One Hundred[1] Solved[2] Exercises[3] for the subject:
Stochastic Processes I[4]

Takis Konstantopoulos[5]

1.

In the Dark Ages, Harvard, Dartmouth, and Yale admitted only male students. Assume that, at that time, 80 percent of the sons of Harvard men went to Harvard and the rest went to Yale, 40 percent of the sons of Yale men went to Yale, and the rest split evenly between Harvard and Dartmouth; and of the sons of Dartmouth men, 70 percent went to Dartmouth, 20 percent to Harvard, and 10 percent to Yale. (i) Find the probability that the grandson of a man from Harvard went to Harvard. (ii) Modify the above by assuming that the son of a Harvard man always went to Harvard. Again, find the probability that the grandson of a man from Harvard went to Harvard.

Solution. We first form a Markov chain with state space $S = \{H, D, Y\}$ and the following transition probability matrix :

$$\mathsf{P} = \begin{pmatrix} .8 & 0 & .2 \\ .2 & .7 & .1 \\ .3 & .3 & .4 \end{pmatrix}.$$

Note that the columns and rows are ordered: first H, then D, then Y. Recall: the ij^{th} entry of the matrix P^n gives the probability that the Markov chain starting in state i will be in state j after n steps. Thus, the probability that the grandson of a man from Harvard went to Harvard is the upper-left element of the matrix

$$\mathsf{P}^2 = \begin{pmatrix} .7 & .06 & .24 \\ .33 & .52 & .15 \\ .42 & .33 & .25 \end{pmatrix}.$$

It is equal to $.7 = .8^2 + .2 \times .3$ and, of course, one does not need to calculate all elements of P^2 to answer this question.
If all sons of men from Harvard went to Harvard, this would give the following matrix for the new Markov chain with the same set of states:

$$\mathsf{P} = \begin{pmatrix} 1 & 0 & 0 \\ .2 & .7 & .1 \\ .3 & .3 & .4 \end{pmatrix}.$$

The upper-left element of P^2 is 1, which is not surprising, because the offspring of Harvard men enter this very institution only.

2.

Consider an experiment of mating rabbits. We watch the evolution of a particular

[1] More or less

[2] Most of them

[3] Some of these exercises are taken verbatim from Grinstead and Snell; some from other standard sources; some are original; and some are mere repetitions of things explained in my lecture notes.

[4] The subject covers the basic theory of Markov chains in discrete time and simple random walks on the integers

[5] Thanks to Andrei Bejan for writing solutions for many of them

gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid–the order is irrelevant, so gG is the same as Gg) or gg (recessive). In mating two rabbits, the offspring inherits a gene from each of its parents with equal probability. Thus, if we mate a dominant (GG) with a hybrid (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2.
Start with a rabbit of given character (GG, Gg, or gg) and mate it with a hybrid. The offspring produced is again mated with a hybrid, and the process is repeated through a number of generations, always mating with a hybrid.
(i) Write down the transition probabilities of the Markov chain thus defined.
(ii) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\mu_n(GG), \mu_n(Gg), \mu_n(gg)$ are the probabilities that the n-th generation rabbit is GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Can you do the same for μ_n for general n?

Solution. (i) The set of states is $S = \{GG, Gg, gg\}$ with the following transition probabilities:

	GG	Gg	gg
GG	.5	.5	0
Gg	.25	.5	.25
gg	0	.5	.5

We can rewrite the transition matrix in the following form:

$$\mathsf{P} = 2^{-1} \begin{pmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}.$$

(ii) The elements from the second row of the matrix P^n will give us the probabilities for a hybrid to give dominant, hybrid or recessive species in $(n-1)^{\text{th}}$ generation in this experiment, respectively (reading this row from left to right). We first find

$$\mathsf{P}^2 = 2^{-2} \begin{pmatrix} 1.5 & 2 & 0 \\ 1 & 2 & 1 \\ 0.5 & 2 & 1.5 \end{pmatrix},$$

$$\mathsf{P}^3 = 2^{-3} \begin{pmatrix} 2.5 & 4 & 1.5 \\ 2 & 4 & 2 \\ 1.5 & 4 & 2.5 \end{pmatrix},$$

$$\mathsf{P}^4 = 2^{-4} \begin{pmatrix} 4.5 & 8 & 3.5 \\ 4 & 8 & 4 \\ 3.5 & 8 & 4.5 \end{pmatrix},$$

so that

$$\mu_i(GG) = .25, \mu_i(Gg) = .5, \mu_i(gg) = .25, \quad i = 1, 2, 3.$$

Actually the probabilities are the same for any $i \in \mathbb{N}$. If you obtained this result before 1858 when Gregor Mendel started to breed garden peas in his monastery garden and analysed the offspring of these matings, you would probably be very famous because it definitely looks like a law! This is what Mendel found when he crossed mono-hybrids.

In a more general setting, this law is known as Hardy-Weinberg law.
As an exercise, show that

$$\mathsf{P}^n = 2^{-n}\begin{pmatrix} \frac{3}{2}+(2^{n-2}-1) & 2^{n-1} & \frac{1}{2}+(2^{n-2}-1) \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ \frac{1}{2}+(2^{n-2}-1) & 2^{n-1} & \frac{3}{2}+(2^{n-2}-1) \end{pmatrix}.$$

Try!

3.

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?
Now draw a tree and assign probabilities assuming that the process begins in state 0 and moves through two stages of transmission. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit)?

Solution. Taking as states the digits 0 and 1 we identify the following Markov chain (by specifying states and transition probabilities):

$$\begin{array}{ccc} & 0 & 1 \\ 0 & q & p \\ 1 & p & q \end{array}$$

where $p + q = 1$. Thus, the transition matrix is as follows:

$$\mathsf{P} = \begin{pmatrix} q & p \\ p & q \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}.$$

It is clear that the probability that that the machine will produce 0 if it starts with 0 is $p^2 + q^2$.

4.

Assume that a man's profession can be classified as professional, skilled labourer, or unskilled labourer. Assume that, of the sons of professional men, 80 percent are professional, 10 percent are skilled labourers, and 10 percent are unskilled labourers. In the case of sons of skilled labourers, 60 percent are skilled labourers, 20 percent are professional, and 20 percent are unskilled. Finally, in the case of unskilled labourers, 50 percent of the sons are unskilled labourers, and 25 percent each are in the other two categories. Assume that every man has at least one son, and form a Markov chain by following the profession of a randomly chosen son of a given family through several generations. Set up the matrix of transition probabilities. Find the probability that a randomly chosen grandson of an unskilled labourer is a professional man.

Solution. The Markov chain in this exercise has the following set states

$$S = \{\text{Professional}, \text{Skilled}, \text{Unskilled}\}$$

with the following transition probabilities:

	Professional	Skilled	Unskilled
Professional	.8	.1	.1
Skilled	.2	.6	.2
Unskilled	.25	.25	.5

so that the transition matrix for this chain is

$$\mathsf{P} = \begin{pmatrix} .8 & .1 & .1 \\ .2 & .6 & .2 \\ .25 & .25 & .5 \end{pmatrix}$$

with

$$\mathsf{P}^2 = \begin{pmatrix} 0.6850 & 0.1650 & 0.1500 \\ 0.3300 & 0.4300 & 0.2400 \\ 0.3750 & 0.3000 & 0.3250 \end{pmatrix},$$

and thus the probability that a randomly chosen grandson of an unskilled labourer is a professional man is 0.375.

5.

