pole" is meant a "pole of the system" (or an eigenvalue of the "A" matrix). This is to be distinguished from the "poles of the transfer function."

Note that all matrices A, B, and F are real, so the coefficients of the polynomial $\det[sI - (A + BF)]$ are also real. This imposes the restriction that the complex roots of this polynomial must appear in conjugate pairs. Also, note that if (A, B) is not fully controllable, then (9.6) can be used together with the methods described a little later, to assign all the controllable eigenvalues; the uncontrollable ones will remain fixed.

It is assumed in the following discussion that B has full column rank; i.e.,

$$\operatorname{rank} B = m. \tag{9.9}$$

This means that the system $\dot{x} = Ax + Bu$ has m independent inputs. If $\operatorname{rank} B = r < m$, this would imply that one could achieve the same result by manipulating only r inputs (instead of $m > r$). To assign eigenvalues in this case, one can proceed by writing

$$A + BF = A + (BM)(M^{-1}F) = A + [B_1, 0] \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = A + B_1F_1, \tag{9.10}$$

where M is chosen so that $BM = [B_1, 0]$ with $B_1 \in R^{n\times r}$ and $\operatorname{rank} B_1 = r$. Then $F_1 \in R^{r\times n}$ can be determined to assign the eigenvalues of $A + B_1F_1$, using any one of the methods presented next. Note that (A, B) is controllable implies that (A, B_1) is controllable. The state feedback matrix F is given in this case by

$$F = M \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{9.11}$$

where $F_2 \in R^{(m-r)\times n}$ is arbitrary.

1. Direct Method

Let $F = [f_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and express the coefficients of the characteristic polynomial of $A + BF$ in terms of f_{ij}; i.e.,

$$\det(sI - (A + BF)) = s^n + g_{n-1}(f_{ij})s^{n-1} + \cdots + g_0(f_{ij}).$$

Now if the roots of the polynomial

$$\alpha_d(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0$$

are the n desired eigenvalues, then the f_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$, must be determined so that

$$g_k(f_{ij}) = d_k, \quad k = 0, 1, \ldots, n-1. \tag{9.12}$$

In general, (9.12) constitutes a nonlinear system of algebraic equations; however, it is linear in the single-input case, $m = 1$. The main difficulty in this method is not so much in deriving a numerical solution for the nonlinear system of equation, but in carrying out the symbolic manipulations needed to determine the coefficients g_k in terms of the f_{ij} in (9.12). This difficulty usually restricts this method to the simplest cases, with $n = 2$ or 3 and $m = 1$ or 2 being typical.

Example 9.7. For $A = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we have $\det(sI - A) = s(s - 5/2)$, and therefore, the eigenvalues of A are 0 and $5/2$. We wish to determine F so that the eigenvalues of $A + BF$ are at $-1 \pm j$.

If $F = [f_1, f_2]$, then $\det(sI - (A+BF)) = \det\left(\begin{bmatrix} s-1/2 & -1 \\ -1 & s-2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}[f_1, f_2]\right) = \det \begin{bmatrix} s - 1/2 - f_1, & -1 - f_2 \\ -1 - f_1, & s - 2 - f_2 \end{bmatrix} = s^2 + s(-\frac{5}{2} - f_1 - f_2) + f_1 - \frac{1}{2}f_2$. The desired eigenvalues are the roots of the polynomial

$$\alpha_d(s) = (s - (-1 + j))(s - (-1 - j)) = s^2 + 2s + 2.$$

Equating coefficients, one obtains $-\frac{5}{2} - f_1 - f_2 = 2$, $f_1 - \frac{1}{2}f_2 = 2$, a linear system of equations. Note that it is linear because $m = 1$. In general one must solve a set of nonlinear algebraic equations. We have

$$F = [f_1, f_2] = [-1/6, -13/3]$$

as the appropriate state feedback matrix.

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2. The Use of Controller Forms

Given that the pair (A, B) is controllable, there exists an equivalence transformation matrix P so that the pair $(A_c = PAP^{-1}, B_c = PB)$ is in controller form (see Section 6.4). The matrices $A + BF$ and $P(A + BF)P^{-1} = PAP^{-1} + PBFP^{-1} = A_c + B_cF_c$ have the same eigenvalues, and the problem is to determine F_c so that $A_c + B_cF_c$ has desired eigenvalues. This problem is easier to solve than the original one because of the special structures of A_c and B_c. Once F_c has been determined, then the original feedback matrix F is given by

$$F = F_cP. \tag{9.13}$$

We shall now assume that (A, B) has already been reduced to (A_c, B_c) and describe methods of deriving F_c for eigenvalue assignment.

Single-Input Case $(m = 1)$. We let

$$F_c = [f_0, \ldots, f_{n-1}]. \tag{9.14}$$

In view of Section 6.4, since A_c, B_c are in controller form, we have

$$\begin{aligned} A_{cF} &\triangleq A_c + B_cF_c \\ &= \begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [f_0, \ldots, f_{n-1}] \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -(\alpha_0 - f_0) & -(\alpha_1 - f_1) & \cdots & -(\alpha_{n-1} - f_{n-1}) \end{bmatrix}, \end{aligned} \tag{9.15}$$

where α_i, $i = 0, \ldots, n-1$, are the coefficients of the characteristic polynomial of A_c; i.e.,

$$\det(sI - A_c) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0. \tag{9.16}$$

Notice that A_{cF} is also in companion form, and its characteristic polynomial can be written directly as

$$\det(sI - A_{cF}) = s^n + (\alpha_{n-1} - f_{n-1})s^{n-1} + \cdots + (\alpha_0 - f_0). \tag{9.17}$$

If the desired eigenvalues are the roots of the polynomial

$$\alpha_d(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_0, \tag{9.18}$$

then by equating coefficients, f_i, $i = 0, 1, \ldots, n-1$, must satisfy the relations $d_i = \alpha_i - f_i$, $i = 0, 1, \ldots, n-1$, from which we obtain

$$f_i = \alpha_i - d_i, \quad i = 0, \dots, n-1. \tag{9.19}$$

Alternatively, note that there exists a matrix A_d in companion form, the characteristic polynomial of which is (9.18). An alternative way of deriving (9.19) is then to set $A_{cF} = A_c + B_c F_c = A_d$, from which we obtain

$$F_c = B_m^{-1}[A_{dm} - A_m], \tag{9.20}$$

where $B_m = 1, A_{d_m} = [-d_0, \dots, -d_{n-1}]$ and $A_m = [-\alpha_0, \dots, -\alpha_{n-1}]$. Therefore, B_m, A_{d_m}, and A_m are the nth rows of B_c, A_d, and A_c, respectively (see Section 6.4). Relationship (9.20), which is an alternative formula to (9.19), has the advantage that it is in a form that can be generalized to the multi-input case studied below.

Example 9.8. Consider the matrices $A = \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ of Example 9.7. Determine F so that the eigenvalues of $A + BF$ are $-1 \pm j$, i.e., so that they are the roots of the polynomial $\alpha_d(s) = s^2 + 2s + 2$.

To reduce (A, B) into the controller form, let

$$\mathcal{C} = [B, AB] = \begin{bmatrix} 1 & 3/2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathcal{C}^{-1} = \frac{2}{3} \begin{bmatrix} 3 & -3/2 \\ -1 & 1 \end{bmatrix},$$

from which $P = \begin{bmatrix} q \\ qA \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 & 2 \\ 1 & 2 \end{bmatrix}$ [see (6.38) in Section 6.4]. Then $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1/2 & 1 \end{bmatrix}$ and

$$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 5/2 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, $A_m = [0,\ 5/2]$ and $B_m = 1$. Now $A_d = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$ and $A_{d_m} = [-2,\ -2]$ since the characteristic polynomial of A_d is $s^2 + 2s + 2 = \alpha_d(s)$. Applying (9.20), we obtain that

$$F_c = B_m^{-1}[A_{d_m} - A_m] = [-2, -9/2]$$

and $F = F_c P = [-2, -9/2] \begin{bmatrix} -2/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix} = [-1/6, -13/3]$ assigns the eigenvalues of the closed-loop system at $-1 \pm j$. This is the same result as the one obtained by the direct method given in Example 9.7. If $\alpha_d(s) = s^2 + d_1 s + d_0$, then $A_{d_m} = [-d_0, -d_1]$, $F_c = B_m^{-1}[A_{d_m} - A_m] = [-d_0, -d_1 - 5/2]$, and

$$F = F_c P = \frac{1}{3}[2d_0 - d_1 - \frac{5}{2},\ -2d_0 - 2d_1 - 5].$$

In general the larger the difference between the coefficients of $\alpha_d(s)$ and $\alpha(s)$, $(A_{d_m} - A_m)$, the larger the gains in F. This is as expected, since larger changes require in general larger control action.

Note that (9.20) can also be derived using (6.55) of Section 6.4. To see this, write

$$A_{cF} = A_c + B_c F_c = (\bar{A}_c + \bar{B}_c A_m) + (\bar{B}_c B_m) F_c = \bar{A}_c + \bar{B}_c (A_m + B_m F_c).$$

Selecting $A_d = \bar{A}_c + \bar{B}_c A_{d_m}$ and requiring $A_{cF} = A_d$ implies

$$\bar{B}_c [A_m + B_m F_c] = \bar{B}_c A_{d_m},$$

from which $A_m + B_m F_c = A_{d_m}$, which in turn implies (9.20).

After F_c has been found, to determine F so that $A + BF$ has desired eigenvalues, one should use $F = F_c P$ given in (9.13). Note that P, which reduces (A, B) to the controller form, has a specific form in this $(m = 1)$ case [see (6.38) of Section 6.4]. Combining these results, it is possible to derive a formula for the eigenvalue assigning F in terms of the original pair (A, B) and the coefficients of the desired polynomial $\alpha_d(s)$. In particular, the $1 \times n$ matrix F that assigns the n eigenvalues of $A + BF$ at the roots of $\alpha_d(s)$ is unique and is given by

$$F = -e_n^T \mathcal{C}^{-1} \alpha_d(A), \tag{9.21}$$

where $e_n = [0, \ldots, 0, 1]^T \in R^n$ and $\mathcal{C} = [B, AB, \ldots, A^{n-1}B]$ is the controllability matrix. Relation (9.21) is known as *Ackermann's formula*; for details, see [1, p. 334].

Example 9.9. To the system of Example 9.8, we apply (9.21) and obtain

$$\begin{aligned}
F &= -e_2^T \mathcal{C}^{-1} \alpha_d(A) \\
&= -[0, 1] \begin{bmatrix} 2 & -1 \\ -2/3 & 2/3 \end{bmatrix} \left(\begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix}^2 + 2 \begin{bmatrix} 1/2 & 1 \\ 1 & 2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= -[-2/3, 2/3] \begin{bmatrix} 17/4 & 9/2 \\ 9/2 & 11 \end{bmatrix} = [-1/6, -13/3],
\end{aligned}$$

which is identical to the F found in Example 9.8.

Multi-Input Case $(m > 1)$. We proceed in a way completely analogous to the single-input case. Assume that A_c and B_c are in the controller form, (6.54). Notice that $A_{cF} \triangleq A_c + B_c F_c$ is also in (controller) companion form with an identical block structure as A_c for any F_c. In fact, the pair (A_{cF}, B_c) has the same controllability indices $\mu_i, i = 1, \ldots, m$, as (A_c, B_c). This can be seen directly, since

$$A_c + B_c F_c = (\bar{A}_c + \bar{B}_c A_m) + (\bar{B}_c B_m) F_c = \bar{A}_c + \bar{B}_c (A_m + B_m F_c), \tag{9.22}$$

where $\bar{A}_c$ and $\bar{B}_c$ are defined in (6.55). We can now select an $n \times n$ matrix A_d with desired characteristic polynomial

$$\det(sI - A_d) = \alpha_d(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_0, \tag{9.23}$$

and in companion form, having the same block structure as A_{cF} or A_c; that is, $A_d = \bar{A}_c + \bar{B}_c A_{dm}$. Now if $A_{cF} = A_d$, then in view of (9.22), $\bar{B}_c(A_m + B_m F_c) = \bar{B}_c A_{dm}$. From this, it follows that

$$F_c = B_m^{-1}[A_{dm} - A_m], \tag{9.24}$$

where B_m, A_{dm}, and A_m are the m σ_jth rows of B_c, A_d, and A_c, respectively, and $\sigma_j = \sum_{i=1}^{j} \mu_i$, $j = 1, \ldots, m$. Note that this is a generalization of (9.20) of the single-input case.

We shall now show how to select an $n \times n$ matrix A_d in multivariable companion form to have the desired characteristic polynomial.

One choice is

$$A_d = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & \cdots & -d_{n-1} \end{bmatrix},$$

the characteristic polynomial of which is $\alpha_d(s)$. In this case the $m \times n$ matrix A_{dm} is given by

$$A_{dm} = \begin{bmatrix} 0 & \cdots & 0 & 1 \cdots 0 \cdots & 0 \\ \vdots & & \vdots & \vdots \;\; \vdots & \vdots \\ 0 & \cdots & 0 & 0 \cdots 1 \cdots & 0 \\ -d_0 & & \cdots & & -d_{n-1} \end{bmatrix},$$

where the ith row, $i = 1, \ldots, m-1$, is zero everywhere except at the $\sigma_i + 1$ column location, where it is one.

Another choice is to select $A_d = [A_{ij}]$, $i, j = 1, \ldots, m$, with $A_{ij} = 0$ for $i \neq j$, i.e.,

$$A_d = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix},$$

noting that $\det(sI - A_d) = \det(sI - A_{11}) \ldots \det(sI - A_{mm})$. Then

$$A_{ii} = \begin{bmatrix} 0 & 1 \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & 1 \\ \times & \cdots & \times \end{bmatrix},$$

where the last row is selected so that $\det(sI - A_{ii})$ has desired roots. The disadvantage of this selection is that it may impose unnecessary restrictions on the number of real eigenvalues assigned. For example, if $n = 4, m = 2$ and the dimensions of A_{11} and A_{22}, which are equal to the controllability indices, are $d_1 = 3$ and $d_2 = 1$, then two eigenvalues must be real.

There are of course other selections for A_d, and the reader is encouraged to come up with additional choices. A point that should be quite clear by now is that F_c (or F) is not unique in the present case, since different F_c can be derived for different A_{dm}, all assigning the eigenvalues at the same desired locations. In the single-input case, F_c is unique, as was shown. Therefore, the following result has been established.

Lemma 9.10. *Let (A, B) be controllable, and suppose that n desired real complex conjugate eigenvalues for $A + BF$ have been selected. The state feedback matrix F that assigns all eigenvalues of $A + BF$ to desired locations is not unique in the multi-input case ($m > 1$). It is unique in the single-input case $m = 1$.* ■

Example 9.11. Consider the controllable pair (A, B), where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$. It was shown in Example 6.17, that this pair can be reduced to its controller form

$$A_c = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_c = PB = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$

where $P = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 1 & 0 & -1/2 \end{bmatrix}$. Suppose we desire to assign the eigenvalues of $A+BF$ to the locations $\{-2, -1 \pm j\}$, i.e., at the roots of the polynomial $\alpha_d(s) = (s + 2)(s^2 + 2s + 2) = s^3 + 4s^2 + 6s + 4$. A choice for A_d is

$$A_{d1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix}, \text{ leading to } A_{d_{m_1}} = \begin{bmatrix} 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix},$$

and

$$\begin{aligned} F_{c1} = B_m^{-1}[A_{d_{m1}} - A_m] &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right] \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ -5 & -6 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 5 \\ -5 & -6 & -4 \end{bmatrix}. \end{aligned}$$

Alternatively,

$$A_{d_2} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ from which } A_{d_{m2}} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and

$$\begin{aligned} F_{c_2} &= B_m^{-1}[A_{dm_2} - A_m] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & -1 & 0 \\ -1 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -1 & 2 \\ -1 & 0 & -2 \end{bmatrix}. \end{aligned}$$

Both $F_1 = F_{c1}P = \begin{bmatrix} 5 & 7 & -9/2 \\ -4 & -6 & 5/2 \end{bmatrix}$ and $F_2 = F_{c2}P = \begin{bmatrix} 2 & -1 & -2 \\ -2 & 0 & 1/2 \end{bmatrix}$ assign the eigenvalues of $A + BF$ to the locations $\{-2, -1 \pm j\}$.

