Μάθημα 16

11/05/2015

Εφαρμογή του Θ. Green σε αστρόβιλα δ.π.

$\vec{F}(x,y)=(P(x,y),Q(x,y))$, $(x,y)\in A\ (\subseteq \mathbb{R}^2)$, A=ανοικτό

$\vec{F}=C^2$

$\text{curl}\,\vec{F}(x,y)=\left(\dfrac{\partial Q(x,y)}{\partial x}-\dfrac{\partial P(x,y)}{\partial y}\right)\cdot\vec{k}$, $\vec{k}=(0,0,1)$, $(x,y)\in A$

$\vec{F}$ είναι αστρόβιλο στο A $\Longleftrightarrow$ $\text{curl}\,\vec{F}(x,y)=(0,0,0)$, $(x,y)\in A$

$\Longleftrightarrow \dfrac{\partial P(x,y)}{\partial y}=\dfrac{\partial Q(x,y)}{\partial x}$, $(x,y)\in A$.

Ασκήσεις

1) $\vec{F}(x,y)=\left(\dfrac{-y}{x^2+y^2},\dfrac{x}{x^2+y^2}\right)$ $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ (C^∞)

ΝΔΟ i) $\vec{F}$ αστρόβιλο (κλειστή διαφορίσιμη μορφή)

ii) $I_1=\oint_{\Gamma_1}\vec{F}\cdot d\vec{r}=0$, αν $(0,0)\notin(\varepsilon\sigma.\Gamma_1)\cup\Gamma_1$

Γ_1 τυχαία, $\vec{r}(t)=(x(t),y(t))$ κλειστή + απλή + λεία, C^1

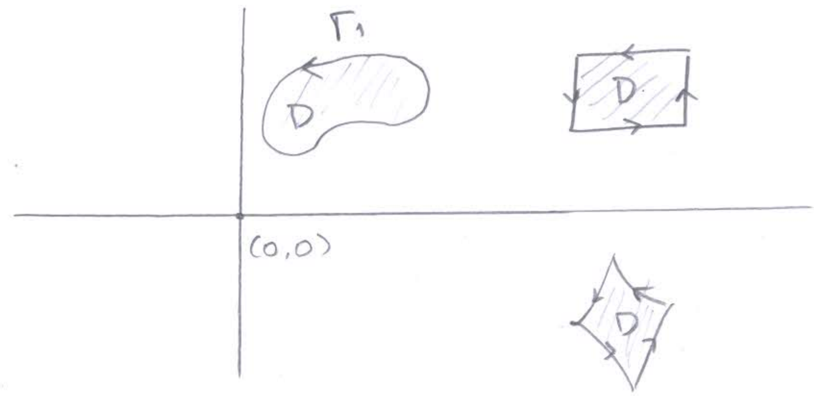
$(\exists\, r'(t)\neq 0)$

iii) $I_2=\oint_{\Gamma_2}\vec{F}\cdot d\vec{r}=2\pi$, αν $(0,0)\in\varepsilon\sigma\Gamma_2$, Γ_2 »

Λύση

i) $$\left.\begin{aligned} P(x,y) &= \frac{-y}{x^2+y^2} \\ Q(x,y) &= \frac{x}{x^2+y^2} \end{aligned}\right\} \quad \frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y), \ (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

ii)



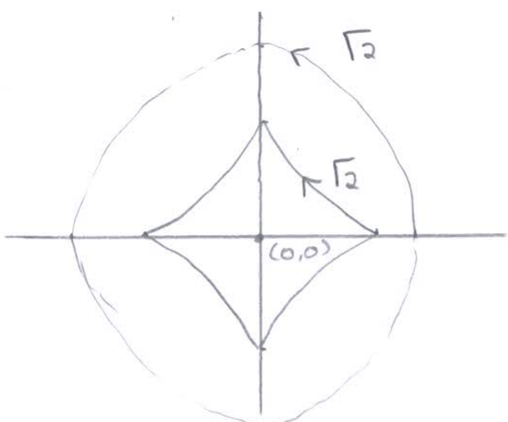
$D = \varepsilon\sigma\Gamma_1 \cup \Gamma_1$, D σύνολο Green

($\partial D = \Gamma_1$, Γ_1 = απλή + κλειστή + C' + λεία)

Άρα μπορούμε να εφαρμόσουμε τον τ. Green

$$\oint_{\Gamma_1 = (\partial D)^+} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy \overset{(i)}{=} \iint_D 0 \, dx\, dy = 0$$

iii)