I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.
1. If the probability of rain is p, what is the probability that I get wet?
2. Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

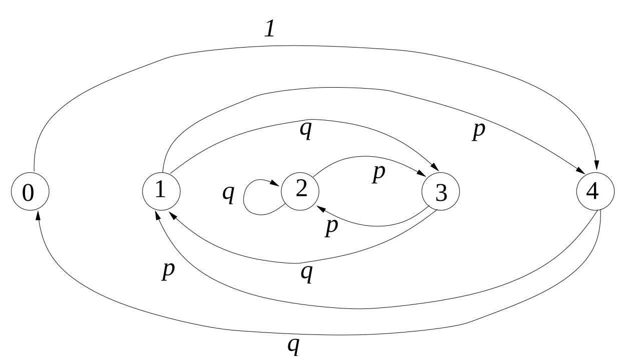
Solution. To solve the problem, consider a Markov chain taking values in the set $S = \{i : i = 0, 1, 2, 3, 4\}$, where i represents the number of umbrellas in the place where I am currently at (home or office). If $i = 1$ and it rains then I take the umbrella, move to the other place, where there are already 3 umbrellas, and, including the one I bring, I have next 4 umbrellas. Thus,

$$p_{1,4} = p,$$

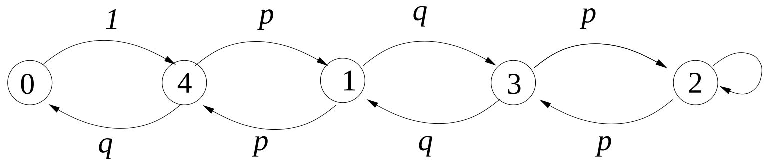
because p is the probability of rain. If $i = 1$ but does not rain then I do not take the umbrella, I go to the other place and find 3 umbrellas. Thus,

$$p_{1,3} = 1 - p \equiv q.$$

Continuing in the same manner, I form a Markov chain with the following diagram:



But this does not look very nice. So let's redraw it:



Let us find the stationary distribution. By equating fluxes, we have:

$$\pi(2) = \pi(3) = \pi(1) = \pi(4)$$
$$\pi(0) = \pi(4)q.$$

Also,

$$\sum_{i=0}^{4} \pi(i) = 1.$$

Expressing all probabilities in terms of $\pi(4)$ and inserting in this last equation, we find

$$\pi(4)q + 4\pi(4) = 1,$$

or

$$\pi(4) = \frac{1}{q+4} = \pi(1) = \pi(2) = \pi(3), \quad \pi(0) = \frac{q}{q+4}.$$

I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is $\pi(0)$. The chance it rains is p. Hence

$$P(WET) = \pi(0) \cdot p = \frac{qp}{q+4}.$$

With $p = 0.6$, i.e. $q = 0.4$, we have

$$P(WET) \approx 0.0545,$$

less than 6%. That's nice.

If I want the chance to be less than 1% then, clearly, I need more umbrellas. So, suppose I need N umbrellas. Set up the Markov chain as above. It is clear that

$$\pi(N) = \pi(N-1) = \cdots = \pi(1),$$
$$\pi(0) = \pi(N)q.$$

Inserting in $\sum_{i=0}^{N} \pi(i)$ we find

$$\pi(N) = \frac{1}{q+N} = \pi(N-1) = \cdots = \pi(1), \quad \pi(0) = \frac{q}{q+N},$$

and so

$$P(WET) = \frac{pq}{q+N}.$$

We want $P(WET) = 1/100$, or $q + N > 100pq$, or

$$N > 100pq - q = 100 \times 0.4 \times 0.6 - 0.4 = 23.6.$$

So to reduce the chance of getting wet from 6% to less than 1% I need 24 umbrellas instead of 4. That's too much. I'd rather get wet.

6.

Suppose that $\xi_0, \xi_1, \xi_2, \ldots$ are independent random variables with common probability function $f(k) = P(\xi_0 = k)$ where k belongs, say, to the integers. Let $S = \{1, \ldots, N\}$. Let X_0 be another random variable, independent of the sequence (ξ_n), taking values in S and let $f : S \times \mathbb{Z} \to S$ be a certain function. Define new random variables $X_1, X_2, \ldots$ by

$$X_{n+1} = f(X_n, \xi_n), \quad n = 0, 1, 2 \ldots$$

(i) Show that the X_n form a Markov chain.
(ii) Find its transition probabilities.

Solution. (i) Fix a time $n \geq 1$. Suppose that you know that $X_n = x$. The goal is to show that PAST$=(X_0, \ldots, X_{n-1})$ is independent of FUTURE$=(X_{n+1}, X_{n+2}, \ldots)$. The variables in the PAST are functions of

$$X_0, \xi_1, \ldots, \xi_{n-2}.$$

The variables in the FUTURE are functions of

$$x, \xi_n, \xi_{n+1}, \ldots$$

But $X_0, \xi_1, \ldots, \xi_{n-2}$ are independent of $\xi_n, \xi_{n+1}, \ldots$. Therefore, the PAST and the FUTURE are independent.

(ii)

$$\begin{aligned} P(X_{n+1} = y | X_n = x) &= P(f(X_n, \xi_n) = y | X_n = x) \\ &= P(f(x, \xi_n) = y | X_n = x) \\ &= P(f(x, \xi_n) = y) \\ &= P(f(x, \xi_0) = y) = P(\xi_0 \in A_{x,y}), \end{aligned}$$

where

$$A_{x,y} := \{\xi : \ f(x, \xi) = y\}.$$

7.

Discuss the topological properties of the graphs of the following Markov chains:

(a) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$ (b) $\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 0 \end{pmatrix}$ (c) $\mathsf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}$

(d) $\mathsf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (e) $\mathsf{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$

Solution. Draw the transition diagram for each case.
(a) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p_{1,1}^{(n)} > 0$ are $1, 2, 3, 4, 5, \ldots$ and their gcd is 1.
(b) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p_{1,1}^{(n)} > 0$ are $1, 2, 3, 4, 5, \ldots$ and their gcd is 1.
(c) Irreducible? NO because starting from state 2 it remains at 2 forever. However, it

can be checked that all states have period 1, simply because $p_{i,i} > 0$ for all $i = 1, 2, 3$.
(d) Irreducible? YES because there is a path from every state to any other state. Aperiodic? NO because the times n for which $p^{(n)}_{1,1} > 0$ are $2, 4, 6, \dots$ and their gcd is 2.
(e) Irreducible? YES because there is a path from every state to any other state. Aperiodic? YES because the times n for which $p^{(n)}_{1,1} > 0$ are $1, 2, 3, 4, 5, \dots$ and their gcd is 1.

8.

Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.
(i) Why is this process a Markov chain?
(ii) What is the state space?
(iii) Is it irreducible? Is it aperiodic?
(iv) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?
(v) Which are the most likely states in steady-state? Which are the least likely ones?

Solution. (i) Part of the problem is to set it up correctly in mathematical terms. When we say that the "knight selects one of the next positions at random independently of the past" we mean that the next position X_{n+1} is a function of the current position X_n and a random choice ξ_n of a neighbour. Hence the problem is in the same form as the one above. Hence (X_n) is a Markov chain.

(ii) The state space is the set of the squares of the chess board. There are $8 \times 8 = 64$ squares. We can label them by a pair of integers. Hence the state space is

$$S = \{(i_1, i_2) : \ 1 \le i_1 \le 8, \ 1 \le i_2 \le 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}.$$

(iii) The best way to see if it is irreducible is to take a knight and move it on a chess board. You will, indeed, realise that you can find a path that takes the knight from any square to any other square. Hence every state communicates with every other state, i.e. it is irreducible.
To see what the period is, find the period for a specific state, e.g. from $(1, 1)$. You can see that, if you start the knight from $(1, 1)$ you can return it to $(1, 1)$ only in even number of steps. Hence the period is 2. So the answer is that the chain is not aperiodic.