The reader should plot the states of the equation $\dot{x} = (A + BF)x$ for $F = F_1$ and $F = F_2$ when $x(0) = [1, \ 1, \ 1]^T$ and should comment on the differences between the trajectories.

Relation (9.24) gives *all feedback matrices*, F_c (or $F = F_cP$), that assign the n eigenvalues of $A_c + B_cF_c$ (or $A + BF$) to desired locations. The freedom in selecting such F_c is expressed in terms of the different A_d, all in companion form, with $A_d = [A_{ij}]$ and A_{ij} of dimensions $\mu_i \times \mu_j$, which have the same characteristic polynomial. Deciding which one of all the possible matrices A_d to select, so that in addition to eigenvalue assignment other objectives can be achieved, is not apparent. This flexibility in selecting F can also be expressed in terms of other parameters, where both eigenvalue and eigenvector assignment are discussed, as will now be shown.

3. Assigning Eigenvalues and Eigenvectors

Suppose now that F was selected so that $A + BF$ has a desired eigenvalue s_j with corresponding eigenvector v_j. Then $[s_jI - (A + BF)]v_j = 0$, which can be written as

$$[s_jI - A, B] \begin{bmatrix} v_j \\ -Fv_j \end{bmatrix} = 0. \tag{9.25}$$

To determine an F that assigns s_j as a closed-loop eigenvalue, one could first determine a basis for the right kernel (null space) of $[s_jI - A, B]$, i.e., one could determine a basis $\begin{bmatrix} M_j \\ -D_j \end{bmatrix}$ such that

$$[s_jI - A, B] \begin{bmatrix} M_j \\ -D_j \end{bmatrix} = 0. \tag{9.26}$$

Note that the dimension of this basis is $(n + m) - \text{rank}[s_jI - A, B] = (n + m) - n = m$, where $\text{rank}[s_jI - A, B] = n$ since the pair (A, B) is controllable. Since it is a basis, there exists a nonzero $m \times 1$ vector a_j so that

$$\begin{bmatrix} M_j \\ -D_j \end{bmatrix} a_j = \begin{bmatrix} v_j \\ -Fv_j \end{bmatrix}. \tag{9.27}$$

Combining the relations $-D_j a_j = -Fv_j$ and $M_j a_j = v_j$, one obtains

$$FM_j a_j = D_j a_j. \tag{9.28}$$

This is the relation that F must satisfy for s_j to be a closed-loop eigenvalue. The nonzero $m \times 1$ vector a_j can be chosen arbitrarily. Note that $M_j a_j = v_j$ is the eigenvector corresponding to s_j. Note also that a_j represents the flexibility one has in selecting the corresponding eigenvector, in addition to assigning an eigenvalue. The $n \times 1$ eigenvector v_j cannot be arbitrarily assigned; rather, the $m \times 1$ vector a_j can be (almost) arbitrarily selected. These mild conditions on a_j are discussed below.

Theorem 9.12. *The pair (s_j, v_j) is an (eigenvalue, eigenvector)-pair of $A + BF$ if and only if F satisfies (9.28) for some nonzero vector a_j such that $v_j = M_j a_j$ with $\begin{bmatrix} M_j \\ -D_j \end{bmatrix}$ a basis of the null space of $[s_j I - A, B]$ as in (9.26).*

Proof. Necessity has been shown. To prove sufficiency, postmultiply $s_j I - (A + BF)$ by $M_j a_j$ and use (9.28) to obtain $(s_j I - A)M_j a_j - BD_j a_j = 0$ in view of (9.26). Thus,

$$[s_j I - (A + BF)]M_j a_j = 0,$$

which implies that s_j is an eigenvalue of $A + BF$ and $M_j a_j = v_j$ is the corresponding eigenvector. ∎

If relation (9.28) is written for n desired eigenvalues s_j, where the a_j *are selected so that the corresponding eigenvectors $v_j = M_j a_j$ are linearly independent*, then

$$FV = W, \tag{9.29}$$

where $V \triangleq [M_1 a_1, \ldots, M_n a_n]$ and $W \triangleq [D_1 a_1, \ldots, D_n a_n]$ uniquely specify F as the solution to these n linearly independent equations. When s_j are distinct, the n vectors $M_j a_j$, $j = 1, \ldots, n$, are linearly independent for almost any nonzero a_j, and so V has full rank. When s_j have repeated values, it may still be possible under certain conditions to select a_j so that $M_j a_j$ are linearly independent; however, in general, for multiple eigenvalues, (9.29) needs to be modified, and the details for this can be found in the literature. Also note that if $s_{j+1} = s_j^*$, the complex conjugate of s_j, then the corresponding eigenvector $v_{i+1} = v_j^* = M_j^* a_j^*$.

Relation (9.29) clearly shows that the F that assigns all n closed-loop eigenvalues is not unique (see also Lemma 9.10). All such F are parameterized by the vectors a_j that in turn characterize the corresponding eigenvectors. If the corresponding eigenvectors have been decided upon—of course within the set of possible eigenvectors $v_j = M_j a_j$—then F is uniquely specified. Note that in the single-input case, (9.28) becomes $FM_j = D_j$, where $v_j = M_j$. In this case, F is unique.

Example 9.13. Consider the controllable pair (A, B) of Example 9.11 given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Again, it is desired to assign the eigenvalues of $A + BF$ at $-2, -1 \pm j$. Let $s_1 = -2, s_2 = -1 + j$, and $s_3 = -1 - j$. Then, in view of (9.26),

$$\begin{bmatrix} M_1 \\ -D_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 0 \\ \hline -1 & -2 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} M_2 \\ -D_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j & 0 \\ 2 & 0 \\ \hline 2+j & -1+j \\ 1 & 1-j \end{bmatrix},$$

and $\begin{bmatrix} M_3 \\ -D_3 \end{bmatrix} = \begin{bmatrix} M_2^* \\ -D_2^* \end{bmatrix}$, the complex conjugate, since $s_3 = s_2^*$.

Each eigenvector $v_i = M_i a_i$, $i = 1, 2, 3$, is a linear combination of the columns of M_i. Note that $v_3 = v_2^*$. If we select the eigenvectors to be

$$V = [v_1, v_2, v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & j & -j \\ 0 & 2 & 2 \end{bmatrix},$$

i.e., $a_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then (9.29) implies that

$$F \begin{bmatrix} 1 & 1 & 1 \\ 0 & j & -j \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2-j & -2+j \\ -2 & -1 & -1 \end{bmatrix},$$

from which we have

$$\begin{aligned} F &= \frac{1}{4j} \begin{bmatrix} 2 & -2-j & -2+j \\ -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 4j & 0 & -2j \\ 0 & 2 & j \\ 0 & -2 & j \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & -2 \\ -2 & 0 & 1/2 \end{bmatrix}. \end{aligned}$$

As it can be verified, this matrix F is such that $A + BF$ has the desired eigenvalues and eigenvectors.

Remarks

At this point, several comments are in order.

1. In Example 9.13, if the eigenvectors were chosen to be the eigenvectors of $A + BF_1$ (instead of $A + BF_2$) of Example 9.11, then from $FV = W$, it follows that F would have been F_1 (instead of F_2).
2. When $s_i = s_{i+1}^*$, then the corresponding eigenvectors are also complex conjugates; i.e., $v_i = v_{i+1}^*$. In this case we obtain from (9.29) that

$$\begin{aligned} FV &= F[\ldots, v_{iR} + jv_{iI}, v_{iR} - jv_{iI}, \ldots] \\ &= [\ldots, w_{iR} + jw_{iI}, w_{iR} - jw_{iI}, \ldots] = W. \end{aligned}$$

   Although these calculations could be performed over the complex numbers (as was done in the example), this is not necessary, since postmultiplication of $FV = W$ by

$$\left[\begin{array}{c|cc|c} I & & & \\ \hline & \frac{1}{2} & -j\frac{1}{2} & \\ & \frac{1}{2} & +j\frac{1}{2} & \\ \hline & & & I \end{array}\right]$$

   shows that the above equation $FV = W$ is equivalent to

$$F[\ldots, v_{iR}, v_{iI}, \ldots] = [\ldots, w_{iR}, w_{iI}, \ldots],$$

   which involves only reals.
3. The bases $\begin{bmatrix} M_j \\ -D_j \end{bmatrix}$, $j = 1, \ldots, n$, in (9.26) can be determined in an alternative way and the calculations can be simplified if the controller form of the pair (A, B) is known. In particular, note that $[sI - A, B] \begin{bmatrix} P^{-1}S(s) \\ D(s) \end{bmatrix} = 0$, where the $n \times m$ matrix $S(s)$ is given by $S(s) = \text{block diag}[1, s, \ldots, s^{\mu_i - 1}]$ and the μ_i, $i = 1, \ldots, m$, are the controllability indices of (A, B). Also, the $m \times m$ matrix $D(s)$ is given by $D(s) = B_m^{-1}[\text{diag}[s^{\mu_1}, \cdots s^{\mu_m}] - A_m S(s)]$. Note that $S(s)$ and $D(s)$ were defined in the Structure Theorem (controllable version) in Section 6.4. It was shown there that $(sI - A_c)S(s) = B_c D(s)$, from which it follows that $(sI - A)P^{-1}S(s) = BD(s)$, where P is a similarity transformation matrix that reduces (A, B) to the controller form ($A_c = PAP^{-1}, B_c = PB$). Since $P^{-1}S(s)$ and $D(s)$ are right coprime polynomial matrices (see Section 7.5), we have rank $\begin{bmatrix} P^{-1}S(s_j) \\ D(s_j) \end{bmatrix} = m$ for any s_j, and therefore, $\begin{bmatrix} P^{-1}S(s_j) \\ D(s_j) \end{bmatrix}$ qualifies as a basis for the null space of the matrix $[s_j I - A, B]$ ($P = I$ when A, B are in controller form; i.e., $A = A_c$ and $B = B_c$.)

Example 9.13 continued. Continuing the above example, the controller form of (A, B) was found in Example 9.11 using

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

Here $S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$, $\quad D(s) = \begin{bmatrix} s^2+s-1 & -s \\ -1 & s \end{bmatrix}$,

and $\begin{bmatrix} M(s) \\ -D(s) \end{bmatrix} = \begin{bmatrix} P^{-1}S(s) \\ -D(s) \end{bmatrix} = \left[\begin{array}{cc} 1 & 1 \\ s+1 & 0 \\ 2 & 0 \\ \hdashline -(s^2+s-1) & s \\ 1 & -s \end{array}\right]$.

Then

$$\begin{bmatrix} M_1 \\ -D_1 \end{bmatrix} = \begin{bmatrix} M(-2) \\ -D(-2) \end{bmatrix} = \left[\begin{array}{cc} 1 & 1 \\ -1 & 0 \\ 2 & 0 \\ \hdashline -1 & -2 \\ 1 & 2 \end{array}\right],$$

$$\begin{bmatrix} M_2 \\ -D_2 \end{bmatrix} = \begin{bmatrix} M(-1+j) \\ -D(-1+j) \end{bmatrix} = \left[\begin{array}{cc} 1 & 1 \\ j & 0 \\ 2 & 0 \\ \hdashline 2+j & -1+j \\ 1 & 1-j \end{array}\right],$$

and $\begin{bmatrix} M_3 \\ -D_3 \end{bmatrix} = \begin{bmatrix} M_2^* \\ -D_2^* \end{bmatrix}$, which are precisely the bases used above.

Remarks (cont.)

4. If in Example 9.13 the only requirement were that $(s_1, v_1) = (-2, (1,\ 0,\ 0)^T)$, then $F(1,\ 0,\ 0)^T = (2,\ -2)^T$; i.e., any $F = \begin{bmatrix} 2 & f_{12} & f_{13} \\ 2 & f_{22} & f_{23} \end{bmatrix}$ will assign the desired values to an eigenvalue of $A+BF$ and its corresponding eigenvector.
5. All possible eigenvectors v_1 and $v_2 (v_3 = v_2^*)$ in Example 9.13 are given by

$$v_1 = M_1 a_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} \text{ and } v_2 = M_2 a_2 = \begin{bmatrix} 1 & 1 \\ j & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a_{21} + ja_{31} \\ a_{22} + ja_{32} \end{bmatrix},$$

where the a_{ij} are such that the set $\{v_1, v_2, v_3\}$ is linearly independent (i.e., $V = [v_1, v_2, v_3]$ is nonsingular) but otherwise arbitrary. Note that in this case (s_j distinct), almost any arbitrary choice for a_{ij} will satisfy the above requirement; see [1, Appendix A.4].

9.2.3 The Linear Quadratic Regulator (LQR): Continuous-Time Case

A linear state feedback control law that is optimal in some appropriate sense can be determined as a solution to the so-called Linear Quadratic Regulator (LQR) problem (also called the H_2 optimal control problem). The LQR problem has been studied extensively, and the interested reader should consult the extensive literature on *optimal control* for additional information on the subject. In the following discussion, we give a brief outline of certain central results of this topic to emphasize the fact that the state feedback gain F can be determined to satisfy, in an optimal fashion, requirements other than eigenvalue assignment, discussed above. The LQR problem has been studied for the time-varying and time-invariant cases. Presently, we will concentrate on the time-invariant optimal regulator problem.

Consider the time-invariant linear system given by

$$\dot{x} = Ax + Bu, \quad z = Mx, \tag{9.30}$$

where the vector $z(t)$ represents the variables to be regulated—to be driven to zero.

We wish to determine $u(t)$, $t \geq 0$, which minimizes the quadratic cost

$$J(u) = \int_0^\infty [z^T(t)Qz(t) + u^T(t)Ru(t)]dt \tag{9.31}$$

for any initial state $x(0)$. The weighting matrices Q, R are real, symmetric, and positive definite; i.e., $Q = Q^T, R = R^T$, and $Q > 0, R > 0$. This is the most common version of the LQR problem. The term $z^TQz = x^T(M^TQM)x$ is nonnegative, and it minimizes its integral forces $z(t)$ to approach zero as t goes to infinity. The matrix M^TQM is in general positive semidefinite, which allows some states to be treated as "do not care" states. The term u^TRu with $R > 0$ is always positive for $u \neq 0$, and it minimizes its integral forces $u(t)$ to remain small. The relative "size" of Q and R enforces tradeoffs between the size of the control action and the speed of response.

Assume that $(A, B, Q^{1/2}M)$ is controllable (-from-the-origin) and observable. It turns out that the solution $u^*(t)$ to this optimal control problem can be expressed in state feedback form, which is independent of the initial condition $x(0)$. In particular, the optimal control u^* is given by

$$u^*(t) = F^*x(t) = -R^{-1}B^TP_c^*x(t), \tag{9.32}$$

where P_c^* denotes the symmetric positive definite solution of the *algebraic Riccati equation*

$$A^T P_c + P_c A - P_c B R^{-1} B^T P_c + M^T Q M = 0. \tag{9.33}$$

This equation may have more than one solution but only one that is positive definite (see Example 9.14). It can be shown that $u^*(t) = F^* x(t)$ is a stabilizing feedback control law and that the minimum cost is given by $J_{\min} = J(u^*) = x^T(0) P_c^* x(0)$.

The assumptions that $(A, B, Q^{1/2}M)$ are controllable and observable may be relaxed somewhat. If $(A, B, Q^{1/2}M)$ is stabilizable and detectable, then the uncontrollable and unobservable eigenvalues, respectively, are stable, and P_c^* is the unique, symmetric, but now positive-semidefinite solution of the algebraic Riccati equation. The matrix F^* is still a stabilizing gain, but it is understood that the uncontrollable and unobservable (but stable) eigenvalues will not be affected by F^*.