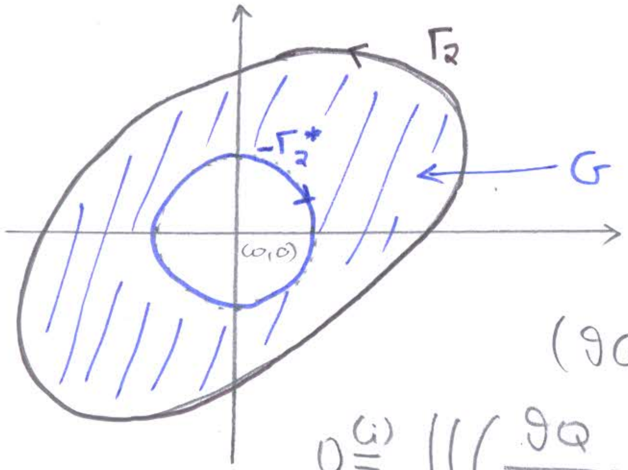
$D = \left((\varepsilon\sigma\Gamma_2) \cup \Gamma_2 \right) \setminus \{(0,0)\}$

δεν είναι σύνολο Green

($\partial D = \Gamma_2 \setminus \{(0,0)\}$ δεν αποτελείται από απλές + κλειστές + C' + λείες καμ.)

Θεωρούμε $\Gamma_2^*: \vec{r}(t) = (a\sigma\upsilon\nu t,\ a\eta\mu t)$, $t \in [0, 2\pi]$ $(a>0)$

$\Gamma_2^* \subseteq \varepsilon\sigma\,\Gamma_2$



$G = (\Gamma_2 \cup \Gamma_2^*) \cup (\varepsilon\sigma\,\Gamma_2 \setminus \varepsilon\sigma\,\Gamma_2)^*$

Απλό σύνολο Green.

($\partial G = \Gamma_2 \cup \Gamma_2^*$, Γ_2, Γ_2^* απλ + κλ + C^1 + λείες)

$$0 \overset{(i)}{=} \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_{(\partial G)^+} \vec{F} \cdot d\vec{r} =$$

$$= \cancel{\oint_{\Gamma_2} \vec{F} \cdot d\vec{r}} = \oint_{\Gamma_2} \vec{F} \cdot d\vec{r} - \int_{\Gamma_2^*} \vec{F} \cdot d\vec{r}$$

Άρα $$\oint_{\Gamma_2} \vec{F} \cdot dr = \int_{\Gamma_2^*} \vec{F} \cdot dr = \int_0^{2\pi} \left(\frac{-a\eta\mu t}{a^2}, \frac{a\sigma\upsilon\nu t}{a^2}\right) \cdot (-a\eta\mu t, a\sigma\upsilon\nu t)\, dt$$

$$= \int_0^{2\pi} \frac{1}{a^2} (a^2\eta\mu^2 t + a^2\sigma\upsilon\nu^2 t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

Σημείωση.

Το $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$, $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

είναι αστρόβιλο, δεν είναι συντηρητικό.

Βλέπε Γ1, Γ2 στο σχήμα

2) $\vec{F}(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right) \quad (x,y)\in \mathbb{R}^2 \setminus \{(0,0)\}$

Να υπολογιστούν:

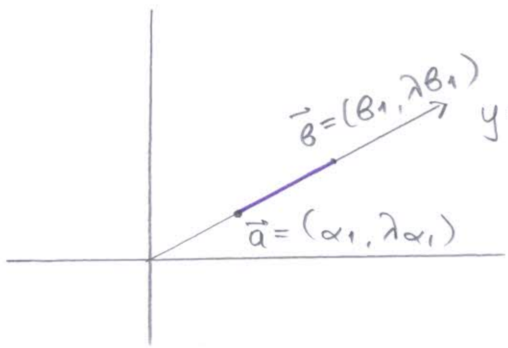
i) $\displaystyle\int_{[\vec{\alpha},\vec{\beta}]} \vec{F}\cdot d\vec{r}$, $\vec{\alpha} \neq \vec{\beta}$, $\vec{a},\vec{\beta}$ βρίσκονται σε ημιευθεία του $\mathbb{R}^2$ με αρχή το $(0,0)$ $(\vec{\alpha},\vec{\beta} \neq (0,0))$

ii) $\displaystyle\int_{\Gamma_1} \vec{F}\cdot d\vec{r}$, $\Gamma_1 \begin{cases} x(t) = a\sigma\upsilon\nu t \\ y(t) = a\eta\mu t \end{cases}$ $t\in[\vartheta,\varphi] \quad 0<\vartheta<\varphi<\frac{\pi}{2}$, $\alpha>0$.

iii) $\displaystyle\int_{\Gamma_2} \vec{F}\cdot d\vec{r}$, $\Gamma_2 = \begin{cases} x(t) = \alpha'\sigma\upsilon\nu t \\ y(t) = \beta'\eta\mu t \end{cases}$ $t\in[\vartheta,\varphi] \quad 0<\vartheta<\varphi<\frac{\pi}{2}$, $\alpha',\beta'>0$.