(iv) You have no chance in solving a set of 64 equations with 64 unknowns, unless you make an educated guess. First, there is a lot of symmetry. So squares (states) that are symmetric with respect to the centre of the chess board must have the probability under the stationary distribution. So, for example, states $(1, 1)$, $(8, 1)$, $(1, 8)$, $(8, 8)$ have the same probability. And so on. Second, you should realise that $(1, 1)$ must be less likely than a square closer to the centre, e.g. $(4, 4)$. The reason is that $(1, 1)$ has fewer next states (exactly 2) than $(4, 4)$ (which has 8 next states). So let us make the guess that if $x = (i_1, i_2)$, then $\pi(x)$ is proportional to the number $N(x)$ of the possible next states of the square x:

$$\pi(x) = CN(x).$$

But we must SHOW that this choice is correct. Let us say that y us a NEIGHBOUR of x if y is a possible next state of x (if it is possible to move the knight from x to y

in one step). So we must show that such a π satisfies the balance equations:

$$\pi(x) = \sum_{y \in S} \pi(y) p_{y,x}.$$

Equivalently, by cancelling C from both sides, we wonder whether

$$N(x) = \sum_{y \in S} N(y) p_{y,x}$$

holds true. But the sum on the right is zero unless x is a NEIGHBOUR of y:

$$N(x) = \sum_{y \in S:\ x \text{ neighbour of } y} N(y) p_{y,x}$$

But the rule of motion is to choose on of the neighbours with equal probability:

$$p_{y,x} = \begin{cases} \frac{1}{N(y)}, & \text{if } x \text{ is a neighbour of } y \\ 0, & \text{otherwise.} \end{cases}$$

Which means that the previous equation becomes

$$\begin{aligned} N(x) = \sum_{y \in S:\ x \text{ neighbour of } y} N(y) \frac{1}{N(y)} &= \sum_{y \in S:\ x \text{ neighbour of } y} 1 \\ &= \sum_{y \in S:\ y \text{ neighbour of } x} 1, \end{aligned}$$

where in the last equality we used the obvious fact that x is a neighbour of y if and only if y is a neighbour of x (symmetry of the relation) and so the last sum equals, indeed, $N(x)$. So our guess is correct!

Therefore, all we have to do is count the neighbours of each square x. Here we go:

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

We have

$$2 \times 4 + 3 \times 8 + 4 \times 20 + 6 \times 16 + 8 \times 16 = 336.$$

So $C = 1/336$, and

$$\pi(1,1) = 2/336, \quad \pi(1,2) = 3/336, \quad \pi(1,3) = 4/336, \quad \ldots, \quad \pi(4,4) = 8/336, \quad \ldots,$$

etc.

Meaning of π. If we start with

$$P(X_0 = x) = \pi(x), \quad x \in S,$$

then, for all times $n \geq 1$,

$$P(X_n = x) = \pi(x), \quad x \in S.$$

(v) The corner ones are the least likely: 2/336. The 16 middle ones are the most likely: 8/336.

9.

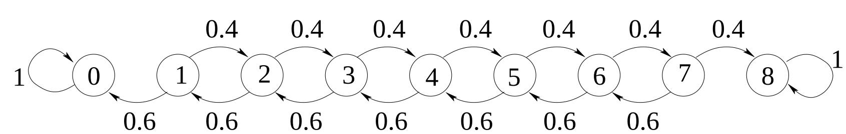
Consider a Markov chain with two states 1, 2. Suppose that $p_{1,2} = a$, $p_{2,1} = b$. For which values of a and b do we obtain an absorbing Markov chain?

Solution. One of them (or both) should be zero. Because, if they are both positive, the chain will keep moving between 1 and 2 forever.

10.

Smith is in jail and has 3 dollars; he can get out on bail if he has 8 dollars. A guard agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability 0.4 and loses A dollars with probability 0.6. Find the probability that he wins 8 dollars before losing all of his money if (a) he bets 1 dollar each time (timid strategy). (b) he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy). (c) Which strategy gives Smith the better chance of getting out of jail?

Solution. (a) The Markov chain $(X_n, n = 0, 1, \ldots)$ representing the evolution of Smith's money has diagram



Let $\varphi(i)$ be the probability that the chain reaches state 8 before reaching state 0, starting from state i. In other words, if S_j is the first $n \geq 0$ such that $X_n = j$,

$$\varphi(i) = P_i(S_8 < S_0) = P(S_8 < S_0 | X_0 = i).$$

Using first-step analysis (viz. the Markov property at time $n = 1$), we have

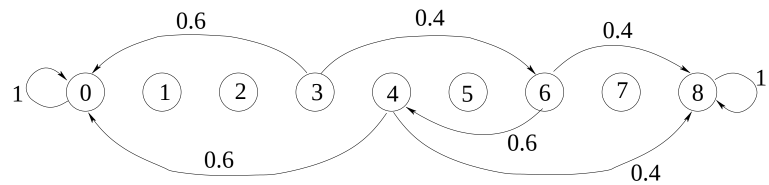
$$\begin{aligned} \varphi(i) &= 0.4\varphi(i+1) + 0.6\varphi(i-1), \quad i = 1, 2, 3, 4, 5, 6, 7 \\ \varphi(0) &= 0 \\ \varphi(8) &= 1. \end{aligned}$$

We solve this system of linear equations and find

$$\begin{aligned} \varphi &= (\varphi(1),\ \varphi(2),\ \varphi(3),\ \varphi(4),\ \varphi(5),\ \varphi(6),\ \varphi(7)) \\ &= (0.0203,\ 0.0508,\ 0.0964,\ 0.1649,\ 0.2677,\ 0.4219,\ 0.6531,\ 1). \end{aligned}$$

E.g., the probability that the chain reaches state 8 before reaching state 0, starting from state 3 is the third component of this vector and is equal to 0.0964. Note that $\varphi(i)$ is increasing in i, which was expected.

(b) Now the chain is



and the equations are:

$$\begin{aligned}
\varphi(3) &= 0.4\varphi(6) \\
\varphi(6) &= 0.4\varphi(8) + 0.6\varphi(4) \\
\varphi(4) &= 0.4\varphi(8) \\
\varphi(0) &= 0 \\
\varphi(8) &= 1.
\end{aligned}$$

We solve and find

$$\varphi(3) = 0.256, \varphi(4) = 0.4, \varphi(6) = 0.64.$$

(c) By comparing the third components of the vector φ we find that the bold strategy gives Smith a better chance to get out jail.

11.

A Markov chain with state space $\{1, 2, 3\}$ has transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that state 3 is absorbing and, starting from state 1, find the expected time until absorption occurs.

Solution. Let $\psi(i)$ be the expected time to reach state 3 starting from state i, where $i \in \{1, 2, 3\}$. We have

$$\begin{aligned}
\psi(3) &= 0 \\
\psi(2) &= 1 + \frac{1}{2}\psi(2) + \frac{1}{2}\psi(3) \\
\psi(1) &= 1 + \frac{1}{3}\psi(1) + \frac{1}{3}\psi(2) + \frac{1}{2}\psi(3).
\end{aligned}$$

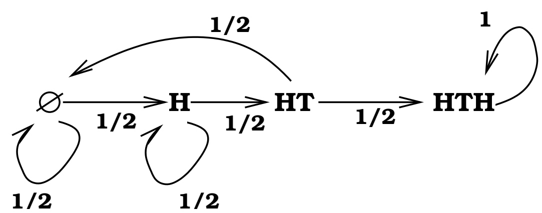
We solve and find

$$\psi(3) = 0, \quad \psi(2) = 2, \quad \psi(1) = 5/2.$$

12.

A fair coin is tossed repeatedly and independently. Find the expected number of tosses till the pattern HTH appears.