Note that if the time interval of interest in the evaluation of the cost goes from 0 to $t_1 < \infty$, instead of 0 to ∞, that is, if

$$J(u) = \int_0^{t_1} [z^T(t) Q z(t) + u^T(t) R u(t)] dt, \tag{9.34}$$

then the optimal control law is time-varying and is given by

$$u^*(t) = -R^{-1} B^T P^*(t) x(t), \quad 0 \le t \le t_1, \tag{9.35}$$

where $P^*(t)$ is the unique, symmetric, and positive-semidefinite solution of the Riccati equation, which is a matrix differential equation of the form

$$-\frac{d}{dt} P(t) = A^T P(t) + P(t) A - P(t) B R^{-1} B^T P(t) + M^T Q M, \tag{9.36}$$

where $P(t_1) = 0$. It is interesting to note that if $(A, B, Q^{1/2}M)$ is stabilizable and detectable (or controllable and observable), then the solution to this problem as $t_1 \to \infty$ approaches the steady-state value P_c^* given by the algebraic Riccati equation; that is, when $t_1 \to \infty$ the optimal control policy is the time-invariant control law (9.32), which is much easier to implement than time-varying control policies.

Example 9.14. Consider the system described by the equations $\dot{x} = Ax + Bu$, $y = Cx$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1,\ 0]$. Then (A, B, C) is controllable and observable and $C(sI - A)^{-1} B = 1/s^2$. We wish to determine the optimal control $u^*(t), t \ge 0$, which minimizes the performance index

$$J = \int_0^{\infty} (y^2(t) + \rho u^2(t)) dt,$$

where ρ is positive and real. Then $R = \rho > 0, z(t) = y(t)$, $M = C$, and $Q = 1 > 0$. In the present case the algebraic Riccati equation (9.33) assumes the form

$$\begin{aligned}
A^T P_c + P_c A - &P_c B R^{-1} B^T P_c + M^T Q M \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P_c + P_c \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{\rho} P_c \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0\ 1] P_c + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1\ 0] \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{\rho} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

where $P_c = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = P_c^T$. This implies that

$$-\frac{1}{\rho} p_2^2 + 1 = 0, \quad p_1 - \frac{1}{\rho} p_2 p_3 = 0, \quad 2p_2 - \frac{1}{\rho} p_3^3 = 0.$$

Now P_c is positive definite if and only if $p_1 > 0$ and $p_1 p_3 - p_2^2 > 0$. The first equation above implies that $p_2 = \pm\sqrt{\rho}$. However, the third equation, which yields $p_3^2 = 2\rho p_2$, implies that $p_2 = +\sqrt{\rho}$. Then $p_3^2 = 2\rho\sqrt{\rho}$ and $p_3 = \pm\sqrt{2\rho\sqrt{\rho}}$. The second equation yields $p_1 = \frac{1}{\rho} p_2 p_3$ and implies that only $p_3 = +\sqrt{2\rho\sqrt{\rho}}$ is acceptable, since we must have $p_1 > 0$ for P_c to be positive definite. Note that $p_1 > 0$ and $p_3 - p_2^2 = 2\rho - \rho = \rho > 0$, which shows that

$$P_c^* = \begin{bmatrix} \sqrt{2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2\rho\sqrt{\rho}} \end{bmatrix}$$

is the positive definite solution of the algebraic Riccati equation. The optimal control law is now given by

$$u^*(t) = F^* x(t) = -R^{-1} B^T P_c^* x(t) = -\frac{1}{\rho} [0,\ 1] P_c^* x(t).$$

The eigenvalues of the compensated system, i.e., the eigenvalues of $A + BF^*$, can now be determined for different ρ. Also, the corresponding $u^*(t)$ and $y(t)$ for given $x(0)$ can be plotted. As ρ increases, the control energy expended to drive the output to zero is forced to decrease. The reader is asked to verify this by plotting $u^*(t)$ and $y(t)$ for different values of ρ when $x(0) = [1,\ 1]^T$. Also, the reader is asked to plot the eigenvalues of $A + BF^*$ as a function of ρ and to comment on the results.

It should be pointed out that the locations of the closed-loop eigenvalues, as the weights Q and R vary, have been studied extensively. Briefly, for the single-input case and for $Q = qI$ and $R = r$ in (9.31), it can be shown that the optimal closed-loop eigenvalues are the stable zeros of $1+(q/r)H^T(-s)H(s)$, where $H(s) = M(sI - A)^{-1}B$. As q/r varies from zero (no state weighting) to infinity (no control weighting), the optimal closed-loop eigenvalues move from the stable poles of $H^T(-s)H(s)$ to the stable zeros of $H^T(-s)H(s)$. Note that the stable poles of $H^T(-s)H(s)$ are the stable poles of $H(s)$ and the stable reflections of its unstable poles with respect to the imaginary axis in the complex plane, whereas its stable zeros are the stable zeros of $H(s)$ and the stable reflections of its unstable zeros.

The solution of the LQR problem relies on solving the Riccati equation. A number of numerically stable algorithms exist for solving the algebraic Riccati equation. The reader is encouraged to consult the literature for computer software packages that implement these methods. A rather straightforward method for determining P_c^* is to use the *Hamiltonian matrix* given by

$$H \triangleq \begin{bmatrix} A & -BR^{-1}B^T \\ -M^TQM & -A^T \end{bmatrix}. \tag{9.37}$$

Let $[V_1^T, V_2^T]^T$ denote the n eigenvectors of H that correspond to the n stable $[Re\ (\lambda) < 0]$ eigenvalues. Note that of the $2n$ eigenvalues of H, n are stable and are the mirror images reflected on the imaginary axis of its n unstable eigenvalues. When $(A, B, Q^{1/2}M)$ is controllable and observable, then H has no eigenvalues on the imaginary axis $[Re(\lambda) = 0]$. In this case the n stable eigenvalues of H are in fact the closed-loop eigenvalues of the optimally controlled system, and the solution to the algebraic Riccati equation is then given by

$$P_c^* = V_2V_1^{-1}. \tag{9.38}$$

Note that in this case the matrix V_1 consists of the n eigenvectors of $A+BF^*$, since for λ_1 a stable eigenvalue of H, and v_1 the corresponding (first) column of V_1, we have

$$\begin{aligned}
[\lambda_1 I - (A + BF^*)]v_1 &= [\lambda_1 I - A + BR^{-1}B^TV_2V_1^{-1}]v_1 \\
&= \left[[\lambda_1 I, 0] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} - [A, -BR^{-1}B^T] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \right] V_1^{-1}v_1 \\
&= \begin{bmatrix} 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{bmatrix} V_1^{-1}v_1 = \begin{bmatrix} 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},
\end{aligned}$$

where the fact that $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$ are eigenvectors of H was used. It is worth reflecting for a moment on the relationship between (9.38) and (9.29). The optimal control F derived by (9.38) is in the class of F derived by (9.29).

9.2.4 Input–Output Relations

It is useful to derive the input–output relations for a closed-loop system that is compensated by linear state feedback. Given the uncompensated or open-loop system $\dot{x} = Ax+Bu, y = Cx+Du$, with initial conditions $x(0) = x_0$, we have

$$\hat{y}(s) = C(sI - A)^{-1}x_0 + H(s)\hat{u}(s), \tag{9.39}$$

where the open-loop transfer function $H(s) = C(sI - A)^{-1}B + D$. Under the feedback control law $u = Fx + r$, the compensated closed-loop system is described by the equations $\dot{x} = (A + BF)x + Br$, $y = (C + DF)x + Dr$, from which we obtain

$$\hat{y}(s) = (C + DF)[sI - (A + BF)]^{-1}x_0 + H_F(s)\hat{r}(s), \tag{9.40}$$

where the closed-loop transfer function $H_F(s)$ is given by

$$H_F(s) = (C + DF)[sI - (A + BF)]^{-1}B + D.$$

Alternative expressions for $H_F(s)$ can be derived rather easily by substituting (9.4), namely,

$$\hat{u}(s) = F[sI - (A + BF)]^{-1}x_0 + [I - F(sI - A)^{-1}B]^{-1}\hat{r}(s),$$

into (9.39). This corresponds to working with an open-loop control law that nominally produces the same results when applied to the system [see the discussion on open- and closed-loop control that follows (9.4)]. Substituting, we obtain

$$\begin{aligned}\hat{y}(s) = {} & [C(sI - A)^{-1} + H(s)F[sI - (A + BF)]^{-1}]x_0 \\ & + H(s)[I - F(sI - A)^{-1}B]^{-1}\hat{r}(s).\end{aligned} \tag{9.41}$$

Comparing with (9.40), we see that $(C + DF)[sI - (A + BF)]^{-1} = C(sI - A)^{-1} + H(s)F[sI - (A + BF)]^{-1}$, and that

$$\begin{aligned}H_F(s) &= (C + DF)[sI - (A + BF)]^{-1}B + D \\ &= [C(sI - A)^{-1}B + D][I - F(sI - A)^{-1}B]^{-1} \\ &= H(s)[I - F(sI - A)^{-1}B]^{-1}.\end{aligned} \tag{9.42}$$

The last relation points out the fact that $\hat{y}(s) = H_F(s)\hat{r}(s)$ can be obtained from $\hat{y}(s) = H(s)\hat{u}(s)$ using the open-loop control $\hat{u}(s) = [I - F(sI - A)^{-1}B]^{-1}\hat{r}(s)$.

Using Matrix Fractional Descriptions

Relation (9.42) can easily be derived in an alternative manner, using fractional matrix descriptions for the transfer function, introduced in Section 7.5. In particular, the transfer function $H(s)$ of the open-loop system $\{A, B, C, D\}$ is given by

$$H(s) = N(s)D^{-1}(s),$$

where $N(s) = CS(s) + DD(s)$, with $S(s)$ and $D(s)$ satisfying $(sI - A)S(s) = BD(s)$ (refer to the proof of the controllable version of the Structure Theorem given in Section 6.4). Notice that it has been assumed, without loss of generality, that the pair (A, B) is in controller form.

Similarly, the transfer function $H_F(s)$ of the compensated system $\{A + BF, B, C + DF, D\}$ is given by

$$H_F(s) = N_F(s)D_F^{-1}(s),$$

where $N_F(s) = (C + DF)S(s) + DD_F(s)$, with $S(s)$ and $D_F(s)$ satisfying $[sI - (A + BF)]S(s) = BD_F(s)$. This relation implies that $(sI - A)S(s) = B[D_F(s) + FS(s)]$, from which we obtain $D_F(s) + FS(s) = D(s)$. Then $N_F(s) = CS(s) + D[FS(s) + D_F(s)] = CS(s) + DD(s) = N(s)$; that is,

$$H_F(s) = N(s)D_F^{-1}(s), \tag{9.43}$$

where $D_F(s) = D(s) - FS(s)$.

Note that $I - F(sI - A)^{-1}B$ in (9.42) is the transfer function of the system $\{A, B, -F, I\}$ and can be expressed as $D_F(s)D^{-1}(s)$, where $D_F(s) = -FS(s) + ID(s)$. Let $M(s) = (D_F(s)D^{-1}(s))^{-1}$. Then (9.43) assumes the form

$$H_F(s) = N(s)D_F^{-1}(s) = (N(s)D^{-1}(s))(D(s)D_F^{-1}(s)) = H(s)M(s). \tag{9.44}$$

Relation $H_F(s) = N(s)D_F^{-1}(s)$ also shows that the zeros of $H(s)$ [in $N(s)$; see also Subsection 7.5.4] are invariant under linear state feedback; they can be changed only via cancellations with poles. Also observe that $M(s) = D(s)D_F^{-1}(s)$ is the transfer function of the system $\{A + BF, B, F, I\}$. This implies that $H_F(s)$ in (9.42) can also be written as

$$H_F(s) = H(s)[F(sI - (A + BF))^{-1}B + I], \tag{9.45}$$

which is a result that could also be shown directly using matrix identities.

Example 9.15. Consider the system $\dot{x} = Ax + Bu, y = Cx$, where

$$A = A_c = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = B_c = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

as in Example 9.11, and let $C = C_c = [1,\ 1,\ 0]$. $H_F(s)$ will now be determined. In view of the Structure Theorem developed in Section 6.4, the transfer function is given by $H(s) = N(s)D^{-1}(s)$, where

$$N(s) = C_cS(s) = [1,\ 1,\ 0] \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} = [s + 1, 0]$$

and

$$D(s) = B_m^{-1}[\Lambda(s) - A_m S(s)] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \left[\begin{bmatrix} s^2 & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s^2+s-2 & 0 \\ -1 & s \end{bmatrix} = \begin{bmatrix} s^2+s-1 & -s \\ -1 & s \end{bmatrix}.$$

Then

$$H(s) = N(s)D^{-1}(s) = [s+1, 0] \begin{bmatrix} s^2+s-1 & -s \\ -1 & s \end{bmatrix}^{-1}$$

$$= [s+1, 0] \begin{bmatrix} s & s \\ 1 & s^2+s-1 \end{bmatrix} \frac{1}{s^3+s^2-2s}$$

$$= \frac{1}{s(s^2+s-2)}[s(s+1), s(s+1)] = \frac{s+1}{s^2+s-2}[1, 1].$$

If $F_c = \begin{bmatrix} 3 & 7 & 5 \\ -5 & -6 & -4 \end{bmatrix}$ (which is F_{c1} of Example 9.11), then

$$D_F(s) = D(s) - F_c S(s) = \begin{bmatrix} s^2+s-1 & -s \\ -1 & s \end{bmatrix} - \begin{bmatrix} 3 & 7 & 5 \\ -5 & -6 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s^2-6s-4 & -s-5 \\ 6s+4 & s+4 \end{bmatrix}.$$

Note that $\det D_F(s) = s^3 + 4s^2 + 6s + 4 = (s+2)(s^2+2s+2)$ with roots $-2, -1 \pm j$, as expected. Now

$$H_F(s) = N(s)D_F^{-1}(s) = [s+1, 0] \begin{bmatrix} s+4 & s+5 \\ -6s-4 & s^2-6s-4 \end{bmatrix} \frac{1}{(s+2)(s^2+2s+2)}$$

$$= \frac{s+1}{(s+2)(s^2+2s+2)}[s+4, s+5].$$

Note that the zeros of $H(s)$ and $H_F(s)$ are identical, located at -1. Then $H_F(s) = H(s)M(s)$, where

$$M(s) = D(s)D_F^{-1}(s) = \begin{bmatrix} s^2+s-1 & -s \\ -1 & s \end{bmatrix} \begin{bmatrix} s+4 & s+5 \\ -6s-4 & s^2-6s-4 \end{bmatrix} \frac{1}{s^3+4s^2+6s+4}$$

$$= \begin{bmatrix} s^3+11s^2+7s-4 & 12s^2+8s-5 \\ -6s^2-5s-4 & s^3-6s^2-5s-5 \end{bmatrix} \frac{1}{s^3+4s^2+6s+4}$$

$$= [I - F_c(sI - A_c)^{-1}B_c]^{-1}.$$

Note that the open-loop uncompensated system is unobservable, with 0 being the unobservable eigenvalue, whereas the closed-loop system is observable; i.e., the control law changed the observability of the system.

9.2.5 Discrete-Time Systems

Linear state feedback control for discrete-time systems is defined in a way that is analogous to the continuous-time case. The definitions are included here for purposes of completeness.

We consider a linear, time-invariant, discrete-time system described by equations of the form

$$x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k) + Du(k), \tag{9.46}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$, and $k \geq k_0$, with $k \geq k_0 = 0$ being typical.