Λύση.

i)



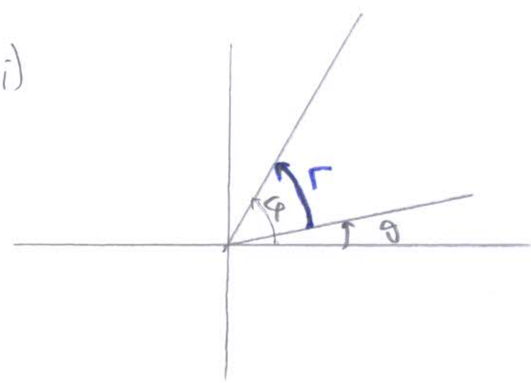
$y=\lambda x$, $\lambda>0$, $x\in(0,+\infty)$

$(0<)\,\alpha_1<\beta_1$

$\Gamma_1: \vec{r}(t) = (t,\lambda t)$, $t\in[a_1,\beta_1]$, $\vec{r}'(t) = (1,\lambda)$

$$\int_{[\vec{\alpha},\vec{\beta}]} \vec{F}\cdot d\vec{r} = \int_{\alpha_1}^{\beta_1} \left(\frac{-\lambda t}{t^2+\lambda^2 t^2}, \frac{t}{t^2+\lambda^2 t^2}\right)\cdot(1,\lambda)\,dt = \int_{\alpha_1}^{\beta_1} \frac{(-\lambda t+\lambda t)}{t^2+\lambda^2 t^2}\,dt = 0$$

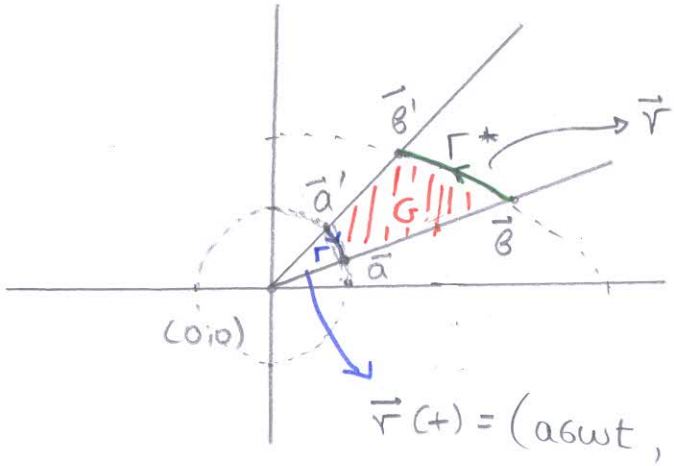
ii)



$\Gamma: \vec{r}'(t) = (-a\eta\mu t, a\sigma\upsilon\nu t)$, $t\in[\vartheta,\varphi]$

$$\int_\Gamma \vec{F}\cdot d\vec{r} = \int_\vartheta^\varphi 1\,dt = \underline{\varphi-\vartheta}$$

iii)



$\vec{r}^*(t) = (\alpha' \sigma\upsilon\nu t, \beta' \eta\mu t)$ $t \in [\vartheta, \varphi]$

$\alpha' \neq \beta'$, $\alpha', \beta' > 0$

$\vec{r}(t) = (\alpha \sigma\upsilon\nu t, \alpha \eta\mu t)$, $t \in [\vartheta, \varphi]$, $(\alpha > 0)$

$(\partial G) = \Gamma \cup \Gamma^* \cup [\vec{a}, \vec{\beta}] \cup [\vec{\alpha}', \vec{\beta}']$ / G = σύνολο Green

$$0 \overset{\text{Ασκ 1}}{\underset{(i)}{=}} \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = -\int_\Gamma \vec{F} \cdot d\vec{r} + \int_{[\vec{\alpha}, \vec{\beta}]} \vec{F} \cdot d\vec{r} + \int_{\Gamma^*} \vec{F} \cdot d\vec{r} - \int_{[\vec{\alpha}', \vec{\beta}']} \vec{F} \cdot d\vec{r}$$