Solution. Call HTH our target. Consider a chain that starts from a state called nothing $\emptyset$ and is eventually absorbed at HTH. If we first toss H then we move to state H because this is the first letter of our target. If we toss a T then we move back to $\emptyset$ having expended 1 unit of time. Being in state H we either move to a new state HT if we bring T and we are 1 step closer to the target or, if we bring H, we move back to H: we have expended 1 unit of time, but the new H can be the beginning of a target. When in state HT we either move to HTH and we are done or, if T occurs then we move to $\emptyset$. The transition diagram is



Rename the states $\emptyset$, H, HT, HTH as $0, 1, 2, 3$, respectively. Let $\psi(i)$ be the expected number of steps to reach HTH starting from i. We have

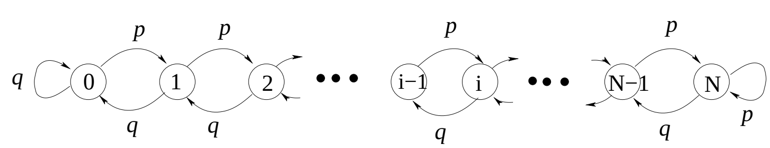
$$\psi(2) = 1 + \frac{1}{2}\psi(0)$$
$$\psi(1) = 1 + \frac{1}{2}\psi(1) + \frac{1}{2}\psi(2)$$
$$\psi(0) = 1 + \frac{1}{2}\psi(0) + \frac{1}{2}\psi(1).$$

We solve and find $\psi(0) = 10$.

13.

Consider a Markov chain with states $S = \{0, \dots, N\}$ and transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = q$, for $1 \le i \le N-1$, where $p + q = 1$, $0 < p < 1$; assume $p_{0,1} = 1$, $p_{N,N-1} = 1$.
1. Draw the graph (= transition diagram).
2. Is the Markov chain irreducible?
3. Is it aperiodic?
4. What is the period of the chain?
5. Find the stationary distribution.

Solution. 1. The transition diagram is:



2. Yes, it is possible to go from any state to any other state.

3. Yes, because $p_{0,0} > 0$.

4. One.

5. We write balance equations by equating fluxes:

$$\pi(i)q = \pi(i-1)p,$$

as long as $1 \le i \le N$. Hence

$$\pi(i) = \frac{p}{q}\pi(i-1) = \left(\frac{p}{q}\right)^2 \pi(i-2) = \cdots = \left(\frac{p}{q}\right)^i \pi(0), \quad 0 \le i \le N.$$

Since

$$\pi(0) + \pi(1) + \ldots + \pi(N-1) + \pi(N) = 1,$$

we find

$$\pi(0)\left[1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^N\right] = 1,$$

which gives

$$\pi(0) = \left[1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^N\right]^{-1} = \frac{(p/q)^N - 1}{(p/q) - 1},$$

as long as $p \neq q$. Hence, if $p \neq q$,

$$\pi(i) = \frac{(p/q)^N - 1}{(p/q) - 1}\left(\frac{p}{q}\right)^i, \quad 0 \le i \le N.$$

If $p = q = 1/2$, then

$$\pi(0) = \left[1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots + \left(\frac{p}{q}\right)^N\right]^{-1} \frac{1}{N+1},$$

and so

$$\pi(i) = \frac{1}{N+1}, \text{ for all } i.$$

Thus, in this case, $\pi(i)$ is the uniform distribution on the set of states.

14.

A. Assume that an experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is

$$\frac{m^k - 1}{m - 1}.$$

Hint: Form an absorbing Markov chain with states $1, 2, \ldots, k$ *with state* i *representing the length of the current run. The expected time until a run of* k *is 1 more than the expected time until absorption for the chain started in state 1.*
B. It has been found that, in the decimal expansion of $\pi = 3.14159\ldots$, starting with the 24,658,601st digit, there is a run of nine 7's. What would your result say about the expected number of digits necessary to find such a run if the digits are produced randomly?

Solution. A. Let the outcomes be $a, b, c, \ldots$ (m of them in total). Suppose that a is the desirable outcome. We set up a chain as follows. Its states are

$$\emptyset, \quad (a), \quad (aa), \quad (aaa), \quad \ldots, \quad \underbrace{(aa\cdots a)}_{m \text{ times}}$$

Or, more simply, $0, 1, 2, \ldots, m$. State k means that you are currently at the end of a run of k a's. If you see an extra a (with probability $1/m$) you go to state $k+1$. Otherwise, you go to $\emptyset$. Let $\psi(k)$ be the expected number of steps till state m is reached, starting from state k:

$$\psi(k) := E_k S_m.$$

We want to find $\psi(0)$. We have

$$\psi(k) = 1 + (1 - 1/m)\psi(0) + (1/m)\psi(k+1).$$

Solving these, we find

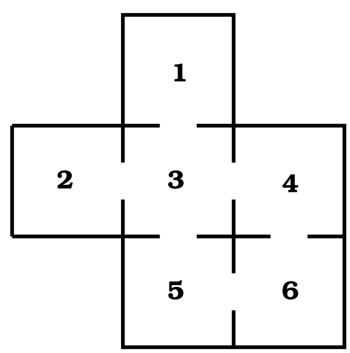
$$\psi(0) = 1 + m + m^2 + \cdots + m^{k-1} = \frac{m^k - 1}{m - 1}.$$

B. So to get 10 consecutive sixes by rolling a die, you need more than 12 million rolls on the average (12, 093, 235 rolls to be exact).

C. They are not random. If they were, we expect to have to pick $(10^9 - 1)/9$ digits before we see nine consecutive sevens. That's about 100 million digits. The actual position (24 million digits) is one fourth of the expected one.

15.

A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room.



(a) Give the transition matrix P for this Markov chain. (b) Show that it is irreducible but not aperiodic. (c) Find the stationary distribution (d) Now suppose that a piece of mature cheddar is placed on a deadly trap in Room 5. The mouse starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1. (e) Find the expected time to return to room 1.

Solution

(a) The transition matrix P for this Markov chain is as follows:

$$\mathsf{P} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}.$$

(b) The chain is irreducible, because it is possible to go from any state to any other state. However, it is not aperiodic, because for any n even $p^{(n)}_{6,1}$ will be zero and for any n odd $p^{(n)}_{6,5}$ will also be zero (why?). This means that there is no power of P that would have all its entries strictly positive.
(c) The stationary distribution is

$$\pi = (\frac{1}{12}, \frac{1}{12}, \frac{4}{12}, \frac{2}{12}, \frac{2}{12}, \frac{2}{12}).$$

You should carry out the calculations and check that this is correct.
(d) We find from π that the mean recurrence time (i.e. the expected time to return) for the room 1 is $1/\pi(1)$=12.
(e) Let

$$\psi(i) = E(\text{number of steps to reach state 5} \mid X_0 = i).$$

We have

$$\begin{aligned}
\psi(5) &= 0 \\
\psi(6) &= 1 + (1/2)\psi(5) + (1/2)\psi(4) \\
\psi(4) &= 1 + (1/2)\psi(6) + (1/2)\psi(3) \\
\psi(3) &= 1 + (1/4)\psi(1) + (1/4)\psi(2) + (1/4)\psi(4) + (1/4)\psi(5) \\
\psi(1) &= 1 + \psi(3) \\
\psi(2) &= 1 + \psi(3).
\end{aligned}$$

We solve and find $\psi(1) = 7$.

16.

Show that if P is the transition matrix of an irreducible chain with finitely many states, then $Q := (1/2)(I + \mathsf{P})$ is the transition matrix of an irreducible and aperiodic chain. (Note that I stands for the identity matrix, i.e. the matrix which has 1 everywhere on its diagonal and 0 everywhere else.)
Show that P and $(1/2)(I + \mathsf{P})$ have the same stationary distributions.
Discuss, physically, how the two chains are related.