Definition 9.16. *The linear (discrete-time, time-invariant) state feedback control law is defined by*

$$u(k) = Fx(k) + r(k), \tag{9.47}$$

where $F \in R^{m\times n}$ is a gain matrix and $r(k) \in R^m$ is the external input vector. ■

The compensated closed-loop system is now given by

$$\begin{aligned} x(k+1) &= (A+BF)x(k) + Br(k),\\ y(k) &= (C+DF)x(k) + Dr(k). \end{aligned} \tag{9.48}$$

In view of Section 3.5, the system $x(k+1) = (A+BF)x(k)$ is asymptotically stable if and only if the eigenvalues of $A+BF$ satisfy $|\lambda_i| < 1$, i.e., if they lie strictly within the unit disk of the complex plane. The stabilization problem for the time-invariant case therefore becomes a problem of shifting the eigenvalues of $A+BF$, which is precisely the problem studied before for the continuous-time case. Theorem 9.2 and Lemmas 9.3 and 9.6 apply without change, and the methods developed before for eigenvalue assignment can be used here as well. The only difference in this case is the location of the desired eigenvalues: They are assigned to be within the unit circle to achieve stability. We will not repeat here the details for these results.

Input–output relations for discrete-time systems, which are in the spirit of the results developed in the preceding subsection for continuous-time systems, can be derived in a similar fashion, this time making use of the z-transform of $x(k+1) = Ax(k) + Bu(k), x(0) = x_0$ to obtain

$$\hat{x}(z) = z(zI - A)^{-1}x_0 + (zI - A)^{-1}B\hat{u}(z). \tag{9.49}$$

[Compare expression (9.49) with $\hat{x}(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\hat{u}(s)$.]

9.2.6 The Linear Quadratic Regulator (LQR): Discrete-Time Case

The formulation of the LQR problem in the discrete-time case is analogous to the continuous-time LQR problem. Consider the time-invariant linear system

$$x(k+1) = Ax(k) + Bu(k), z(k) = Mx(k), \tag{9.50}$$

where the vector $z(t)$ represents the variables to be regulated. The LQR problem is to determine a control sequence $\{u^*(k)\}$, $k \geq 0$, which minimizes the cost function

$$J(u) = \sum_{k=0}^{\infty} [z^T(k)Qz(k) + u^T(k)Ru(k)] \tag{9.51}$$

for any initial state $x(0)$, where the weighting matrices Q and R are real symmetric and positive definite.

Assume that $(A, B, Q^{1/2}M)$ is reachable and observable. Then the solution to the *LQR* problem is given by the linear state feedback control law

$$u^*(k) = F^*x(k) = -[R + B^T P_c^* B]^{-1} B^T P_c^* Ax(k), \tag{9.52}$$

where P_c^* is the unique, symmetric, and positive definite solution of the *(discrete-time) algebraic Riccati equation*, given by

$$P_c = A^T[P_c - P_c B[R + B^T P_c B]^{-1} B^T P_c]A + M^T QM. \tag{9.53}$$

The minimum value of J is $J(u^*) = J_{\min} = x^T(0)P_c^* x(0)$.

As in the continuous-time case, it can be shown that the solution P_c^* can be determined from the eigenvectors of the *Hamiltonian matrix*, which in this case is

$$H = \begin{bmatrix} A + BR^{-1}B^T A^{-T} M^T QM & -BR^{-1}B^T A^{-T} \\ -A^{-T} M^T QM & A^{-T} \end{bmatrix}, \tag{9.54}$$

where it is assumed that A^{-1} exists. Variations of the above method that relax this assumption exist and can be found in the literature. Let $[V_1^T, V_2^T]^T$ be n eigenvectors corresponding to the n stable ($|\lambda| < 1$) eigenvalues of H. Note that out of the $2n$ eigenvalues of H, n of them are stable (i.e., within the unit circle) and are the reciprocals of the remaining n unstable eigenvalues (located outside the unit circle). When $(A, B, Q^{1/2}M)$ is controllable (-from-the-origin) and observable, then H has no eigenvalues on the unit circle ($|\lambda| = 1$). In fact the n stable eigenvalues of H are in this case the closed-loop eigenvalues of the optimally controlled system.

The solution to the algebraic Riccati equation is given by

$$P_c^* = V_2 V_1^{-1}. \tag{9.55}$$

As in the continuous-time case, we note that V_1 consists of the n eigenvectors of $A + BF^*$.

Example 9.17. We consider the system $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k)$, where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = [1, 0]$ and we wish to determine the optimal control sequence $\{u^*(k)\}$, $k \geq 0$, that minimizes the performance index

$$J(u) = \sum_{k=0}^{\infty} (y^2(k) + \rho u^2(k)),$$

where $\rho > 0$. In (9.51), $z(k) = y(k)$, $M = C$, $Q = 1$, and $R = \rho$. The reader is asked to determine $u^*(k)$ given in (9.52) by solving the discrete-time algebraic Riccati equation (9.53) in a manner analogous to the solution in Example 9.14 (for the continuous-time algebraic Riccati equation).

9.3 Linear State Observers

Since the states of a system contain a great deal of useful information, there are many applications where knowledge of the state vector over some time interval is desirable. It may be possible to measure states of a system by appropriately positioned sensors. This was in fact assumed in the previous section, where the state values were multiplied by appropriate gains and then fed back to the system in the state feedback control law. Frequently, however, it may be either impossible or simply impractical to obtain measurements for all states. In particular, some states may not be available for measurement at all (as in the case, for example, with temperatures and pressures in inaccessible parts of a jet engine). There are also cases where it may be impractical to obtain state measurements from otherwise available states because of economic reasons (e.g., some sensors may be too expensive) or because of technical reasons (e.g., the environment may be too noisy for any useful measurements). Thus, there is a need to be able to estimate the values of the state of a system from available measurements, typically outputs and inputs (see Figure 9.2). Given the system parameters A, B, C, D and the values of the inputs and outputs over a time interval, it is possible to estimate the state when the system is observable. This problem, a problem in state estimation, is discussed in this section. In particular, we will address the so-called full-order and reduced-order asymptotic estimators, which are also called full-order and reduced-order observers.

9.3.1 Full-Order Observers: Continuous-Time Systems

We consider systems described by equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{9.56}$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, and $D \in R^{p \times m}$.

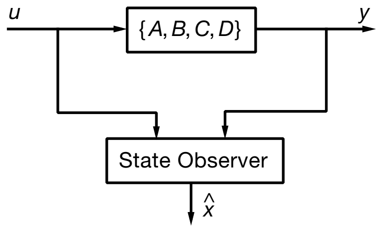

Figure 9.2. Linear state observer configuration

Full-State Observers: The Identity Observer

An estimator of the full state $x(t)$ can be constructed in the following manner. We consider the system

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \tag{9.57}$$

where $\hat{y} \triangleq C\hat{x} + Du$. Note that (9.57) can be written as

$$\dot{\hat{x}} = (A - KC)\hat{x} + [B - KD, K]\begin{bmatrix} u \\ y \end{bmatrix}, \tag{9.58}$$

which clearly reveals the role of u and y (see Figure 9.3). The error between the actual state $x(t)$ and the estimated state $\hat{x}(t), e(t) = x(t) - \hat{x}(t)$, is governed by the differential equation

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = [Ax + Bu] - [A\hat{x} + Bu + KC(x - \hat{x})]$$

or

$$\dot{e}(t) = [A - KC]e(t). \tag{9.59}$$

Solving (9.59), we obtain

$$e(t) = \exp[(A - KC)t]e(0). \tag{9.60}$$

Now if the eigenvalues of $A - KC$ are in the left half-plane, then $e(t) \to 0$ as $t \to \infty$, independently of the initial condition $e(0) = x(0) - \hat{x}(0)$. This *asymptotic state estimator* is known as the *Luenberger observer*.

Lemma 9.18. *There exists $K \in R^{n\times p}$ so that the eigenvalues of $A - KC$ are assigned to arbitrary real or complex conjugate locations if and only if (A, C) is observable.*

Proof. The eigenvalues of $(A - KC)^T = A^T - C^T K^T$ are arbitarily assigned via K^T if and only if the pair (A^T, C^T) is controllable (see Theorem 9.2 of the previous section) or, equivalently, if and only if the pair (A, C) is observable. ∎

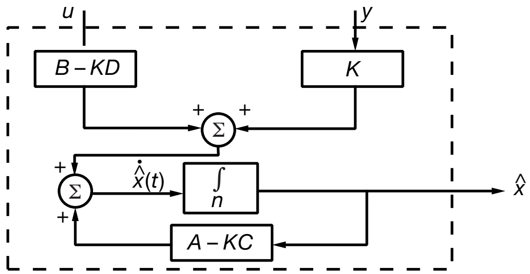

Figure 9.3. Full-state identity observer configuration

Discussion

If (A, C) is not observable, but the unobservable eigenvalues are stable, i.e., (A, C) is detectable, then the error $e(t)$ will still tend to zero asymptotically. However, the unobservable eigenvalues will appear in this case as eigenvalues of $A - KC$, and they may affect the speed of the response of the estimator in an undesirable way. For example, if the unobservable eigenvalues are stable but are located close to the imaginary axis, then their corresponding modes will tend to dominate the response, most likely resulting in a state estimator that converges too slowly to the actual value of the state.

Where should the eigenvalues of $A - KC$ be located? This problem is dual to the problem of closed-loop eigenvalue placement via state feedback and is equally difficult to resolve. On the one hand, the observer must estimate the state sufficiently fast, which implies that the eigenvalues should be placed sufficiently far from the imaginary axis so that the error $e(t)$ will tend to zero sufficiently fast. On the other hand, this requirement may result in a high gain K, which tends to amplify existing noise, thus reducing the accuracy of the estimate. Note that in this case, noise is the only limiting factor of how fast an estimator may be, since the gain K is realized by an algorithm and is typically implemented by means of a digital computer. Therefore, gains of any size can easily be introduced. Compare this situation with the limiting factors in the control case, which is imposed by the magnitude of the required control action (and the limits of the corresponding actuator). Typically, the faster the compensated system, the larger the required control magnitude.

One may of course balance the tradeoffs between speed of response of the estimator and effects of noise by formulating an *optimal estimation problem* to derive the best K. To this end, one commonly assumes certain probabilistic properties for the process. Typically, the measurement noise and the initial condition of the plant are assumed to be Gaussian random variables, and one tries to minimize a quadratic performance index. This problem is typically referred to as the *Linear Quadratic Gaussian (LQG)* estimation problem. This optimal estimation or filtering problem can be seen to be the dual of the quadratic optimal control problem of the previous section, a fact that will be exploited in deriving its solution. Note that the well-known *Kalman filter* is such an estimator. In the following discussion, we shall briefly discuss the optimal estimation problem. First, however, we shall address the following related issues.

1. Is it possible to take $K = 0$ in the estimator (9.57)? Such a choice would eliminate the information contained in the term $y - \hat{y}$ from the estimator, which would now be of the form
$$\dot{\hat{x}} = A\hat{x} + Bu. \tag{9.61}$$
In this case, the estimator would operate without receiving any information on how accurate the estimate $\hat{x}$ actually is. The error $e(t) = x(t) - \hat{x}(t)$ would go to zero only when A is stable. There is no mechanism to affect the speed by which $\hat{x}(t)$ would approach $x(t)$ in this case, and this is undesirable. One could perhaps determine $x(0)$, using the methods in Section 5.4, assuming that the system is observable. Then, by setting $\hat{x}(0) = x(0)$, presumably $\hat{x}(t) = x(t)$ for all $t \geq 0$, in view of (9.61). This of course is not practical for several reasons. First, the calculated $\hat{x}(0)$ is never exactly equal to the actual $x(0)$, which implies that $e(0)$ would be nonzero. Therefore, the method would rely again on A being stable, as before, with the advantage here that $e(0)$ would be small in some sense and so $e(t) \to 0$ faster. Second, this scheme assumes that sufficient data have been collected in advance to determine (an approximation to) $x(0)$ and to initialize the estimator, which may not be possible. Third, it is assumed that this initialization process is repeated whenever the estimator is restarted, which may be impractical.
2. If derivatives of the inputs and outputs are available, then the state $x(t)$ may be determined directly (see Exercise 5.12 in Chapter 5). The estimate $\hat{x}(t)$ is in this case produced instantaneously from the values of the inputs and outputs and their derivatives. Under these circumstances, $\hat{x}(t)$ is the output of a static state estimator, as opposed to the above dynamic state estimator, which leads to a state estimate $\hat{x}(t)$ that only approaches the actual state $x(t)$ asymptotically as $t \to \infty$ [$e(t) = x(t) - \hat{x}(t) \to 0$ as $t \to \infty$]. Unfortunately, this approach is in general not viable since noise present in the measurements of $u(t)$ and $y(t)$ makes accurate calculations of the derivatives problematic, and since errors in $u(t), y(t)$ and their derivatives are not smoothed by the algebraic equations of the static estimator (as opposed to the smoothing effects introduced by integration in dynamic systems). It follows that in this case the state estimates may be erroneous.

Example 9.19. Consider the observable pair
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad C = [1,\ 0,\ 0].$$

We wish to assign the eigenvalues of $A-KC$ in a manner that enables us to design a full-order/full-state asymptotic observer. Let the desired characteristic polynomial be $\alpha_d(s) = s^3 + d_2 s^2 + d_1 s + d_0$, and consider

$$A_D = A^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B_D = C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

To reduce (A_D, B_D) to controller form, we consider

$$\mathcal{C} = [B_D, A_D B_D, A_D^2 B_D] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathcal{C}^{-1}.$$

Then $P = \begin{bmatrix} q \\ qA_D \\ qA_D^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,
from which we obtain

$$A_{D_c} = P A_D P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B_{D_c} = P B_D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The state feedback is then given by $F_{D_c} = B_m^{-1}[A_{d_m} - A_m] = [-d_o, -d_1 - 2, -d_2 + 1]$ and $F_D = F_{D_c} P = [-d_2 + 1, d_2 - d_1 - 3, d_1 - d_0 - 3d_2 + 5]$. Then

$$K = -F_D^T = [d_2 - 1, d_1 - d_2 + 3, d_0 - d_1 + 3d_2 - 5]^T$$

assigns the eigenvalues of $A-KC$ at the roots of $\alpha_d(s) = s^3 + d_2 s^2 + d_1 s + d_0$. Note that the same result could also have been derived using the direct method for eigenvalue assignment, using $|sI - (A - (k_0, k_1, k_2)^T C)| = \alpha_d(s)$. Also, the result could have been derived using the *observable version of Ackermann's formula*, namely,

$$K = -F_D^T = \alpha_d(A)\mathcal{O}^{-1} e_n,$$

where $F_D = -e_n^T \mathcal{C}_D^{-1} \alpha_d(A_D)$ from (9.21). Note that the given system has eigenvalues at $0, 1, -2$ and is therefore unstable. The observer derived in this case will be used in the next section (Example 9.25) in combination with state feedback to stabilize the system $\dot{x} = Ax + Bu, y = Cx$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = [1, 0, 0]$$

(see Example 9.11), using only output measurements.

Example 9.20. Consider the system $\dot{x} = Ax$, $y = Cx$, where $A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$ and $C = [0,\ 1]$, and where (A, C) is in observer form. It is easy to show that $K = [d_0 - 2, d_1 - 2]^T$ assigns the eigenvalues of $A - KC$ at the roots of $s^2 + d_1 s + d_0$. To verify this, note that

$$\det(sI - (A - KC)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -d_0 \\ 1 & -d_1 \end{bmatrix}\right) = s^2 + d_1 + d_0.$$

The error $e(t) = x(t) - \hat{x}(t)$ is governed by the equation $\dot{e}(t) = (A - KC)e(t)$ given in (9.59). Noting that the eigenvalues of A are $-1 \pm j$, select different sets of eigenvalues for the observer and plot the states $x(t), \hat{x}(t)$ and the error $e(t)$ for $x(0) = [2, 2]^T$ and $\hat{x}(0) = [0, 0]^T$. The further away the eigenvalues of the observer are selected from the imaginary axis (with negative real parts), the larger the gains in K will become and the faster $\hat{x}(t) \to x(t)$.