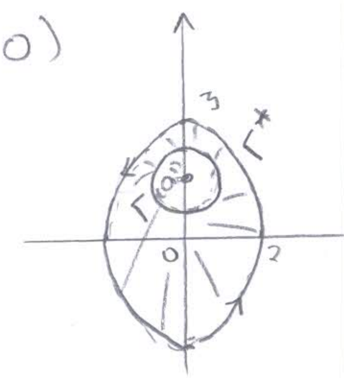
Από το i) $\int_{\Gamma^*} \vec{F} \cdot d\vec{r} = \int_\Gamma \vec{F} \cdot d\vec{r} \overset{(ii)}{=} \varphi - \vartheta$

3) $\vec{F}(x,y) = \dfrac{x^2}{[x^2 + (y-1)^2]^2} (y-1, -x)$, $(x,y) \in \mathbb{R}^2 \setminus \{(0,1)\}$.

i) Να αποδειχθεί ότι το $\vec{F}$ είναι αστρόβιλο.

ii) Να υπολογιστεί $\oint_\Gamma \vec{F} \cdot d\vec{r}$, Γ, $x^2 + (y-1) = \alpha^2$ $(\alpha > 0)$

iii) $\oint_{\Gamma^*} \vec{F} \cdot d\vec{r}$, Γ^*, $9x^2 + 4y^2 = 36$



i) $\vec{F}(x,y) = \left(\dfrac{x^2}{[x^2 + (y-1)^2]^2} (y-1), \dfrac{-x^3}{[x^2 + (y-1)^2]^2} \right)$

$$\frac{\partial P}{\partial y}(x,y) = \frac{x^2 [x^2 - 3(y-1)^2]}{[x^2 + (y-1)^2]^3} = \frac{\partial Q}{\partial x}(x,y)$$ Άρα $\vec{F}$ αστρόβιλο.

ii) Γ: $\vec{r}(t) = (a\cos t, a\sin t + 1)$, $t \in [0, 2\pi]$

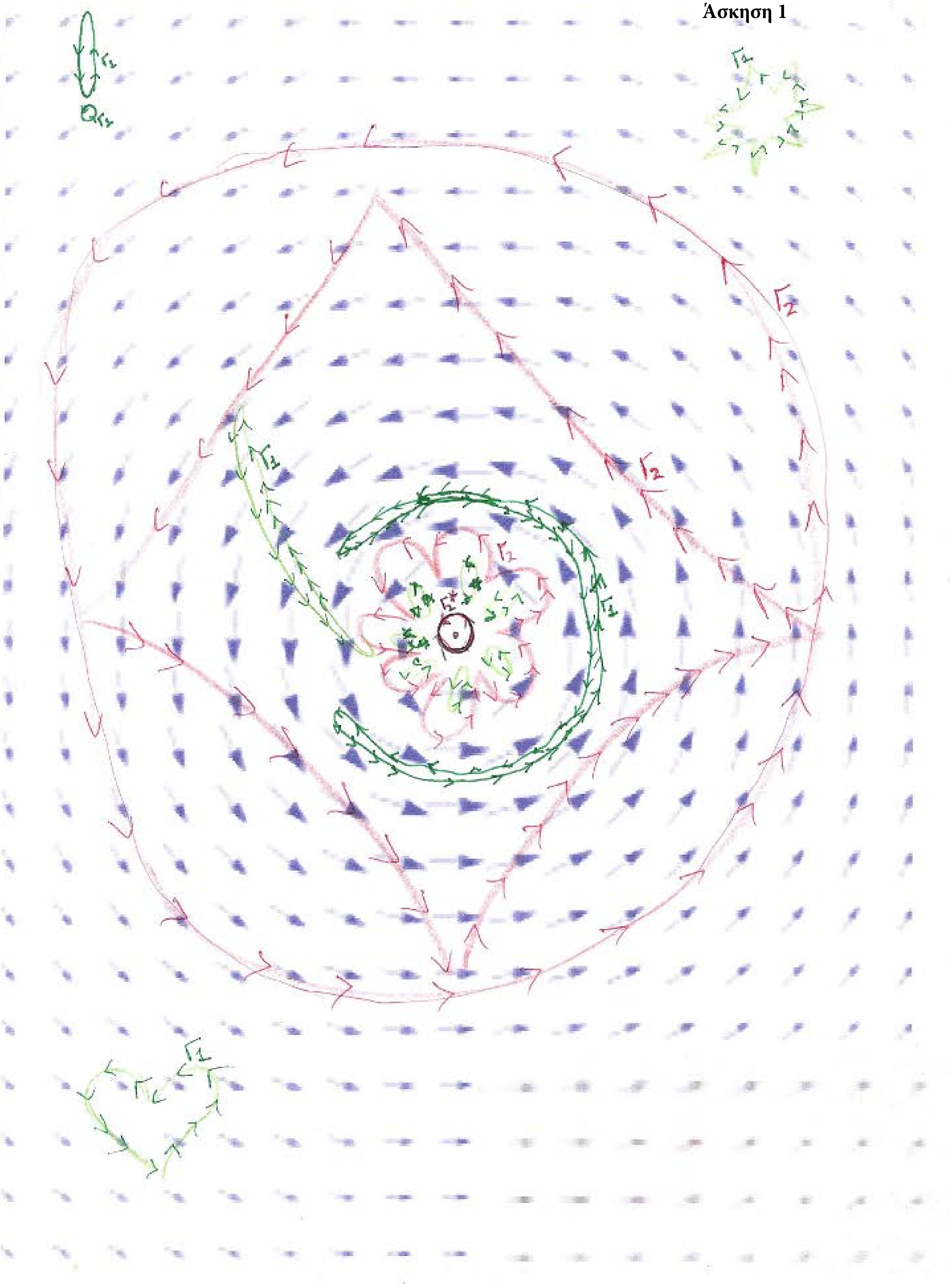
$$\oint_\Gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \frac{1}{a^2} (a\cos^2 t \cdot \sin t, -a\cos^3 t) \cdot (-a\sin t, a\cos t)\,dt = -\pi$$