Solution. Let p_{ij} be the entries of P. Then the entries q_{ij} of Q are

$$\begin{aligned}
q_{ij} &= \frac{1}{2} p_{ij}, \quad \text{if } i \neq j, \\
q_{ii} &= \frac{1}{2}(1 + p_{ii}).
\end{aligned}$$

The graph of the new chain has more arrows than the original one. Hence it is also irreducible. But the new chain also has self-loops for each i because $q_{ii} > 0$ for all i. Hence it is aperiodic.

Let π be a stationary distribution for P. Then

$$\pi P = \pi.$$

We must show that

$$\pi Q = \pi.$$

But

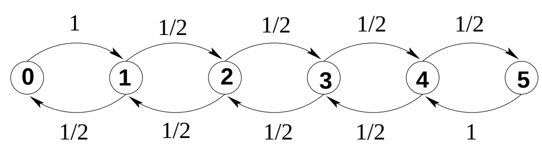
$$\pi Q = \frac{1}{2}(\pi I + \pi \mathsf{P}) = \frac{1}{2}(\pi + \pi) = \pi.$$

The physical meaning of the new chain is that it represents a slowing down of the original one. Indeed, all outgoing probabilities have been halved, while the probability of staying at the same state has been increased. The chain performs the same transitions as the original one but stays longer at each state.

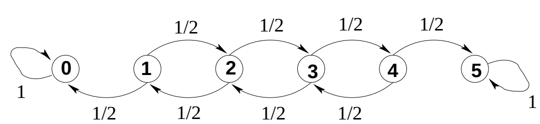
17.

Two players, A and B, play the game of matching pennies: at each time n, each player has a penny and must secretly turn the penny to heads or tails. The players then reveal their choices simultaneously. If the pennies match (both heads or both tails), Player A wins the penny. If the pennies do not match (one heads and one tails), Player B wins the penny. Suppose the players have between them a total of 5 pennies. If at any time one player has all of the pennies, to keep the game going, he gives one back to the other player and the game will continue. (a) Show that this game can be formulated as a Markov chain. (b) Is the chain regular (irreducible + aperiodic?) (c) If Player A starts with 3 pennies and Player B with 2, what is the probability that A will lose his pennies first?

Solution (a) The problem is easy: The probability that two pennies match is 1/2. The probability they do not match is 1/2. Let x be the number of pennies that A has. Then with probability 1/2 he will next have $x+1$ pennies or with probability 1/2 he will next have $x-1$ pennies. The exception is when $x = 0$, in which case, he gets, for free, a penny from B and he next has 1 penny. Also, if $x = 5$ he gives a penny to B and he next has 4 pennies. Thus:



(b) The chain is clearly irreducible. But the period is 2. Hence it is not regular.
(c) To do this, modify the chain and make it stop once one of the players loses his pennies. After all, we are NOT interested in the behaviour of the chain after this time. The modification is an absorbing chain:



We then want to compute the absorbing probability $\varphi_{01}(3)$ where

$$\varphi_{01}(i) = P_i(\text{hit 0 before 1}).$$

Write $\varphi(i) = \varphi_{01}(i)$, for brevity, and apply first-step analysis:

$$\begin{aligned}
\varphi(0) &= 1 \\
\varphi(1) &= \frac{1}{2}\varphi(0) + \frac{1}{2}\varphi(1) \\
\varphi(2) &= \frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(2) \\
\varphi(3) &= \frac{1}{2}\varphi(2) + \frac{1}{2}\varphi(3) \\
\varphi(4) &= \frac{1}{2}\varphi(3) + \frac{1}{2}\varphi(4) \\
\varphi(5) &= 0.
\end{aligned}$$

Six equations with six unknowns. Solve and find: $\varphi(3) = 2/5$.
Alternatively, observe, from Thales' theorem,[6] that φ must be a straight line:

$$\varphi(x) = ax + b.$$

From $\varphi(0) = 1$, $\varphi(5) = 0$, we find $a = -1/5$, $b = 1$, i.e.

$$\varphi(i) \equiv 1 - (i/5),$$

which agrees with the above.

18.

A process moves on the integers 1, 2, 3, 4, and 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five.

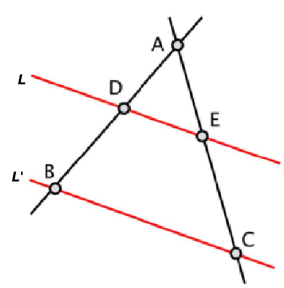
Solution. A Markov chain is defined and its transition probability matrix is as follows:

$$\mathsf{P} = \begin{pmatrix} 0 & \frac14 & \frac14 & \frac14 & \frac14 \\ 0 & 0 & \frac13 & \frac13 & \frac13 \\ 0 & 0 & 0 & \frac12 & \frac12 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We apply first step analysis for the function

$$\psi(i) := E_i S_5, \quad 1 \le i \le 5,$$

[6] Thales' theorem says (proved around the year 600 BCE) says that if the lines L, L' are parallel then $\frac{DE}{BC} = \frac{AE}{AC} = \frac{AD}{AB}$.



where $S_5 = \inf\{n \geq 0 : \; X_n = 5\}$. One of the equations is $\psi(5) = 0$ (obviously). Another is

$$\psi(1) = 1 + \frac{1}{4}\psi(2) + \frac{1}{4}\psi(3) + \frac{1}{4}\psi(3) + \frac{1}{4}\psi(5).$$

It's up to you to write the remaining equations and solve to find

$$\psi(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.0833.$$

19. ---

Generalise the previous exercise, by replacing 5 by a general positive integer n. Find the expected number of steps to reach state n, when starting from state 1. Test your conjecture for several different values of n. Can you conjecture an estimate for the expected number of steps to reach state n, for large n?

Solution. The answer here is

$$E_1 S_n = \sum_{k=1}^{n-1} \frac{1}{k}.$$

We here recognise the harmonic series:

$$\sum_{k=1}^{n} \frac{1}{k} \approx \log n,$$

for large n, in the sense that the difference of the two sides converges to a constant. So,

$$E_1 S_n \approx \log n,$$

when n is large.

20. ---

A gambler plays a game in which on each play he wins one dollar with probability p and loses one dollar with probability $q = 1 - p$. The **Gambler's Ruin Problem** is the problem of finding

$$\begin{aligned}\varphi(x) :=& \text{the probability of winning an amount } b \\ & \text{before losing everything, starting with state } x \\ =& P_x(S_b < S_0).\end{aligned}$$

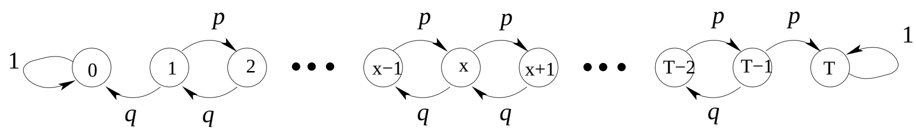
1. Show that this problem may be considered to be an absorbing Markov chain with states $0, 1, 2, \ldots, b$, with 0 and b absorbing states.
2. Write down the equations satisfied by $\varphi(x)$.
3. If $p = q = 1/2$, show that

$$\varphi(x) = x/b.$$

4. If $p \neq q$, show that

$$\varphi(x) = \frac{(q/p)^x - 1}{(q/p)^b - 1}.$$

Solution. 1. If the current fortune is x the next fortune will be either $x+1$ or $x-1$, with probability p or 1, respectively, as long as x is not b or x is not 0. We assume independence between games, so the next fortune will not depend on the previous ones; whence the Markov property. If the fortune reaches 0 then the gambler must stop playing. So 0 is absorbing. If it reaches b then the gambler has reached the target hence the play stops again. So both 0 and T are absorbing states. The transition diagram is:



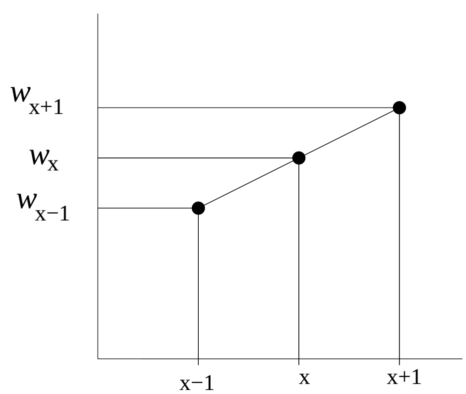
2. The equations are:

$$\begin{aligned}
\varphi(0) &= 0 \\
\varphi(b) &= 1 \\
\varphi(x) &= p\varphi(x+1) + q\varphi(x-1), \quad x = 1, 2, \ldots, b-1.
\end{aligned}$$

3. If $p = q = 1/2$, we have

$$\varphi(x) = \frac{\varphi(x+1) + \varphi(x-1)}{2}, \quad x = 1, 2, \ldots, b-1.$$

This means that the point $(x, \varphi(x))$ in the plane is in the middle of the segment with endpoints $(x-1, \varphi(x-1))$, $(x+1, \varphi(x+1))$. Hence the graph of the function $\varphi(x)$ must be on a straight line (Thales' theorem):



In other words,

$$\varphi(x) = Ax + B.$$

We determine the constants A, B from $\varphi(0) = 0$, $\varphi(b) = 1$. Thus, $\varphi(x) = x/b$.

4. If $p \neq q$, then this nice linear property does not hold. However, if we substitute the given function to the equations, we see that they are satisfied.

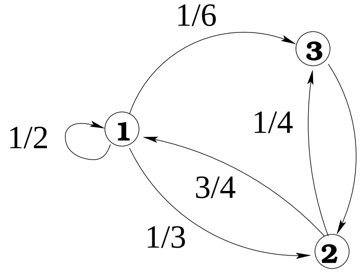
21.

Consider the Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 3/4 & 0 & 1/4 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Show that this is irreducible and aperiodic.
(b) The process is started in state 1; find the probability that it is in state 3 after two steps.
(c) Find the matrix which is the limit of P^n as $n \to \infty$.

Solution



(a) Draw the transition diagram and observe that there is a path from every state to any other state. Hence it is irreducible. Now consider a state, say state $i = 1$ and the times n at which $p_{1,1}^{(n)} > 0$. These times are $1, 2, 3, 4, 5, \dots$ and their gcd is 1. Hence it is aperiodic. So the chain is regular.
(b)

$$\begin{aligned} P_1(X_2 = 3) = p_{1,3}^{(2)} &= \sum_{i=1}^{3} p_{1,i}\ p_{i,3} \\ &= p_{1,1}p_{1,3} + p_{1,2}p_{2,3} + p_{1,3}p_{3,3} \\ &= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot 0 = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}. \end{aligned}$$

(c) The limit exists because the chain is regular. It is given by

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \pi(1) & \pi(2) & \pi(3) \\ \pi(1) & \pi(2) & \pi(3) \\ \pi(1) & \pi(2) & \pi(3) \end{pmatrix}$$

where $\boldsymbol{\pi} = (\pi(1), \pi(2), \pi(3))$ is the stationary distribution which is found by solving the balance equations

$$\pi \mathsf{P} = \pi,$$

together with

$$\pi(1) + \pi(2) + \pi(3) = 1.$$

The balance equations are equivalent to

$$\begin{aligned} \pi(1)\frac{1}{6} + \pi(1)\frac{1}{3} &= \pi(2)\frac{3}{4} \\ \pi(3) &= \pi(2)\frac{1}{4} + \pi(1)\frac{1}{6}. \end{aligned}$$

Solving the last 3 equations with 3 unknowns we find

$$\pi(1) = \frac{3}{6}, \quad \pi(2) = \frac{2}{6}, \quad \pi(3) = \frac{1}{6}.$$

Hence

$$\lim_{n\to\infty} \mathsf{P}^n = \begin{pmatrix} 3/6 & 2/6 & 1/6 \\ 3/6 & 2/6 & 1/6 \\ 3/6 & 2/6 & 1/6 \end{pmatrix}.$$

22.

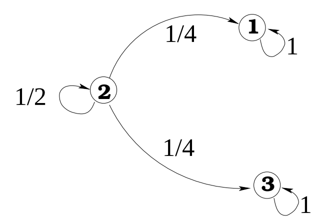
Show that a Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix}$$

has more than one stationary distributions. Find the matrix that P^n converges to, as $n \to \infty$, and verify that it is not a matrix all of whose rows are the same.

You should work out this exercise by direct methods, without appealing to the general limiting theory of Markov chains–see lecture notes.

Solution. The transition diagram is:



Write the balance equations $\pi\mathsf{P} = \pi$:

$$\begin{pmatrix} \pi(1) & \pi(2) & \pi(3) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi(1) & \pi(2) & \pi(3) \end{pmatrix}$$

or

$$\pi(1) \cdot 1 + \pi(2) \cdot (1/4) + \pi(3) \cdot 0 = \pi(1) \tag{1}$$

$$\pi(1) \cdot 0 + \pi(2) \cdot (1/2) + \pi(3) \cdot 0 = \pi(2) \tag{2}$$

$$\pi(1) \cdot 0 + \pi(2) \cdot (1/4) + \pi(3) \cdot 1 = \pi(3), \tag{3}$$

together with the normalisation condition $\sum \pi(i) = 1$, i.e.

$$\pi(1) + \pi(2) + \pi(3) = 1. \tag{4}$$

and solve for $\pi(1), \pi(2), \pi(3)$. Equation (1) gives

$$\pi(2) = 0.$$

Equation (2) gives

$$\pi(2) = \pi(2),$$

i.e. it is useless. Equation (3) gives

$$\pi(3) = \pi(3),$$

again, obviously true. Equation (4) gives

$$\pi(1) + \pi(3) = 1.$$

Therefore, equations (1)–(4) are EQUIVALENT TO:

$$\pi(2) = 0, \quad \pi(1) + \pi(3) = 1.$$

Hence we can set $\pi(1)$ to ANY value we like between 0 and 1, say, $\pi(1) \equiv p$, and then let $\pi(3) = 1-p$. Thus there is not just one stationary distribution but infinitely many. For each value of $p \in [0,1]$, any $\boldsymbol{\pi}$ of the form

$$\pi = \begin{pmatrix} p & 0 & 1-p \end{pmatrix}$$

is a stationary distribution.

To find the limit of P^n as $n \to \infty$, we compute the entries of the matrix P^n. Notice that the (i,j)-entry of P^n equals

$$p_{i,j}^{(n)} = P_i(X_n = j).$$

If $i = 1$ we have

$$P_1(X_n = 1) = 1, \quad P_1(X_n = 2) = 0, \quad P_1(X_n = 3) = 0,$$

because state 1 is absorbing. Similarly, state 3 is absorbing:

$$P_3(X_n = 1) = 0, \quad P_3(X_n = 2) = 0, \quad P_3(X_n = 3) = 1.$$

We thus know the first and third rows of P^n:

$$\mathsf{P}^n = \begin{pmatrix} 1 & 0 & 0 \\ p_{2,1}^{(n)} & p_{2,2}^{(n)} & p_{2,3}^{(n)} \\ 0 & 0 & 1 \end{pmatrix}.$$

We now compute the missing entries of the second row by simple observations, based on the fact that the chain, started in state 2, will remain at 2 for some time and then will leave it and either go to 1 or 3:

$$P_2(X_n = 2) = P_2(\text{chain has stayed in state 2 for } n \text{ consecutive steps}) = (1/2)^n.$$

$$\begin{aligned} P_2(X_n = 1) &= \sum_{m=1}^{n} P_2(X_{m-1} = 2, X_m = 1) \\ &= \sum_{m=1}^{n} (1/2)^{m-1} \cdot (1/4) \\ &= \frac{1-(1/2)^n}{1-(1/2)} \cdot \frac{1}{4} = \frac{1-0.5^n}{2}. \end{aligned}$$

$$P_2(X_n = 3) = 1 - P_2(X_n = 2) - P_2(X_n = 1) = \frac{1-0.5^n}{2}.$$

Therefore,

$$\mathsf{P}^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1-0.5^n}{2} & (0.5)^n & \frac{1-0.5^n}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $0.5^n \to 0$ as $n \to \infty$, we have

$$\mathsf{P}^n \to \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{as } n \to \infty.$$

23.