Partial or Linear Functional State Observers

The state estimator studied above is a full-state estimator or observer; i.e., $\hat{x}(t)$ is an estimate of the full-state vector $x(t)$. There are cases where only part of the state vector, or a linear combination of the states, is of interest. In control problems, for example, $F\hat{x}(t)$ is used and fed back, instead of $Fx(t)$, where F is an $m \times n$ state feedback gain matrix. An interesting question that arises at this point is as follows: Is it possible to estimate directly a linear combination of the state, say, Tx, where $T \in R^{\tilde{n} \times n}$, $\tilde{n} \leq n$? For details of this problem see materials starting with [1, p. 354].

9.3.2 Reduced-Order Observers: Continuous-Time Systems

Suppose that p states, out of the n state, can be measured directly. This information can then be used to reduce the order of the full-state estimator from n to $n - p$. Similar results are true for the estimator of a linear function of the state, but this problem will not be addressed here. To determine a full-state estimator of order $n - p$, first consider the case when $C = [I_p, 0]$. In particular, let

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$z = [I_p, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{9.62}$$

where $z = x_1$ represents the p measured states. Therefore, only $x_2(t) \in R^{(n-p)\times 1}$ is to be estimated. The system whose state is to be estimated is now given by

$$\begin{aligned}\dot{x}_2 &= A_{22}x_2 + [A_{21}, B_2]\begin{bmatrix} x_1 \\ u \end{bmatrix} \\ &= A_{22}x_2 + \widetilde{B}\tilde{u},\end{aligned} \tag{9.63}$$

where $\widetilde{B} \triangleq [A_{21}, B_2]$ and $\tilde{u} \triangleq \begin{bmatrix} x_1 \\ u \end{bmatrix} = \begin{bmatrix} z \\ u \end{bmatrix}$ is a known signal. Also,

$$\tilde{y} \triangleq \dot{x}_1 - A_{11}x_1 - B_1 u = A_{12}x_2, \tag{9.64}$$

where $\tilde{y}$ is known. An estimator for x_2 can now be constructed. In particular, in view of (9.57), we have that the system

$$\begin{aligned}\dot{\hat{x}}_2 &= A_{22}\hat{x}_2 + \widetilde{B}\tilde{u} + \widetilde{K}(\tilde{y} - A_{12}\hat{x}_2) \\ &= (A_{22} - \widetilde{K}A_{12})\hat{x}_2 + (A_{21}z + B_2 u) + \widetilde{K}(\dot{z} - A_{11}z - B_1 u)\end{aligned} \tag{9.65}$$

is an asymptotic state estimator for x_2. Note that the error e satisfies the equation

$$\dot{e} = \dot{x}_2 - \dot{\hat{x}}_2 = (A_{22} - \widetilde{K}A_{12})e, \tag{9.66}$$

and if (A_{22}, A_{12}) is observable, then the eigenvalues of $A_{22} - \widetilde{K}A_{12}$ can be arbitrarily assigned making use of $\widetilde{K}$. It can be shown that if the pair $(A = [A_{ij}], C = [I_p, 0])$ is observable, then (A_{22}, A_{12}) is also observable (prove this using the eigenvalue observability test of Section 6.3). System (9.65) is an estimator of order $n - p$, and therefore, the estimate of the entire state x is $\begin{bmatrix} z \\ \hat{x}_2 \end{bmatrix}$. To avoid using $\dot{z} = \dot{x}_1$ in $\tilde{y}$ given by (9.64), one could use $\hat{x}_2 = w + \widetilde{K}z$ and obtain from (9.65) an estimator in terms of w, z, and u. In particular,

$$\dot{w} = (A_{22} - \widetilde{K}A_{12})w + [(A_{22} - \widetilde{K}A_{12})\widetilde{K} + A_{21} - \widetilde{K}A_{11}]z + [B_2 - \widetilde{K}B_1]u. \tag{9.67}$$

Then w is an estimate of $\hat{x}_2 - \widetilde{K}z$ and of course $w + \widetilde{K}z$ is an estimate for $\hat{x}_2$.

In the above derivation, it was assumed for simplicity that a part of the state x_1, is measured directly; i.e., $C = [I_p, 0]$. One could also derive a reduced-order estimator for the system

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

To see this, let $\operatorname{rank} C = p$ and define a similarity transformation matrix $P = \begin{bmatrix} C \\ \widehat{C} \end{bmatrix}$, where $\widehat{C}$ is such that P is nonsingular. Then

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x} = [I_p, 0]\bar{x}, \tag{9.68}$$

where $\bar{x} = Px, \bar{A} = PAP^{-1}, \bar{B} = PB$, and $\bar{C} = CP^{-1} = [I_p, 0]$. The transformed system is now in an appropriate form for an estimator of order $n - p$ to be derived, using the procedure discussed above. The estimate of $\bar{x}$ is

$\begin{bmatrix} y \\ \hat{\bar{x}}_2 \end{bmatrix}$, and the estimate of the original state x is $P^{-1}\begin{bmatrix} y \\ \hat{\bar{x}}_2 \end{bmatrix}$. In particular, $\bar{x}_2 = w + \widetilde{K}y$, where w satisfies (9.67) with $z = y$, $[A_{ij}] = \bar{A} = PAP^{-1}$, and $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \bar{B} = PB$. The interested reader should verify this result.

Example 9.21. Consider the system $\dot{x} = Ax + Bu$, $y = Cx$, where $A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [0, 1]$. We wish to design a reduced $n - p = n - 1 = 2 - 1 = 1$, a first-order asymptotic state estimator.

The similarity transformation matrix $P = \begin{bmatrix} C \\ \widehat{C} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ leads to (9.68), where $\bar{x} = Px$ and $\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$, $\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\bar{C} = CP^{-1} = [1, 0]$. The system $\{\bar{A}, \bar{B}, \bar{C}\}$ is now in an appropriate form for use of (9.67). We have $\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and (9.67) assumes the form

$$\dot{w} = (-\widetilde{K})w + [-\widetilde{K}^2 + (-2) - \widetilde{K}(-2)]y + (-\widetilde{K})u,$$

which is a system observer of order 1.

For $\widetilde{K} = -10$ we have $\dot{w} = 10w - 122y + 10u$, and $w + \widetilde{K}y = w - 10y$ is an estimate for $\hat{\bar{x}}_2$. Therefore, $\begin{bmatrix} y \\ w - 10y \end{bmatrix}$ is an estimate of $\bar{x}$, and

$$P^{-1}\begin{bmatrix} y \\ w - 10y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ w - 10y \end{bmatrix} = \begin{bmatrix} w - 10y \\ y \end{bmatrix}$$

is an estimate of $x(t)$ for the original system.

9.3.3 Optimal State Estimation: Continuous-Time Systems

The gain K in the estimator (9.57) above can be determined so that it is optimal in an appropriate sense. This is discussed briefly below. The interested reader should consult the extensive literature on *filtering theory* for additional information, in particular, the literature on the *Kalman–Bucy filter*.

In addressing optimal state estimation, noise with certain statistical properties is introduced in the model and an appropriate cost functional is set up that is then minimized. In the following discussion, we shall introduce some of the key equations of the *Kalman–Bucy filter* and we will point out the duality between the optimal control and estimation problems. We concentrate on the time-invariant case, although, as in the LQR control problem discussed earlier, more general results for the time-varying case do exist.

We consider the linear time-invariant system

$$\dot{x} = Ax + Bu + \Gamma w, y = Cx + v, \tag{9.69}$$

where w and v represent process and measurement noise terms. Both w and v are assumed to be white, zero-mean Gaussian stochastic processes; i.e., they are uncorrelated in time and have expected values $E[w] = 0$ and $E[v] = 0$. Let

$$E[ww^T] = W, \quad E[vv^T] = V \tag{9.70}$$

denote their covariances, where W and V are real, symmetric, and positive definite matrices, i.e., $W = W^T, W > 0$, and $V = V^T, V > 0$. Assume that the noise processes w and v are independent; i.e., $E[wv^T] = 0$. Also assume that the initial state $x(0)$ of the plant is a Gaussian random variable of known mean, $E[x(0)] = x_0$, and known covariance, $E[(x(0) - x_0)(x(0) - x_0)^T] = P_{e0}$. Assume also that $x(0)$ is independent of w and v. Note that all these are typical assumptions made in practice.

Consider now the estimator (9.57), namely,

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) = (A - KC)\hat{x} + Bu + Ky, \tag{9.71}$$

and let $(A, \Gamma W^{1/2}, C)$ be controllable (-from-the-origin) and observable. It turns out that the error covariance $E[(x - \hat{x})(x - \hat{x})^T]$ is minimized when the filter gain is given by

$$K^* = P_e^* C^T V^{-1}, \tag{9.72}$$

where P_e^* denotes the symmetric, positive definite solution of the quadratic *(dual) algebraic Riccati equation*

$$P_e A^T + AP_e - P_e C^T V^{-1} C P_e + \Gamma W \Gamma^T = 0. \tag{9.73}$$

Note that P_e^*, which is in fact the minimum error covariance, is the positive semidefinite solution of the above Riccati equation if $(A, \Gamma W^{1/2}, C)$ is stabilizable and detectable. The optimal estimator is asymptotically stable.

The above algebraic Riccati equation is the dual to the Riccati equation given in (9.33) for optimal control and can be obtained from (9.33) making use of the substitutions

$$A \to A^T, B \to C^T, M \to \Gamma^T \quad \text{and} \quad R \to V, Q \to W. \tag{9.74}$$

Clearly, methods that are analogous to the ones developed by solving the control Riccati equation (9.33) may be applied to solve the Riccati equation (9.71) in filtering. These methods are not discussed here.

Example 9.22. Consider the system $\dot{x} = Ax$, $y = Cx$, where $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = [0,\ 1]$, and let $\Gamma = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V = \rho > 0$, $W = 1$. We wish to derive the optimal filter gain $K^* = P_e^* C^T V^{-1}$ given in (9.72). In this case, the Riccati equation (9.73) is precisely the Riccati equation of the control problem given in Example 9.14. The solution of this equation was determined to be

$$P_e^* = \begin{bmatrix} \sqrt{2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2\rho\sqrt{\rho}} \end{bmatrix}.$$

We note that this was expected, since our example was chosen to satisfy (9.74). Therefore,

$$K^* = P_e^* \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{\rho} = \begin{bmatrix} \sqrt{\rho} \\ \sqrt{2\rho\sqrt{\rho}} \end{bmatrix} \frac{1}{\rho}.$$

9.3.4 Full-Order Observers: Discrete-Time Systems

We consider systems described by equations of the form

$$x(k+1) = Ax(k) + Bu(k), \quad y = Cx(k) + Du(k), \tag{9.75}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times m}$, and $D \in R^{p\times m}$.

The construction of state estimators for discrete-time systems is mostly analogous to the continuous-time case, and the results that we established above for such systems are valid here as well, subject to obvious adjustments and modifications. There are, however, some notable differences. For example, *in discrete-time systems, it is possible to construct a state estimator that converges to the true value of the state in finite time, instead of infinite time as in the case of asymptotic state estimators.* This is the estimator known as the *deadbeat observer*. Furthermore, in discrete-time systems it is possible to talk about *current state estimators, in addition to prediction state estimators.* In what follows, a brief description of the results that are analogous to the continuous-time case is given. Current estimators and deadbeat observers that are unique to the discrete-time case are discussed at greater length.

Full-State Observers: The Identity Observer

As in the continuous-time case, following (9.57) we consider systems described by equations of the form

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + K[y(k) - \hat{y}(k)], \tag{9.76}$$

where $\hat{y}(k) \triangleq C\hat{x}(k) + Dx(k)$. This can also be written as

$$\hat{x}(k+1) = (A - KC)\hat{x}(k) + [B - KD, K] \begin{bmatrix} u(k) \\ y(k) \end{bmatrix}. \tag{9.77}$$

It can be shown that the error $e(k) \triangleq x(k) - \hat{x}(k)$ obeys the equation $e(k+1) = (A - KC)e(k)$. Therefore, if the eigenvalues of $A - KC$ are inside the open unit disk of the complex plane, then $e(k) \to 0$ as $k \to \infty$. There exists K so that the eigenvalues of $A - KC$ can be arbitrarily assigned if and only if the pair (A, C) is observable (see Lemma 9.18).

The discussion following Lemma 9.18 for the case when (A, C) is not completely observable, although detectable, is still valid. Also, the remarks on appropriate locations for the eigenvalues of $A - KC$ and noise being the limiting factor in state estimators are also valid in the present case. Note that the latter point should seriously be considered when deciding whether to use the deadbeat observer described next.

To balance the tradeoffs between speed of the estimator response and noise amplification, one may formulate an *optimal estimation problem* as was done in the continuous-time case, the *Linear Quadratic Gaussian (LQG)* design being a common formulation. The Kalman filter (discrete-time case) that is based on the "current estimator" described below is such a quadratic estimator. The LQG optimal estimation problem can be seen to be the dual of the quadratic optimal control problem discussed in the previous section. As in the continuous-time case, optimal estimation in the discrete-time case will be discussed only briefly as follows. First, however, several other related issues are addressed.

Deadbeat Observer

If the pair (A, C) is observable, it is possible to select K so that all the eigenvalues of $A - KC$ are at the origin. In this case $e(k) = x(k) - \hat{x}(k) = (A - KC)^k e(0) = 0$, for some $k \leq n$; i.e., the error will be identically zero within at most n steps. The minimum value of k for which $(A - KC)^k = 0$ depends on the size of the largest block on the diagonal of the Jordan canonical form of $A - KC$. (Refer to the discussion on the modes of discrete-time systems in Subsection 3.5.5.)

Example 9.23. Consider the system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$, where

$$A = \left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad C = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

is in observer form. We wish to design a deadbeat observer. It is rather easy to show (compare with Example 9.11) that

$$K = \left[A_{dm}^T - \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix},$$

which was determined by taking the dual $A_D = A^T, B_D = C^T$ in controller form, using $F_D = B_m^{-1}[A_{d_m} - A_m]$ and $K = -F_D^T$.

The matrix $A_{d_m}^T$ consists of the second and third columns of a matrix $A_d = \left[\begin{array}{c|cc} 0 & \times & \times \\ 1 & \times & \times \\ 0 & \times & \times \end{array}\right]$ in observer (companion) form with all its eigenvalues at 0.

For $A_{d_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we have

$$K_1 = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ -1 & 0 \end{bmatrix},$$

and for $A_{d_2} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we obtain

$$K_2 = \left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $A - K_1C = A_{d_1}$, $A_{d_1}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $A_{d_1}^3 = 0$, and that $A - K_2C = A_{d_2}$ and $A_{d_2}^2 = 0$. Therefore, for the observer gain K_1, the error $e(k)$ in the deadbeat observer will become zero in $n = 3$ steps, since $e(3) = (A - K_1C)^3 e(0) = 0$. For the observer gain K_2, the error $e(k)$ in the deadbeat observer will become zero in $2 < n$ steps, since $e(2) = (A - K_2C)^2 e(0) = 0$. The reader should determine the Jordan canonical forms of A_{d_1} and A_{d_2} and verify that the dimension of the largest block on the diagonal is 3 and 2, respectively.

The comments in the discussion following Lemma 9.18 on taking $K = 0$ are valid in the discrete-time case as well. Also, the approach of determining the state instantaneously in the continuous-time case, using the derivatives of the input and output, corresponds in the discrete-time case to determining the state from current and future input and output values (see Exercise 5.12 in Chapter 5). This approach was in fact used to determine $x(0)$ when studying observability in Section 5.4. The disadvantage of this method is that it requires future measurements to calculate the current state. This issue of using future or past measurements to determine the current state is elaborated upon next.

Current Estimator

The estimator (9.76) is called a *prediction estimator*. The state estimate $\hat{x}(k)$ is based on measurements up to and including $y(k-1)$. It is often of interest in applications to determine the state estimate $\hat{x}(k)$ based on measurements up to and including $y(k)$. This may seem rather odd at first; however, if the computation time required to calculate $\hat{x}(k)$ is short compared with the sample period in a sampled-data system, then it is certainly possible practically to determine the estimate $\hat{x}(k)$ before $x(k+1)$ and $y(k+1)$ are generated by the system. If this state estimate, which is based on current measurements of $y(k)$, is to be used to control the system, then the unavoidable computational delays should be taken into consideration.