iii) $D = \{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 \geq \frac{1}{2}$ και $9x^2 + 4y^2 \leq 36\}$

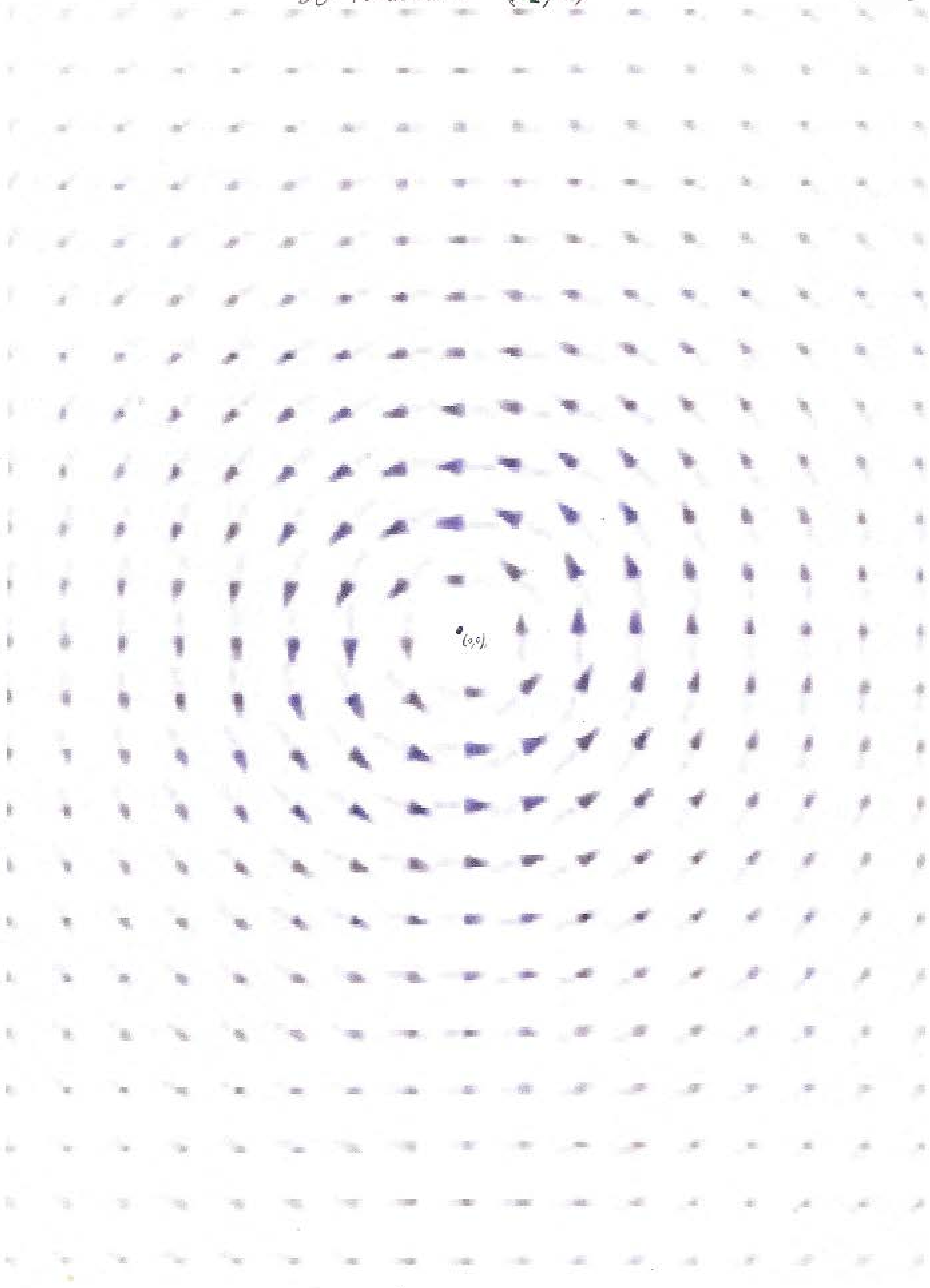
D = σύνολο Green $\int_{(\partial D)^+} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_D 0\,dx\,dy = 0 \Rightarrow$