Toss a fair die repeatedly. Let S_n denote the total of the outcomes through the nth toss. Show that there is a limiting value for the proportion of the first n values of S_n that are divisible by 7, and compute the value for this limit.
Hint: The desired limit is a stationary distribution for an appropriate Markov chain with 7 states.

Solution. An integer $k \geq 1$ is divisible by 7 if it leaves remainder 0 when divided by 7. When we divide an integer $k \geq 1$ by 7, the possible remainders are

$$0, 1, 2, 3, 4, 5, 6.$$

Let $X_1, X_2, \dots$ be the outcomes of a fair die tossing. These are i.i.d. random variables uniformly distributed in $\{1, 2, 3, 4, 5, 6\}$. We are asked to consider the sum

$$S_n = X_1 + \cdots + X_n.$$

Clearly, S_n is an integer. We are interested in the remainder of S_n when divided by 7. Call this R_n. So:

$$R_n := \text{ the remainder of the division of } S_n \text{ by } 7.$$

Note that the random variables $R_1, R_2, R_3, \dots$ form a Markov chain because if we know the value of R_n, all we have to do to find the next value R_{n+1} is to add X_n to R_n, divide by 7, and take the remainder of this division, as in elementary-school arithmetic:

$$R_{n+1} = \text{ the remainder of the division of } R_n + X_n \text{ by } 7.$$

We need to find the transition probabilities

$$\begin{aligned} p_{i,j} :=& P(R_{n+1} = j | R_n = i) \\ =& P(\text{ the remainder of the division of } i + X_n \text{ by } 7 \text{ equals } j) \end{aligned}$$

for this Markov chain, for all $i, j \in \{0, 1, 2, 3, 4, 5, 6\}$. But X_n takes values in $\{1, 2, 3, 4, 5, 6\}$ with equal probabilities 1/6. If to an i we add an x chosen from $\{1, 2, 3, 4, 5, 6\}$ and then divide by 7 we are going to obtain *any* j in $\{0, 1, 2, 3, 4, 5, 6\}$. Therefore,

$$p_{i,j} = 1/6, \qquad \text{for all } i \text{ and all } j \in \{0, 1, 2, 3, 4, 5, 6\}.$$

We are asked to consider the proportion of the first n values of S_n that are divisible by 7, namely the quantity

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(R_k = 0).$$

This quantity has a limit from the Strong Law of Large Numbers for Markov chains and the limit is the stationary distribution at state 0:

$$P\left(\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}(R_k = 0) = \pi(0) \right) = 1$$

Therefore we need to compute π for the Markov chain (R_n). This is very easy. From symmetry, all states i must have the same $\pi(i)$. Therefore

$$\pi(i) = 1/7, \quad i = 0, 1, 2, 3, 4, 5, 6.$$

Hence

$$P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}(R_k = 0) = 1/7\right) = 1.$$

In other words, *if you toss a fair die* $10,000$ *times then approximately* 1667 *times* n *you had a sum* S_n *that was divisible by* 7*, and this is true with probability very close to* 1*.*

24.

(i) Consider a Markov chain on the vertices of a triangle: the chain moves from one vertex to another with probability 1/2. Find the probability that, in n steps, the chain returns to the vertex it started from.
(ii) Suppose that we alter the probabilities as follows:

$$p_{12} = p_{23} = p_{31} = 2/3, \quad p_{21} = p_{32} = p_{13} = 1/3.$$

Answer the same question as above.

Solution. (i) The transition matrix is

$$\mathsf{P} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial is

$$\det(xI - \mathsf{P}) = x^3 - \frac{1}{2}x^2 - \frac{1}{2}x$$

whose roots are

$$x_1 := 0, \quad x_2 := 1, \quad x_3 := -\frac{1}{2}.$$

Therefore,

$$p_{11}^{(n)} = C_1 x_1^n + C_2 x_2^n + C_3 x_3^n = C_2 x_2^n + C_3 x_3^n,$$

where C_2, C_3 are constants. Since, clearly, $p_{11}^{(0)} = 1$, $p_{11}^{(1)} = p_{11} = 0$, we have

$$C_2 + C_3 = 1$$
$$C_2 x_2 + C_3 x_3 = 0.$$

Solving, we find $C_2 = 1/3$, $C_3 = 2/3$. So

$$p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3}(-1/2)^n.$$

(ii) We now have

$$\mathsf{P} = \frac{1}{3}\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

The characteristic polynomial is

$$\det(xI - \mathsf{P}) = x^3 - \frac{2}{3}x - \frac{1}{3} = \frac{1}{3}(3x^3 - 2x - 1) := \frac{1}{3}f(x).$$

Checking the divisors of the constant (1 or 2), we are lucky because we see that 1 is a zero:

$$f(1) = 3 - 2 - 1 = 0.$$

So we divide $f(x)$ with $x - 1$. Since

$$3x^2(x-1) = 3x^3 - 3x^2,$$

we have

$$f(x) - 3x^2(x-1) = 3x^2 - 2x - 1.$$

Since

$$3x(x-1) = 3x^2 - 3x,$$

we have

$$3x^2 - 2x - 1 - 3x(x-1) = x - 1.$$

Therefore,

$$\begin{aligned} f(x) &= 3x^2(x-1) + 3x^2 - 2x - 1 \\ &= 3x^2(x-1) + 3x(x-1) + (x-1) \\ &= (3x^2 + 3x + 1)(x-1). \end{aligned}$$

So the other roots of $f(x) = 0$ are the roots of $3x^2 + 3x + 1 = 0$. The discriminant of this quadratic is

$$3^2 - 4 \cdot 3 \cdot 1 = -3 < 0,$$

so the roots are complex:

$$x_1 = \frac{-1}{2} + \frac{\sqrt{-3}}{6}, \quad x_2 = \frac{-1}{2} - \frac{\sqrt{-3}}{6}.$$

Letting $x_3 = 1$ (the first root we found), we now have

$$p_{11}^{(n)} = C_1 x_1^n + C_2 x_2^n + C_3.$$

We need to determine the constants C_1, C_2, C_3. But we have

$$\begin{aligned} 1 &= p_{11}^{(0)} = C_1 + C_2 + C_3 \\ 0 &= p_{11}^{(1)} = C_1 x_1 + C_2 x_2 + C_3 x_3 \\ \frac{2}{9} &= p_{11}^{(2)} = C_1 x_1^2 + C_2 x_2^2 + C_3 x_3^2 \end{aligned}$$

Solving for the constants, we find

$$p_{11}^{(n)} = \frac{1}{3} + \frac{2}{3}(-1/\sqrt{3})^n \cos(n\pi/6).$$

25.

A certain experiment is believed to be described by a two-state Markov chain with the transition matrix P, where

$$\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ p & 1-p \end{pmatrix},$$

and the parameter p is not known. When the experiment is performed many times, the chain ends in state one approximately 20 percent of the time and in state two approximately 80 percent of the time. Compute a sensible estimate for the unknown parameter p and explain how you found it.