Now let $\bar{x}(k)$ denote the current state estimate based on measurements up through $y(k)$. Consider the current estimator

$$\bar{x}(k) = \hat{x}(k) + K_c(y(k) - C\hat{x}(k)), \tag{9.78}$$

where

$$\hat{x}(k) = A\bar{x}(k-1) + Bu(k-1); \tag{9.79}$$

i.e., $\hat{x}(k)$ denotes the estimate based on model prediction from the previous time estimate, $\bar{x}(k-1)$. Note that in (9.78), the error is $y(k) - \hat{y}(k)$, where $\hat{y}(k) = C\hat{x}(k)$ $(D = 0)$, for simplicity.

Combining the above, we obtain

$$\hat{x}(k) = (I - K_cC)A\bar{x}(k-1) + [(I - K_cC)B, -K_c]\begin{bmatrix} u(k-1) \\ y(k) \end{bmatrix}. \tag{9.80}$$

The relation to the prediction estimator (9.76) can be seen by substituting (9.78) into (9.79) to obtain

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + AK_c[y(k) - C\hat{x}(k)]. \tag{9.81}$$

Comparison with the prediction estimator (9.76) (with $D = 0$) shows that it is clear that if

$$K = AK_c, \tag{9.82}$$

then (9.81) is indeed the prediction estimator, and the estimate $\hat{x}(k)$ used in the current estimator (9.78) is indeed the prediction state estimate. In view of this, we expect to obtain for the error $\hat{e}(k) = x(k) - \hat{x}(k)$ the difference equation

$$\hat{e}(k+1) = (A - AK_cC)\hat{e}(k). \tag{9.83}$$

To determine the error $\bar{e}(k) = x(k) - \bar{x}(k)$ we note that $\bar{e}(k) = \hat{e}(k) - (\bar{x}(k) - \hat{x}(k))$. Equation (9.78) now implies that $\bar{x}(k) - \hat{x}(k) = K_cCe(k)$. Therefore,

$$\bar{e}(k) = (I - K_cC)\hat{e}(k). \tag{9.84}$$

This establishes the relationship between errors in current and prediction estimators.

Premultiplying (9.81) by $I - K_cC$ (assuming $|I - K_cC| \neq 0$), we obtain

$$\bar{e}(k+1) = (A - K_cCA)\bar{e}(k), \tag{9.85}$$

which is the current estimator error equation. The gain K_c is chosen so that the eigenvalues of $A - K_cCA$ are within the open unit disk of the complex plane. The pair (A, CA) must be observable for arbitrary eigenvalue assignment. Note that the two error equations (9.83) and (9.85) have identical eigenvalues.

Example 9.24. Consider the system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$, where $A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$, $C = [0,\ 1]$, which is in observer form (see also Example 9.20). We wish to design a current estimator. In view of the error equation (9.85), we consider

$$\begin{aligned} \det(sI - (A - K_cCA)) &= \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} k_0 \\ k_1 \end{bmatrix} [1 \ -2]\right)\right) \\ &= \det\begin{bmatrix} s + k_0 & 2 - 2k_0 \\ k_1 - 1 & s + 2 - 2k_1 \end{bmatrix} \\ &= s^2 + s(2 - 2k_1 + k_0) + (2 - 2k_1) \\ &= s^2 + d_1 s + d_0 = \alpha_d(s), \end{aligned}$$

a desired polynomial, from which $K_c = [k_0, k_1]^T = [d_1 - d_0, \frac{1}{2}(2 - d_0)]^T$. Note that $AK_c = [d_0 - 2, d_1 - 2]^T = K$, found in Example 9.20, as noted in (9.82).

The current estimator (9.80) is now given by $\bar{x}(k) = (A - K_cCA)\bar{x}(k-1) - K_cCBu(k-1) + K_cy(k)$, or

$$\bar{x}(k) = \begin{bmatrix} -k_0 & -2 + 2k_0 \\ 1 - k_1 & -2 + 2k_1 \end{bmatrix} \bar{x}(k-1) + \begin{bmatrix} k_0 \\ k_1 \end{bmatrix} y(k).$$

Partial or Linear Functional State Observers

The problem of estimating a linear function of the state, $Tx(k)$, $T \in R^{\tilde{n} \times n}$, where $\tilde{n} \leq n$, using a prediction estimator, is completely analogous to the continuous-time case.

9.3.5 Reduced-Order Observers: Discrete-Time Systems

It is possible to estimate the full state $x(k)$ using an estimator of order $n - p$, where $p = \operatorname{rank} C$. If a prediction estimator is used for that part of the state that needs to be estimated, then the problem in the discrete-time case is completely analogous to the continuous-time case, discussed before.

9.3.6 Optimal State Estimation: Discrete-Time Systems

The formulation of the Kalman filtering problem in discrete-time is analogous to the continuous-time case.

Consider the linear time-invariant system given by

$$x(k+1) = Ax(k) + Bu(k) + \Gamma w(k), \quad y(k) = Cx(k) + v, \tag{9.86}$$

where the process and measurement noises w, v are white, zero-mean Gaussian stochastic processes; i.e., they are uncorrelated in time with $E[w] = 0$ and $E[v] = 0$. Let the covariances be given by

$$E[ww^T] = W, \quad E[vv^T] = V, \tag{9.87}$$

where $W = W^T, W > 0$ and $V = V^T, V > 0$. Assume that w, v are independent, that the initial state $x(0)$ is Gaussian of known mean ($E[x(0)] = x_0$), that $E[(x(0) - x_0)(x(0) - x_0)^T] = P_{e0}$, and that $x(0)$ is independent of w and v.

Consider now the current estimator (9.76), namely,

$$\bar{x}(k) = \hat{x}(k) + K_c[y(k) - C\hat{x}(k)],$$

where $\hat{x}(k) = A\bar{x}(k-1) + Bu(k-1)$ and $\hat{x}(k)$ denotes the prior estimate of the state at the time of a measurement.

It turns out that the state error covariance is minimized when the filter gain is

$$K_c^* = P_e^* C^T (C P_e^* C^T + V)^{-1}, \tag{9.88}$$

where P_e^* is the unique, symmetric, positive definite solution of the Riccati equation

$$P_e = A[P_e - P_e C^T [C P_e C^T + V]^{-1} C P_e] A^T + \Gamma W \Gamma^T. \tag{9.89}$$

It is assumed here that $(A, \Gamma W^{1/2}, C)$ is reachable and observable. This *algebraic Riccati equation* is the dual to the Riccati equation (9.53) that arose in the discrete-time LQR problem and can be obtained by substituting

$$A \to A^T, B \to C^T, M \to \Gamma^T \quad \text{and} \quad R \to V, Q \to W. \tag{9.90}$$

It is clear that, as in the case of the LQR problem, the solution of the algebraic Riccati equation can be determined using the eigenvectors of the (dual) Hamiltonian.

The filter derived above is called the *discrete-time Kalman filter*. It is based on the current estimator (9.78). Note that AK_c yields the gain K of the prediction estimator [see (9.82)].

9.4 Observer-Based Dynamic Controllers

State estimates, derived by the methods described in the previous section, may be used in state feedback control laws to compensate given systems. This section addresses this topic.

In Section 9.2, the linear state feedback control law was introduced. There it was implicitly assumed that the state vector $x(t)$ is available for measurement. The values of the states $x(t)$ for $t \geq t_0$ were fed back and used to generate a control input in accordance with the relation $u(t) = Fx(t) + r(t)$. There are cases, however, when it may be either impossible or impractical to measure the states directly. This has provided the motivation to develop methods for estimating the states. Some of these methods were considered in Section 9.3. A natural question that arises at this time is the following: What would happen to system performance if, in the control law $u = Fx + r$, the state estimate $\hat{x}$ were used in place of x as in Figure 9.4? How much, if any, would the compensated system response deteriorate? What are the difficulties in designing such estimator-(observer-)based linear state feedback controllers? These questions are addressed in this section. Note that observer-based controllers of the type described in the following are widely used.

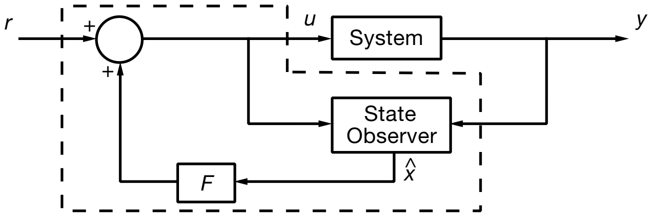

Figure 9.4. Observer-based controller

In the remainder of this section we will concentrate primarily on full-state/full-order observers and (static) linear state feedback, as applied to linear time-invariant systems. The analysis of partial-state and/or reduced-order observers with static or dynamic state feedback is analogous; however, it is more complex. In this section, continuous-time systems are addressed. The discrete-time case is completely analogous and will be omitted.

9.4.1 State-Space Analysis

We consider systems described by equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{9.91}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, and $D \in R^{p\times m}$. For such systems, we determine an estimate $\hat{x}(t) \in R^n$ of the state $x(t)$ via the (full-state/full-order) *state observer* (9.57) given by

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K(y - \hat{y}) \\ &= (A - KC)\hat{x} + [B - KD, K]\begin{bmatrix} u \\ y \end{bmatrix}, \\ z &= \hat{x}, \end{aligned} \tag{9.92}$$

where $\hat{y} = C\hat{x} + Du$. We now compensate the system by *state feedback* using the control law

$$u = F\hat{x} + r, \tag{9.93}$$

where $\hat{x}$ is the output of the state estimator and we wish to analyze the behavior of the compensated system. To this end we first eliminate y in (9.92) to obtain

$$\dot{\hat{x}} = (A - KC)\hat{x} + KCx + Bu. \tag{9.94}$$

The state equations of the compensated system are then given by

$$\begin{aligned} \dot{x} &= Ax + BF\hat{x} + Br, \\ \dot{\hat{x}} &= KCx + (A - KC + BF)\hat{x} + Br, \end{aligned} \tag{9.95}$$

and the output equation assumes the form

$$y = Cx + DF\hat{x} + Dr, \tag{9.96}$$

where u was eliminated from (9.91) and (9.94), using (9.93). Rewriting in matrix form, we have

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BF \\ KC & A - KC + BF \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} r, \\ y &= [C,\ DF] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + Dr, \end{aligned} \tag{9.97}$$

which is a representation of the compensated closed-loop system. Note that (9.97) constitutes a $2n$th-order system. Its properties are more easily studied if an appropriate similarity transformation is used to simplify the representation. Such a transformation is given by

$$P \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix}, \tag{9.98}$$

where the error $e(t) = x(t) - \hat{x}(t)$. Then the equivalent representation is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} A + BF & -BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \\ y &= [C + DF,\ -DF] \begin{bmatrix} x \\ e \end{bmatrix} + Dr. \end{aligned} \tag{9.99}$$

It is now quite clear that the closed-loop system is not fully controllable with respect to r (this can be explained in view of Subsection 6.2.1). In fact, $e(t)$ does not depend on r at all. This is of course as it should be, since the error $e(t) = x(t) - \hat{x}(t)$ should converge to zero independently of the externally applied input r.

The closed-loop eigenvalues are the roots of the polynomial

$$|sI_n - (A + BF)||sI_n - (A - KC)|. \tag{9.100}$$

Recall that the roots of $|sI_n - (A + BF)|$ are the eigenvalues of $A + BF$ that can arbitrarily be assigned via F provided that the pair (A, B) is controllable. These are in fact the closed-loop eigenvalues of the system when the state x is available and the linear state feedback control law $u = Fx + r$ is used (see Section 9.2). The roots of $|sI_n - (A-KC)|$ are the eigenvalues of $(A-KC)$ that can arbitrarily be assigned via K provided that the pair (A, C) is observable. These are the eigenvalues of the estimator (9.92).

The above discussion points out that the *design of the control law (9.93) can be carried out independently of the design of the estimator (9.92).* This is referred to as the *Separation Property* and is generally not true for more complex systems. The separation property indicates that the linear state feedback control law may be designed as if the state x were available and the eigenvalues of $A + BF$ are assigned at appropriate locations. The feedback matrix F can also be determined by solving an optimal control problem (LQR). If state measurements are not available for feedback, a state estimator is employed. The eigenvalues of a full-state/full-order estimator are given by the eigenvalues of $A-KC$. These are typically assigned so that the error $e(t) = x(t)-\hat{x}(t)$ becomes adequately small in a short period of time. For this reason, the eigenvalues of $A - KC$ are (empirically) taken to be about 6 to 10 times further away from the imaginary axis (in the complex plane, for continuous-time systems) than the eigenvalues of $A + BF$. The estimator gain K may also be determined by solving an optimal estimation problem (the Kalman filter). In fact, under the assumption of Gaussian noise and initial conditions given earlier (see Section 9.3), F and K can be found by solving, respectively, optimal control and estimation problems with quadratic performance criteria. In particular, the deterministic LQR problem is first solved to determine the optimal control gain F^*, and then the stochastic Kalman filtering problem is solved to determine the optimal filter gain K^*. The separation property (i.e., Separation Theorem—see any optimal control textbook) guarantees that the overall *(state estimate feedback) Linear Quadratic Gaussian (LQG) control design* is optimal in the sense that the control law $u^*(t) = F^*\hat{x}(t)$ minimizes the quadratic performance index $E[\int_0^\infty (z^T Qz+u^T Ru)dt]$. As was discussed in previous sections, the gain matrices F^* and K^* are evaluated in the following manner.

Consider

$$\dot{x} = Ax + Bu + \Gamma w, \quad y = Cx + v, z = Mx \tag{9.101}$$

with $E\{ww^T\} = W > 0$ and $E[vv^T] = V > 0$ and with $Q > 0, R > 0$ denoting the matrix weights in the performance index $E[\int_0^\infty (z^T Qx+u^T Ru)dt]$. Assume that both $(A, B, Q^{1/2}M)$ and $(A, \Gamma W^{1/2}, C)$ are controllable and observable. Then the optimal control law is given by

$$u^*(t) = F^*\hat{x}(t) = -R^{-1}B^T P_c^*\hat{x}(t), \tag{9.102}$$

where $P_c^* > 0$ is the solution of the algebraic Riccati equation (9.33) given by

$$A^T P_c + P_c A - P_c B R^{-1} B^T P_c + M^T Q M = 0. \tag{9.103}$$

The estimate $\hat{x}$ is generated by the optimal estimator

$$\dot{\hat{x}} = A\hat{x} + Bu + K^*(y - C\hat{x}), \tag{9.104}$$

where

$$K^* = P_c^* C^T V^{-1}, \tag{9.105}$$

in which $P_e^* > 0$ is the solution to the dual algebraic Riccati equation (9.71) given by

$$P_e A^T + A P_e - P_e C^T V^{-1} C P_e + \Gamma W \Gamma^T = 0. \tag{9.106}$$

Designing observer-based dynamic controllers by the LQG control design method has been quite successful, especially when the plant model is accurately known. In this approach the weight matrices Q, R and the covariance matrices W, V are used as design parameters. Unfortunately, this method does not necessarily lead to robust designs when uncertainties are present. This limitation has led to an enhancement of this method, called the LQR/LTR (Loop Transfer Recovery) method, where the design parameters W and V are selected (iteratively) so that the robustness properties of the LQR design are recovered.

Finally, as mentioned earlier, the discrete-time case is analogous to the continuous-time case and its discussion will be omitted.