$$\int_\Gamma \vec{F} \cdot d\vec{r} + \int_{\Gamma^*} \vec{F} \cdot d\vec{r} = 0 \Longleftrightarrow \int_{\Gamma^*} \vec{F} \cdot d\vec{r} = \int_\Gamma \vec{F} \cdot d\vec{r} = -\pi.$$

Άσκηση 1



DO YOURSELF (Γ_1, Γ_2)



Άσκηση / Κάθετη μορφή Θ. Green.

$\vec{F} = (P, Q) : D\ (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2,\quad C^1$ δ.π.

$\partial D =$ απλή + κλειστή + λεία + C^1, Γ, $\vec{r}(t) = (x(t), y(t))$ $t \in [\alpha, \beta]$.

$$\int_\Gamma \vec{F}(x(t), y(t)) \cdot (y'(t), -x'(t))\, dt$$

$$= \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx\, dy = \iint_D (\operatorname{div} \vec{F})\, dx\, dy .$$

Λύση $\vec{F}_1 = (-Q, P)$ $\quad (\vec{F} \cdot \vec{F}_1 = 0)$

$ \quad\quad\quad \overset{\parallel}{P_1} \quad \overset{\parallel}{Q_1}$

$$\iint_D \left(\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y} \right) dx\, dy = \oint_\Gamma \vec{F}_1(\vec{r}(t)) \cdot (\vec{r}'(t))\, dt \quad (\text{Θ. Green})$$

$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \oint_\Gamma \underset{(\vec{r}(t))}{(P_1, Q_1)} \cdot \vec{r}'(t)\, dt =$$

$$= \int_\Gamma (-Q(x(t), y(t)), P(x(t), y(t)) \cdot (x'(t), y'(t))\, dt =$$

$$= \int_\Gamma (P y' - Q x')\, dt = \int_\Gamma (P, Q) \cdot (y' - x')\, dt .$$

Διανυσματική μορφή του Θ. Green

$\vec{F}=(P,Q): D(\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}^2,\ C^1$

$\partial D \ \Big|\ \vec{r}(t)=(x(t),y(t)),\ t\in[a,b]$
απλή + κλειστή + λεία + C^1

$\vec{r}\,'(t)=(x'(t),y'(t)),\quad \vec{T}(t)=\dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ μον. εφαπτομένη

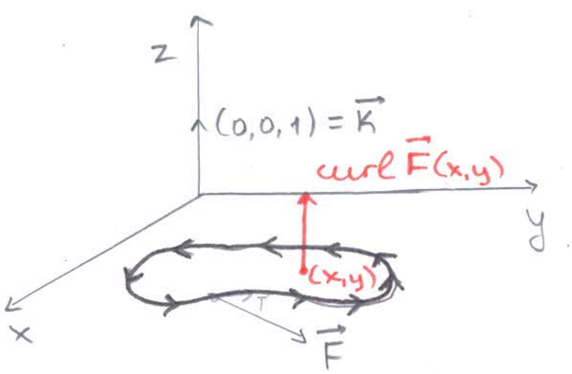
$(x'(t),-x'(t)) \perp \vec{r}\,'(t),\quad \vec{n}(t)=\dfrac{(y'(t),-x'(t))}{\|\vec{r}\,'(t)\|}$ μον. κάθετο.

$S(t)=\int_a^b \|\vec{r}\,'(\tau)\|\,d\tau \qquad \dfrac{ds(t)}{dt}=\|\vec{r}\,'(t)\|$

I → Εφαπτομενική μορφή τ. Green, $\displaystyle\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\oint_{(\partial D)^+}\vec{F}\cdot\vec{r}\,'(t)\,dt$

II → Κάθετη μορφή τ. Green $\displaystyle\iint_D\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy=\oint_{(\partial D)^+}\vec{F}\cdot(y'(t),-x'(t))\,dt$

I Η ροή του $\vec{F}$ κατά μήκος της Γ = "ολικό" στροβιλισμό που περνά δια μέσου της επιφάνειας $D(\subseteq x\text{-}y)$



$$\oint_{(\partial D)^+}(\vec{F}\cdot\vec{T})\,ds=\iint_D(\operatorname{curl}\vec{F})\cdot\vec{k}\,dx\,dy$$

II $\displaystyle\oint(\vec{F}\cdot\vec{n})\,ds=\iint_D(\operatorname{div}\vec{F}(x,y))\,dx\,dy$ Η εξερχόμενη ροή του $\vec{F}$ = "ολική" απόκλιση

I Για $d=2$ Θ. Stokes

II Θ. Gauss, της Απόκλισης, της Ροής (Συμπλ. Υλικό, Παν. Αιγαίου)

Ταυτότητες Green για δύο μεταβλητές. * SOS

$f, g : D (\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}$, C^2

∂D = απλή + κλειστή + C^1 + λεία καμπύλη / Γ

i) 1η Ταυτότητα Green

$$\iint_D f \nabla^2 g + \iint_D \nabla f \cdot \nabla g = \oint_{(\partial D)^+} -f \frac{\partial g}{\partial y} dx + f \frac{\partial g}{\partial x} dy = \oint_{(\partial D)^+} (f \nabla g) \circ \vec{n} \, ds.$$

ii) 2η ταυτότητα Green.

$$\iint_D f \nabla^2 f + \iint_D \| \nabla f \|^2 = \oint_{(\partial D)^+} (f \nabla f) \circ \vec{n} \, ds$$

iii) Έστω f αρμονική συνάρτηση ($\nabla^2 f = 0$ στο D)

και $f(x,y) = 0$, $(x,y) \in \partial D$

Τότε $f(x,y) = 0$, $(x,y) \in D$

(f_1, f_2 αρμονικές και $f_1 = f_2$ στο $\partial D \Rightarrow f_1 = f_2$ στο D).

Λύση

$$\vec{F} = \left(-f \frac{\partial g}{\partial y}, f \frac{\partial g}{\partial x}\right), \quad P = -f \frac{\partial g}{\partial y}, \quad Q = f \frac{\partial g}{\partial x}$$

Τύπος Green: $\displaystyle \oint_{(\partial D)^+} -f \frac{\partial g}{\partial y} dx + f \frac{\partial g}{\partial x} dy = \iint_D \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + f \frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} + f \frac{\partial^2 g}{\partial y^2} \right)$

$$= \iint_D \nabla f \cdot \nabla g + \iint_D f \nabla^2 g \quad / \text{ii) } f = g \quad / \text{iii) } \overbrace{\int_{(\partial D)^+} -f \frac{\partial f}{\partial y} dx + f \frac{\partial f}{\partial x} dy}^{(*)} =$$

$$= \iint_D f \nabla^2 f + \iint_D \| \nabla f \|^2, \quad \nabla^2 f = 0, \quad \iint_D f \nabla^2 f = 0$$

$f(x,y) = 0$, $(x,y) \in \partial D$ $(*) = 0$ / $\displaystyle \iint_D \| \nabla f \|^2 dx dy = 0$

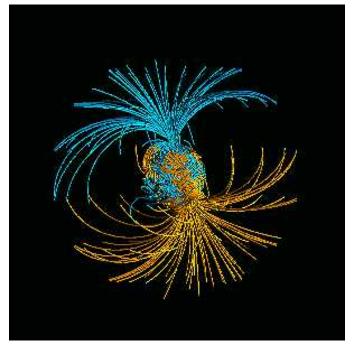
$\| \nabla f \| \geq 0$

∇f = συνεχής.

$\nabla f(x,y) = (0,0)\ ,\ (x,y)\in D$

$$\begin{cases} df(x,y)(h_1,h_2)=0, & (x,y)\in D \\ D = \text{πολυγωνικά συνεκτικό} \end{cases}$$

<u>**ΘΜΤ**</u>
$$\left.\begin{array}{l} f(x,y)=c,\ (x,y)\in D \\ f=\text{συν},\ f(x,y)=0,\ (x,y)\in \partial D \end{array}\right\} \Rightarrow c=0$$



Γραμμές Ροής
Μαγνητικού Πεδίου
Γης

Γραμμές Ροής δ.π.