Example 9.25. Consider the system $\dot{x} = Ax + Bu$, $y = Cx$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = [1, 0, 0].$$

This is a controllable and observable but unstable system with eigenvalues of A equal to $0, -2, 1$. A linear state feedback control $u = Fx + r$ was derived in Example 9.11 to assign the eigenvalues of $A+BF$ at $-2, -1 \pm j$. An appropriate F to accomplish this was shown to be

$$F = \begin{bmatrix} 2 & -1 & -2 \\ -2 & 0 & 1/2 \end{bmatrix}.$$

If the state $x(t)$ is not available for measurement, then an estimate $\hat{x}(t)$ is used instead; i.e., the control law $u = F\hat{x} + r$ is employed. In Example 9.19, a full-order/full-state observer, given by

$$\dot{\hat{x}} = (A - KC)\hat{x} + [B, K] \begin{bmatrix} u \\ y \end{bmatrix},$$

was derived [see (9.58)] with the eigenvalues of $A - KC$ determined as the roots of the polynomial $\alpha_d(s) = s^3 + d_2 s^2 + d_1 s + d_0$. It was shown that the (unique) K is in this case

$$K = [d_2 - 1, d_1 - d_2 + 3, d_0 - d_1 + 3d_2 - 5]^T,$$

and the observer is given by

$$\dot{\hat{x}} = \begin{bmatrix} 1 - d_2 & 1 & 0 \\ -d_1 + d_2 - 3 & 0 & 1 \\ -d_0 + d_1 - 3d_2 + 5 & 2 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & 1 & d_2 - 1 \\ 1 & 1 & d_1 - d_2 + 3 \\ 0 & 0 & d_0 - d_1 + 3d_2 - 5 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}.$$

Using the estimate $\hat{x}$ in place of the control state x in the feedback control law causes some deterioration in the behavior of the system. This deterioration can be studied experimentally. (See the next subsection for analytical results.) To this end, let the eigenvalues of the observer be at, say, $-10, -10, -10$; let $x(0) = [1,\ 1,\ 1]^T$ and $\hat{x}(0) = [0,\ 0,\ 0]^T$; plot $x(t)$, $\hat{x}(t)$, and $e(t) = x(t) - \hat{x}(t)$; and compare these with the corresponding plots of Example 9.11, where no observer was used. Repeat the above with observer eigenvalues closer to the eigenvalues of $A + BF$ (say, at $-2, -1 \pm j$) and also further away. In general the faster the observer, the faster $e(t) \to 0$, and the smaller the deterioration of response; however, in this case, care should be taken if noise is present in the system.

9.4.2 Transfer Function Analysis

For the compensated system (9.99) [or (9.97)], the closed-loop transfer function $T(s)$ between y and r is given by

$$\tilde{y}(s) = T(s)\tilde{r}(s) = [(C + DF)[sI - (A + BF)]^{-1}B + D]\tilde{r}(s), \qquad (9.107)$$

where $\tilde{y}(s)$ and $\tilde{r}(s)$ denote the Laplace transforms of $y(t)$ and $r(t)$, respectively. The function $T(s)$ was found from (9.99), using the fact that the uncontrollable part of the system does not appear in the transfer function (see Section 7.2). Note that $T(s)$ is the transfer function of $\{A+BF, B, C+DF, D\}$; i.e., $T(s)$ is precisely the transfer function of the closed-loop system $H_F(s)$ when no state estimation is present (see Section 9.2). Therefore, the compensated system behaves to the outside world as if there were no estimator present. *Note that this statement is true only after sufficient time has elapsed from the initial time, allowing the transients to become negligible.* (Recall what the transfer function represents in a system.) Specifically, taking Laplace transforms in (9.99) and solving, we obtain

$$\begin{bmatrix} \tilde{x}(s) \\ \tilde{e}(s) \end{bmatrix} = \begin{bmatrix} [sI-(A+BF)]^{-1} & -[sI-(A+BF)]^{-1}BF[sI-(A-KC)]^{-1} \\ 0 & [sI-(A+BF)]^{-1} \end{bmatrix} \begin{bmatrix} x(0) \\ e(0) \end{bmatrix} + \begin{bmatrix} [sI-(A+BF)]^{-1}B \\ 0 \end{bmatrix} \tilde{r}(s),$$

$$\tilde{y}(s) = [C + DF, -DF] \begin{bmatrix} \tilde{x}(s) \\ \tilde{e}(s) \end{bmatrix} + D\tilde{r}(s). \qquad (9.108)$$

Therefore,

$$
\begin{aligned}
\tilde{y}(s) = {} & (C+DF)[sI-(A+BF)]^{-1}x(0) \\
& -[(C+DF)[sI-(A+BF)]^{-1}BF[sI-(A-KC)]^{-1} \\
& +DF[sI-(A+BF)]^{-1}]e(0)+T(s)\tilde{r}(s),
\end{aligned}
\tag{9.109}
$$

which indicates the effects of the estimator on the input–output behavior of the closed-loop system. Notice how the initial conditions for the error $e(0) = x(0) - \hat{x}(0)$ influence the response. Specifically, when $e(0) \neq 0$, its effect can be viewed as a disturbance that will become negligible at steady state. The speed by which the effect of $e(0)$ on y will diminish depends on the location of the eigenvalues of $A+BF$ and $A-KC$, as can be easily seen from relation (9.109).

Two-Input Controller

In the following discussion, we will find it of interest to view the observer-based controller discussed previously as a one-vector output (u) and a two-vector input (y and r) controller. In particular, from $\dot{\hat{x}} = (A-KC)\hat{x} + (B-KD)u + Ky$ given in (9.92) and $u = F\hat{x} + r$ given in (9.93), we obtain the equations

$$
\begin{aligned}
\dot{\hat{x}} &= (A-KC+BF-KDF)\hat{x} + [K, B-KD]\begin{bmatrix} y \\ r \end{bmatrix}, \\
u &= F\hat{x} + r.
\end{aligned}
\tag{9.110}
$$

This is the description of the (nth order) controller shown in Figure 9.4. The state $\hat{x}$ is of course the state of the estimator, and the transfer function between u and y, r is given by

$$
\begin{aligned}
\tilde{u}(s) = {} & F[sI-(A-KC+BF-KDF)]^{-1}K\tilde{y}(s) \\
& +[F[sI-(A-KC+BF-KDF)]^{-1}(B-KD)+I]\tilde{r}(s).
\end{aligned}
\tag{9.111}
$$

If we are interested only in "loop properties," then r can be taken to be zero; in which case, (9.111) (for $r = 0$) yields the output feedback compensator, which accomplishes the same control objectives (that are typically only "loop properties") as the original observer-based controller. This fact is used in the LQG/LTR design approach. When $r \neq 0$, (9.111) is not appropriate for the realization of the controller since the transfer function from r, which must be outside the loop, may be unstable. Note that an expression for this controller that leads to a realization of a stable closed-loop system is given by

$$
\tilde{u}(s) = [F[sI-(A-KC+BF-KDF)]^{-1}[K, B-KD]+[0, I]]\begin{bmatrix} \tilde{y}(s) \\ \tilde{r}(s) \end{bmatrix} \tag{9.112}
$$

(see Figure 9.5). This was also derived from (9.110). The stability of general two-input controllers (with two degrees of freedom) is discussed at length in Chapter 10.

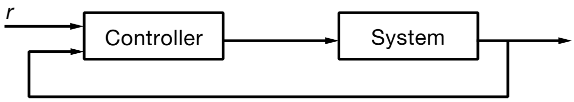

Figure 9.5. Two-input controller

At this point, we find it of interest to determine the relationship of the observer-based controller and the conventional block controller configuration of Figure 9.6. Here, the requirement is to maintain the same transfer functions between inputs y and r and output u. (For further discussion of stability and attainable response maps in system controlled by output feedback controllers, refer to Chapter 10.) We proceed by considering once more (9.92) and (9.93) and by writing

$$\begin{aligned}\tilde{u}(s) &= F[sI-(A-KC)]^{-1}(B-KD)\tilde{u}(s)\\ &\quad + F[sI-(A-KC)]^{-1}K\tilde{y}(s)+\tilde{r}(s) = G_u\tilde{u}(s)+G_y\tilde{y}(s)+\tilde{r}(s).\end{aligned}$$

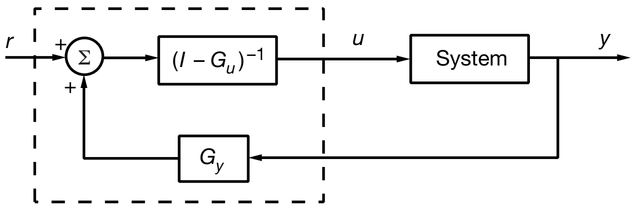

Figure 9.6. Conventional block controller configuration

This yields

$$\tilde{u}(s) = (I-G_u)^{-1}[G_y\tilde{y}(s)+\tilde{r}(s)] \tag{9.113}$$

(see Figure 9.6). Notice that

$$G_y = F[sI-(A-KC)]^{-1}K; \tag{9.114}$$

i.e., the controller in the feedback path is stable. The matrix $(I-G_u)^{-1}$ is not necessarily stable; however, it is inside the loop and therefore the internal stability of the compensated system is preserved. Comparing with (9.111), we obtain

$$(I-G_u)^{-1} = F[sI-(A-KC+BF-KDF)]^{-1}(B-KD)+I. \tag{9.115}$$

Also, as expected, we have

$$(I-G_u)^{-1}G_y = F[sI-(A-KC+BF-KDF)]^{-1}K. \tag{9.116}$$

These relations could have been derived directly as well by the use of matrix identities; however, such derivation is quite involved.

Example 9.26. For the system $\dot{x} = Ax + Bu$, $y = Cx$ with $A = \begin{bmatrix} 0 & -2 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [0, 1]$, we have $H(s) = C(sI - A)^{-1}B = \frac{s}{s^2+2s+2}$. In Example 9.20, it was shown that the gain matrix $K = [d_0 - 2, d_1 - 2]^T$ assigns the eigenvalues of the asymptotic observer (of $A-KC$) at the roots of $s^2+d_1 s+d_0$. In fact $sI - (A - KC) = \begin{bmatrix} s & d_0 \\ -1 & s + d_1 \end{bmatrix}$. It is straightforward to show that $F = [\frac{1}{2}a_0 - 1, 2 - a_1]$ will assign the eigenvalues of the closed-loop system (of $A+BF$) at the roots of $s^2+a_1 s+a_0$. Indeed, $sI-(A+BF) = \begin{bmatrix} s & 2 \\ -\frac{1}{2}a_0 & s + a_1 \end{bmatrix}$. Now in (9.113) we have

$$
\begin{aligned}
G_y(s) &= F(sI - (A - KC))^{-1}K \\
&= \tfrac{s((d_0-2)(\frac{1}{2}a_0-1)+(d_1-2)(2-a_1))+((d_0-d_1)(a_0-2)+(d_0-2)(2-a_1))}{s^2+d_1 s+d_0},
\end{aligned}
$$

$$
\begin{aligned}
G_u(s) = F(sI - (A - KC))^{-1}\ B &= \tfrac{s(2-a_1)-d_0(\frac{1}{2}a_0-1)}{s^2+d_1 s+d_0}, \\
(1 - G_u)^{-1} &= \tfrac{s^2+d_1 s+d_0}{s^2+s(d_1+a_1-2)+\frac{1}{2}a_0 d_0}.
\end{aligned}
$$

9.5 Summary and Highlights

Linear State Feedback

- Given $\dot{x} = Ax + Bu$, $y = Cx + Du$ and the linear state feedback control law $u = Fx + r$, the closed-loop system is

$$\dot{x} = (A + BF)x + Br, \quad y = (C + DF)x + Dr. \tag{9.3}$$

- If u were implemented via open-loop control, it would be given by

$$\hat{u} = F[sI - (A + BF)]^{-1}x(0) + [I - F(sI - A)^{-1}B]^{-1}\hat{r}. \tag{9.4}$$

- The eigenvalues of $A+BF$ can be assigned to arbitrary real and/or complex conjugate locations by selecting F if and only if the system [or (A, B)] is controllable. The uncontrollable eigenvalues of (A, B) cannot be shifted (see Theorem 9.2 and Lemma 9.3).
- Methods to select F to assign the closed-loop eigenvalues in $A+BF$ include
  1. the direct method (see (9.12)), and

2. using controller forms (see (9.20 and (9.24)).
   Controller forms are used to derive Ackermann's formula ($m = 1$)

$$F = -e_n^T \mathcal{C}^{-1} \alpha_d(A), \tag{9.21}$$

   where $e_n = [0, \ldots, 0, 1]^T$, $\mathcal{C} = [B, \ldots, A^{n-1}B]$ is the controllability matrix and $\alpha_d(s)$ is the desired closed-loop characteristic polynomial (its roots are the desired eigenvalues).
3. Assigning eigenvalues and eigenvectors (see Theorem 9.12).
   The flexibility in choosing F that assigns the n closed-loop eignvalues (when $m > 1$) is expressed in terms of desired closed-loop eigenvectors that can be partially assigned,

$$FV = W, \tag{9.29}$$

   where $V \triangleq [M_1 a_1, \ldots, M_n a_n]$ and $W \triangleq [D_1 a_1, \ldots, D_n a_n]$ uniquely specify F as the solution to these n linearly independent equations. When s_j are distinct, the n vectors $M_j a_j$, $j = 1, \ldots, n$, are linearly independent for almost any nonzero a_j, and V has full rank.

- Optimal Control Linear Quadratic Regulator. Given $\dot{x} = Ax + Bu$, $z = Mx$, find $u(t)$ that minimizes the quadratic cost

$$J(u) = \int_0^\infty [z^T(t) Q z(t) + u^T(t) R u(t)] dt. \tag{9.31}$$

  Under controllability and observability conditions, the solution is unique and it is given as a linear state feedback control law

$$u^*(t) = F^* x(t) = -R^{-1} B^T P_c^* x(t), \tag{9.32}$$

  where P_c^* is the symmetric, positive definite solution of the algebraic Riccati equation

$$A^T P_c + P_c A - P_c B R^{-1} B^T P_c + M^T Q M = 0. \tag{9.33}$$

  The corresponding discrete-time case optimal control is described in (9.51), (9.52), and (9.53).
- The closed-loop transfer function $H_f(s)$ is given by

$$\begin{aligned} H_F(s) &= (C + DF)[sI - (A + BF)]^{-1} B + D \\ &= [C(sI - A)^{-1} B + D][I - F(sI - A)^{-1} B]^{-1} \\ &= H(s)[I - F(sI - A)^{-1} B]^{-1} \\ &= H(s)[F(sI - (A + BF))^{-1} B + I]. \end{aligned}$$

  See also (9.42) and (9.45). Also

$$\begin{aligned} H_F(s) &= N(s) D_F^{-1}(s) = [N(s) D^{-1}(s)][D(s) D_F^{-1}(s)] \\ &= H(s)[D(s) D_F^{-1}(s)]. \end{aligned} \tag{9.44}$$

Linear State Observers

- Given $\dot{x} = Ax + Bu$, $y = Cx + Du$, the Luenberger observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - \hat{y}), \tag{9.57}$$

where $\hat{y} = C\hat{y} + D$ or

$$\dot{\hat{x}} = (A - KC)\hat{x} + [B - KD, K] \begin{bmatrix} u \\ y \end{bmatrix}, \tag{9.58}$$

where K is chosen so that all the eigenvalues of $A - KC$ have negative real parts. Then the error $e(t) = x(t) - \hat{x}(t)$ will go to zero asymptotically.
- The eigenvalues of $A - KC$ can be assigned to arbitrary real and/or complex conjugate locations by selecting K if and only if the system [or (A, C)] is observable. The unobservable eigenvalues of (A, C) cannot be shifted. This is the dual problem to the control problem of assigning eigenvalues in $A + BF$, and the same methods can be used (see Lemma 9.18).
- Optimal State Estimation. Consider $\dot{x} = Ax + Bu + \Gamma w$, $y = Cx + v$, where w, v are process and measurement noise. Let the state estimator be

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \tag{9.117}$$
$$= (A - KC)\hat{x} + Bu + Ky, \tag{9.71}$$

and consider minimizing the error covariance $E[(x - \hat{x})(x - \hat{x})^T]$. Under certain controllability and observability conditions, the solution is unique and it is given by

$$K^* = P_e^* C^T V^{-1}, \tag{9.72}$$

where P_e^* is the symmetric, positive definite solution of the quadratic (dual) algebraic Riccati equation

$$P_e A^T + AP_e - P_e C^T V^{-1} C P_e + \Gamma W \Gamma^T = 0. \tag{9.73}$$

This problem is the dual to the Linear Quadratic Regulator problem.
- The discrete-time case is analogous to the continuous-time case [see (9.76)]. The current estimator is given in (9.78)–(9.80). The optimal current estimator is given by (9.88) and (9.89).