- $\vec{F}=(P,Q,R)$, C^1

Γ, $\vec{r}(t) = (x(t), y(t), z(t))$, $t\in I$ (I=διάστημα)

γραμμή ροής $\Longleftrightarrow \vec{r}\,'(t) = \dfrac{d\vec{r}(t)}{dt} = \vec{F}(\vec{r}(t)) \Longleftrightarrow$

$\vec{F}(\vec{r}(t)) \times \vec{r}\,'(t) = \vec{0}$

- $d=2$, $\vec{F} = (P,Q)$, $\vec{r}(t) = (x(t), y(t))$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P & Q & R^{=0} \\ x'(t) & y'(t) & z'(t)^{=0} \end{vmatrix} = (0,0,0) \Longleftrightarrow P\cdot y'(t) - Q\,x'(t) = 0$$

$$(P(x(t),y(t)),\ Q(x(t),y(t)))\underbrace{(y'(t),-x'(t))}_{\vec{u}(t)} = 0$$

<u>Άσκηση</u>

$\vec{F}_1(x,y) = (x,-y)$ $\quad \vec{F}_3(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$ $\quad (x,y)\in \mathbb{R}^2\setminus\{(0,0)\}$

$\vec{F}_2(x,y) = (-y,x)$ $\quad$ i) Γραμμές Ροής

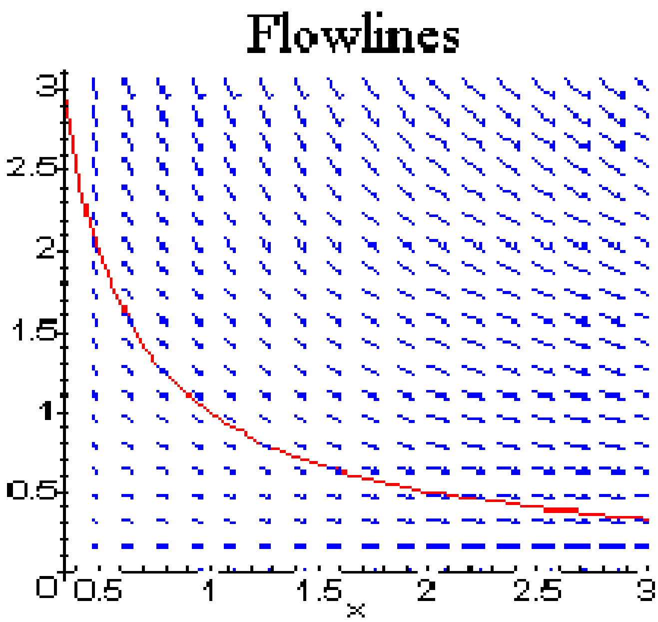
$(x,y)\in\mathbb{R}^2$ $\quad$ ii) Ποια από τα $\vec{F}_i$ είναι συντηρητικά.

Λύση

i) $x(t)\cdot y'(t) - (-y(t))\cdot x'(t) = 0$

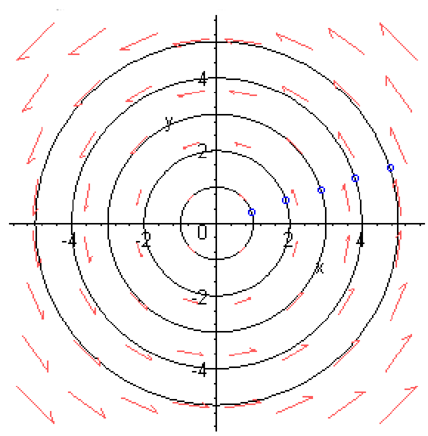
$\frac{d}{dt}(x(t)\cdot y(t)) = 0 \qquad P\cdot y'(t) - Q\cdot x'(t) = 0$

$x\cdot y = c$



$(-y(t))\, y'(t) - x(t)\, x'(t) = 0$

$x^2 + y^2 = c \quad (c>0).$



ii) • $f(x,y) = \frac{x^2}{2} - \frac{y^2}{2} \qquad (x,-y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ / $\vec{F} = \nabla f$ /

$\vec{F}_1(x,y) = (x,-y)$ συντηρητικό

• $F_2(x,y)$, curl $\vec{F}(x,y) = (0,0,2) \neq (0,0,0)$ δεν είναι αστρόβιλο

$\Rightarrow$ δεν είναι συντηρητικό.

• F_3 αστρόβιλο, αλλά όχι συντηρητικό