Observer-Based Dynamic Controllers

- Given $\dot{x} = Ax + Bu$, $y = Cx + Du$ with the state feedback $u = Fx + r$, if the state is estimated via a Luenberger observer, then the closed-loop system is

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \tag{9.118}$$

$$y = [C + DF, \ -DF] \begin{bmatrix} x \\ e \end{bmatrix} + Dr. \tag{9.99}$$

The error $e = x - \hat{x}$.

- The design of the control law (F) can be carried out independently of the design of the estimator (K) [see (9.101)–(9.106]. (Separation property)
- The compensated system behaves to the outside world as if there were no estimator present—after sufficient time so the transients have become negligible [see (9.109)].
- The observer based dynamic controller is a special case of a two degrees of freedom controller [see (9.113)–(9.115)].

9.6 Notes

The fact that if a system is (state) controllable, then all its eigenvalues can arbitrarily be assigned by means of linear state feedback has been known since the 1960s. Original sources include Rissanen [20], Popov [18], and Wonham [24]. (See also remarks in Kailath [11, pp. 187, 195].)

The present approach for eigenvalue assignment via linear state feedback, using the controller form, follows the development in Wolovich [23]. Ackermann's formula first appeared in Ackermann [2].

The development of the eigenvector formulas for the feedback matrix that assign all the closed-loop eigenvalues and (in part) the corresponding eigenvectors follows Moore [17]. The corresponding development that uses (A, B) in controller (companion) form and polynomial matrix descriptions follows Antsaklis [4]. Related results on static output feedback and on polynomial and rational matrix interpolation can be found in Antsaklis and Wolovich [5] and Antsaklis and Gao [6]. Note that the flexibility in assigning the eigenvalues via state feedback in the multi-input case can be used to assign the invariant polynomials of $sI-(A+BF)$; conditions for this are given by Rosenbrock [21].

The Linear Quadratic Regulator (LQR) problem and the Linear Quadratic Gaussian (LQG) problem have been studied extensively, particularly in the 1960s and early 1970s. Sources for these topics include the books by Anderson and Moore [3], Kwakernaak and Sivan [12], Lewis [13], and Dorato et al. [10]. Early optimal control sources include Athans and Falb [7] and Bryson and Ho [9]. A very powerful idea in optimal control is the *Principle of Optimality*, Bellman [8], which can be stated as follows: "An optimal trajectory has the property that at any intermediate point, no matter how it was reached, the remaining part of a trajectory must coincide with an optimal trajectory, computed from the intermediate point as the initial point." For historical remarks on this topic, refer, e.g., to Kailath [11, pp. 240–241].

The most influential work on state observers is the work of Luenberger. Although the asymptotic observer presented here is generally attributed to him, Luenberger's Ph.D. thesis work in 1963 was closer to the reduced-order observer presented above. Original sources on state observers include Luenberger [14], [15], and [16]. For an extensive overview of observers, refer to the book by O'Reilly [19].

When linear quadratic optimal controllers and observers are combined in control design, a procedure called LQG/LTR (Loop Transfer Recovery) is used to enhance the robustness properties of the closed-loop system. For a treatment of this procedure, see Stein and Athans [22] and contemporary textbooks on multivariable control.

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Exercises

9.1. Consider the system $\dot{x} = Ax + Bu$, where $A = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.02 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -0.25 & 0.75 \end{bmatrix}$ with $u = Fx$.

(a) Verify that the three different state feedback matrices given by

$$F_1 = \begin{bmatrix} -1.1 & -3.7 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ -1.1 & 1.2333 \end{bmatrix}, \quad F_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$

all assign the closed-loop eigenvalues at the same locations, namely, at $-0.1025 \pm j0.04944$. Note that in the first control law (F_1) only the first input is used, whereas in the second law (F_2), only the second input is used. For all three cases, plot $x(t) = [x_1(t), x_2(t)]^T$ when $x(0) = [0, 1]^T$ and comment on your results. This example demonstrates how different the responses can be for different designs even though the eigenvalues of the compensated system are at the same locations.

(b) Use the eigenvalue/eigenvector assignment method to characterize all F that assign the closed-loop eigenvalues at $-0.1025 \pm j0.04944$. Show how to select the free parameters to obtain F_1, F_2, and F_3 above. What are the closed-loop eigenvectors in these cases?

9.2. For the system $\dot{x} = Ax + Bu$ with $A \in R^{n\times n}$ and $B \in R^{n\times m}$, where (A, B) is controllable and $m > 1$, choose $u = Fx$ as the feedback control law. It is possible to assign all eigenvalues of $A + BF$ by first reducing this problem to the case of eigenvalue assignment for single-input systems ($m = 1$). This is accomplished by first reducing the system to a *single-input controllable system*. We proceed as follows.

Let $F = g \cdot f$, where $g \in R^m$ and $f^T \in R^n$ are vectors to be selected. Let g be chosen such that (A, Bg) is controllable. Then f in

$$A + BF = A + (Bg)f$$

can be viewed as the state feedback gain vector for a single-input controllable system (A, Bg), and any of the single-input eigenvalue assignment methods can be used to select f so that the closed-loop eigenvalues are at desired locations.

The only question that remains to be addressed is whether there exists g such that (A, Bg) is controllable. It can be shown that if (A, B) is controllable and A is cyclic, then almost any $g \in R^m$ will make (A, Bg) controllable. (A matrix A is cyclic if and only if its characteristic and minimal polynomials are equal.) In the case when A is not cyclic, it can be shown that if (A, B, C) is controllable and observable, then for almost any real output feedback gain matrix H, $A + BHC$ is cyclic. So initially, by an almost arbitrary choice of H or $F = HC$, the matrix A is made cyclic, and then by employing a g, (A, Bg) is made controllable. The state feedback vector gain f is then selected so that the eigenvalues are at desired locations.

Note that $F = gf$ is always a rank one matrix, and this restriction on F reduces the applicability of the method when requirements in addition to eigenvalue assignment are to be met.

(a) For A, B as in Exercise 9.4, use the method described above to determine F so that the closed-loop eigenvalues are at $-1 \pm j$ and $-2 \pm j$. Comment on your choice for g.
(b) For $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, characterize all g such that the closed-loop eigenvalues are at -1.

9.3. Consider the system $x(k + 1) = Ax(k) + Bu(k)$, where

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Determine a linear state feedback control law $u(k) = Fx(k)$ such that all the eigenvalues of $A + BF$ are located at the origin. To accomplish this, use

(a) reduction to a single-input controllable system,
(b) the controller form of (A, B),
(c) $\det(zI - (A + BF))$ and the resulting nonlinear system of equations.

In each case, plot $x(k)$ with $x(0) = [1,\ 1,\ 1]^T$ and comment on your results. In how many steps does your compensated system go to the zero state?

9.4. For the system $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

determine F so that the eigenvalues of $A+BF$ are at $-1 \pm j$ and $-2 \pm j$. Use as many different methods to choose F as you can.

9.5. Consider the SISO system $\dot{x}_c = A_c x_c + B_c u, y = C_c x_c + D_c u$, where (A_c, B_c) is in controller form with

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C_c = [c_0, c_1, \ldots, c_{n-1}],$$

and let $u = F_c x + r = [f_0, f_1, \ldots, f_{n-1}]x + r$ be the linear state feedback control law. Use the Structure Theorem of Section 6.4 to show that the open-loop transfer function is

$$\begin{aligned} H(s) &= C_c(sI - A_c)^{-1}B_c + D_c = \frac{c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} + D_c \\ &= \frac{n(s)}{d(s)} \end{aligned}$$

and that the closed-loop transfer function is

$$\begin{aligned} H_F(s) &= (C_c + D_c F_c)[sI - (A_c + B_c F_c)]^{-1}B_c + D_c \\ &= \frac{(c_{n-1} + D_c f_{n-1})s^{n-1} + \cdots + (c_1 s + D_c f_1)s + (c_0 + D_c f_0)}{s^n + (\alpha_{n-1} - f_{n-1})s^{n-1} + \cdots + (\alpha_1 - f_1)s + (\alpha_0 - f_0)} + D_c \\ &= \frac{n(s)}{d_F(s)}. \end{aligned}$$

Observe that state feedback does not change the numerator $n(s)$ of the transfer function, but it can arbitrarily assign any desired (monic) denominator polynomial $d_F(s) = d(s) - F_c[1, s, \ldots, s^{n-1}]^T$. Thus, state feedback does not (directly) alter the zeros of $H(s)$, but it can arbitrarily assign the poles of $H(s)$. Note that these results generalize to the MIMO case [see (9.43)].

9.6. Consider the system $\dot{x} = Ax + Bu, y = Cx$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1,\ 2,\ 0].$$

(a) Determine an appropriate linear state feedback control law $u = Fx + Gr$ $(G \in R)$ so that the closed-loop transfer function is equal to a given desired transfer function

$$H_m(s) = \frac{1}{s^2 + 3s + 2}.$$

We note that this is an example of *model matching*, i.e., compensating a given system so that it matches the input–output behavior of a desired model. In the present case, state feedback is used; however, output feedback is more common in model matching.

(b) Is the compensated system in (a) controllable? Is it observable? Explain your answers.

(c) Repeat (a) and (b) by assuming that the state is not available for measurement. Design an appropriate state observer, if possible.

9.7. Design an observer for the oscillatory system $\dot{x}(t) = v(t)$, $\dot{v}(t) = -\omega_0^2 x(t)$, using measurements of the velocity v. Place both observer poles at $s = -\omega_0$.

9.8. Consider the undamped harmonic oscillator $\dot{x}_1(t) = x_2(t)$, $\dot{x}_2(t) = -\omega_0^2 x_1(t) + u(t)$. Using an observation of velocity $y = x_2$, design an observer/state feedback compensator to control the position x_1. Place the state feedback controller poles at $s = -\omega_0 \pm j\omega_0$ and both observer poles at $s = -\omega_0$. Plot $x(t)$ for $x(0) = [1, 1]^T$ and $\omega_0 = 2$.

9.9. A servomotor that drives a load is described by the equation $\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt} = u$, where θ is the shaft position (output) and u is the applied voltage. Choose u so that θ and $\frac{d\theta}{dt}$ will go to zero exponentially (when their initial values are not zero). To accomplish this, proceed as follows.

(a) Derive a state-space representation of the servomotor.

(b) Determine linear state feedback, $u = Fx + r$, so that both closed-loop eigenvalues are at -1. Such F is actually optimal since it minimizes $J = \int_0^\infty [\theta^2 + \left(\frac{d\theta}{dt}\right)^2 + u^2] dt$.

(c) Since only θ and u are available for measurement, design an asymptotic state estimator (with eigenvalues at, say, -3) and use the state estimate $\hat{x}$ in the linear state feedback control law. Write the transfer function and the state-space description of the overall system and comment on stability, controllability, and observability.

(d) Plot θ and $d\theta/dt$ in (b) and (c) for $r = 0$ and initial conditions equal to $[1, 1]^T$.

(e) Repeat (c) and (d), using a reduced-order observer of order 1.

9.10. Consider the LQR problem for the system $\dot{x} = Ax + Bu$, where (A, B) is controllable and the performance index is given by

$$\tilde{J}(u) = \int_0^\infty e^{2\alpha t}[x^T(t)Qx(t) + u^T(t)Ru(t)]dt,$$

where $\alpha \in R, \alpha > 0$ and $Q \geq 0, R > 0$.

(a) Show that u^* that minimizes $\tilde{J}(u)$ is a fixed control law with constant gains on the states, even though the weighting matrices $\tilde{Q} = e^{2\alpha t}Q$, $\tilde{R} = e^{2\alpha t}R$ are time varying. Derive the algebraic Riccati matrix equation that characterizes this control law.

(b) The performance index given above has been used to solve the question of relative stability. In light of your solution, how do you explain this?

Hint: Reformulate the problem in terms of the transformed variables $\tilde{x} = e^{\alpha t}x, \tilde{u} = e^{\alpha t}u$.

9.11. Consider the system $\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ and the performance indices J_1, J_2 given by

$$J_1 = \int_0^\infty (x_1^2 + x_2^2 + u^2)dt \text{ and } J_2 = \int_0^\infty (900(x_1^2 + x_2^2) + u^2)dt.$$

Determine the optimal control laws that minimize J_1 and J_2. In each case, plot $u(t)$, $x_1(t)$, $x_2(t)$ for $x(0) = [1, 1]^T$ and comment on your results.

9.12. Consider the system $\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$, $y = [1,\ 0]x$.

(a) Use state feedback $u = Fx$ to assign the eigenvalues of $A + BF$ at $-0.5 \pm j0.5$. Plot $x(t) = [x_1(t), x_2(t)]^T$ for the open- and closed-loop system with $x(0) = [-0.6, 0.4]^T$.
(b) Design an identity observer with eigenvalues at $-\alpha \pm j$, where $\alpha > 0$. What is the observer gain K in this case?
(c) Use the state estimate $\hat{x}$ from (b) in the linear feedback control law $u = F\hat{x}$, where F was found in (a). Derive the state-space description of the closed-loop system. If $u = F\hat{x} + r$, what is the transfer function between y and r?
(d) For $x(0) = [-0.6, 0.4]^T$ and $\hat{x}(0) = [0, 0]^T$, plot $x(t), \hat{x}(t), y(t)$, and $u(t)$ of the closed-loop system obtained in (c) and comment on your results. Use $\alpha = 1, 2, 5$, and 10, and comment on the effects on the system response.

Remark: This exercise illustrates the deterioration of system response when state observers are used to generate the state estimate that is used in the feedback control law.

9.13. Consider the system

$$x(k+1) = Ax(k) + Bu(k) + Eq(k), \quad y(k) = Cx(k),$$

where $q(k) \in R^r$ is some disturbance vector. It is desirable to completely eliminate the effects of $q(k)$ on the output $y(k)$. This can happen only when E satisfies certain conditions. Presently, it is assumed that $q(k)$ is an arbitrary $r \times 1$ vector.

(a) Express the required conditions on E in terms of the observability matrix of the system.
(b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = [1,\ 1]$, characterize all E that satisfy these conditions.

(c) Suppose $E \in R^{n\times 1}$, $C \in R^{p\times 1}$, and $q(k)$ is a step, and let the objective be to asymptotically reduce the effects of q on the output. Note that this specification is not as strict as in (a), and in general it is more easily satisfied. Use z-transforms to derive conditions for this to happen.
Hint: Express the conditions in terms of poles and zeros of $\{A, E, C\}$.

9.14. Consider the system $\dot{x} = Ax + Bu$, $y = Cx + Du$, where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $u = Fx + r$ be a linear state feedback control law. Determine F so that the eigenvalues of $A + BF$ are $-1, -2$ and are unobservable from y. What is the closed-loop transfer function $H_F(s)$ ($\hat{y} = H_F\hat{r}$) in this case?
Hint: Select the eigenvalues and eigenvectors of $A + BF$.

9.15. Consider the controllable and observable $SISO$ system $\dot{x} = Ax + Bu$, $y = Cx$ with $H(s) = C(sI - A)^{-1}B$.

(a) If λ is not an eigenvalue of A, show that there exists an initial state x_0 such that the response to $u(t) = e^{\lambda t}$, $t \geq 0$, is $y(t) = H(\lambda)e^{\lambda t}$, $t \geq 0$. What happens if λ is a zero of $H(s)$?
(b) Assume that A has distinct eigenvalues. Let λ be an eigenvalue of A and show that there exists an initial state x_0 such that with "no input" ($u(t) \equiv 0$), $y(t) = ke^{\lambda t}$, $t \geq 0$, for some $k \in R$.