Solution. If X_k is the position of the chain at time k, we are being told that when we perform the experiment (i.e. watch the chain), say, n times we see that approximately 20% of the time the chain is in state 1:

$$\frac{1}{n}\sum_{n=1}^{n} \mathbf{1}(X_k = 1) \approx 0.2 \tag{5}$$

We know, from the Strong Law (=Theorem) of Large Numbers that

$$P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{N} \mathbf{1}(X_k = 1) = \pi(1)\right) = 1, \tag{6}$$

where $\pi = (\pi(1),\ \pi(2))$ is the stationary distribution. Combining the observation (5) with the Law of Large Numbers (6) we obtain

$$\pi(1) \approx 0.2.$$

We can easily compute $\boldsymbol{\pi}$ because

$$\pi(1)\frac{1}{2} = \pi(2)p,$$

and, of course,

$$\pi(1) + \pi(2) = 1,$$

whence

$$\pi(1) = \frac{2p}{1+2p}.$$

Solving $\frac{2p}{1+2p} = 0.2$ we find $p = 1/8$.

26.

Here is a trick to try on your friends. Shuffle a deck of cards and deal out one at a time. Count the face cards each as ten. Ask your friend to at one of the first ten cards; if this card is a six, she is to look at the card turns up six cards later; if this card is a three, she is to look at the card turns up three cards later, and so forth. Eventually she will reach a where she is to look at a card that turns up x cards later but there are x cards left. You then tell her the last card that she looked at even though you did not know her starting point. You tell her you do this by watching her, and she cannot disguise the times that she looks at the cards. In fact just do the same procedure and, even though you do not start at the point as she does, you will most likely end at the same point. Why?

Solution. Let X_n denote the value of the n-th card of the experiment when you start from the x-th card from the top. Let Y_n denote the value of the n-th card of another experiment when you start from the y-th card from the top. You use exactly the same deck with the cards in the same order in both experiments. If, for some n and some m we have

$$X_n = Y_m,$$

then $X_{n+1} = Y_{m+1}$, $X_{n+2} = Y_{m+2}$, etc. The point is that the event

$$\{\exists m, n \text{ such that } X_n = Y_m\}$$

has a large probability. In fact, it has probability close to 1.

27. ---

You have N books on your shelf, labelled $1, 2, \ldots, N$. You pick a book j with probability $1/N$. Then you place it on the left of all others on the shelf. You repeat the process, independently. Construct a Markov chain which takes values in the set of all $N!$ permutations of the books.
(i) Discuss the state space of the Markov chain. Think how many elements it has and how are its elements represented.
(ii) Show that the chain is regular (irreducible and aperiodic) and find its stationary distribution.
Hint: You can guess the stationary distribution before computing it.

Solution. (i) The state space is

$$S = \{\text{all function } \sigma : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, N\} \text{ which are one-to-one and onto}\}.$$

These σ are called permutations and there are $N!$ of them:

$$|S| = N!$$

Each σ can be represented by a the list of each values:

$$\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(N))$$

i.e. $\sigma(i)$ is its values at i.

(ii) Let us find the transition probabilities. If σ is the current state and we pick the j-th book and place it in front, then the next state is the same if $j = 1$ or

$$(\sigma(j), \sigma(1), \sigma(2), \ldots, \sigma(j-1), \sigma(j+1), \ldots),$$

if $j \neq 1$. There are N possible next states and each occurs with probability $1/N$. If we denote the next state obtained when picking the j-th book by $\sigma^{(j)}$ then we have

$$p_{\sigma, \sigma^{(j)}} = 1/N, \quad j = 1, \ldots, N.$$

(For example, $\sigma^{(1)} = \sigma$.) And, of course, $p_{\sigma,\tau} = 0$ if τ is not of the form $\sigma^{(j)}$ for some j. The chain is aperiodic because $p_{\sigma,\sigma} = 1/N$ for all σ. It is irreducible because, clearly, it can move from any state (i.e. any arrangement of books) to any other. Hence it is regular.

It does not require a lot of thought to see that there is complete symmetry! Therefore all states must have the same stationary distribution, i.e.

$$\pi(\sigma) = \frac{1}{N!}, \quad \text{for all } \sigma \in S.$$

You can easily verify that

$$\pi(\sigma) = \sum_{\tau} \pi(\tau) p_{\tau,\sigma}, \quad \text{for all } \sigma \in S,$$

i.e. the balance equations are satisfied and so our educated guess was correct.

28.

In unprofitable times corporations sometimes suspend dividend payments. Suppose that after a dividend has been paid the next one will be paid with probability 0.9, while after a dividend is suspended the next one will be suspended with probability 0.6. In the long run what is the fraction of dividends that will be paid?

Solution. We here have a Markov chain with two states:
State 1: "dividend paid"
State 2: "dividend suspended"
We are given the following transition probabilities:

$$p_{1,1} = 0.9, \quad p_{2,2} = 0.6$$

Hence

$$p_{1,2} = 0.1, \quad p_{2,1} = 0.4$$

Let π be the stationary distribution. In the long run the fraction of dividends that will be paid equals $\pi(1)$. But

$$\pi(1) \times 0.1 = \pi(2) \times 0.4$$

and

$$\pi(1) + \pi(2) = 1,$$

whence

$$\pi(1) = 4/5.$$

So, in the long run, 80% of the dividends will be paid.

29.

Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n.
(a) Compute the transition probability for X_n.
(b) Find the stationary distribution and show that it corresponds to picking five balls at random to be in the left urn.

Solution Clearly, $(X_0, X_1, X_2, \ldots)$ is a Markov chain with state space

$$S = \{0, 1, 2, 3, 4, 5\}.$$

(a) If, at some point of time, $X_n = x$ (i.e. the number of white balls in the left urn is x) then there are $5-x$ black balls in the left urn, while the right urn contains x black and $5-x$ white balls. Clearly,

$$\begin{aligned}p_{x,x+1} &= P(X_{n+1} = x+1 | X_n = x) \\ &= P(\text{pick a white ball from the right urn and a black ball from the left urn}) \\ &= \frac{5-x}{5} \times \frac{5-x}{5},\end{aligned}$$

as long as $x < 5$. On the other hand,

$$\begin{aligned}p_{x,x-1} &= P(X_{n+1} = x-1 | X_n = x) \\ &= P(\text{pick a white ball from the left urn and a black ball from the right urn}) \\ &= \frac{x}{5} \times \frac{x}{5},\end{aligned}$$

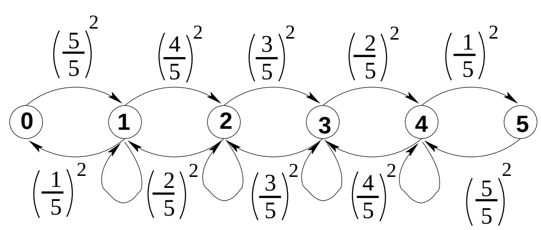
as long as $x > 0$. When $0 < x < 5$, we have

$$p_{x,x} = 1 - p_{x,x+1} - p_{x,x-1},$$

because there is no chance that the number of balls change by more than 1 ball. Summarising, the answer is:

$$p_{x,y} = \begin{cases} \left(\frac{5-x}{5}\right)^2, & \text{if } 0 \le x \le 4, \quad y = x+1 \\ \left(\frac{x}{5}\right)^2, & \text{if } 1 \le x \le 5, \quad y = x-1 \\ 1 - \left(\frac{5-x}{5}\right)^2 - \left(\frac{x}{5}\right)^2, & \text{if } 1 \le x \le 4, \quad y = x, \\ 0, & \text{in all other cases.} \end{cases}$$

If you want, you may draw the transition diagram:



On this diagram, I did not indicate the $p_{x,x}$.
(b) To compute the stationary distribution, cut the diagram between states x and $x-1$ and equate the two flows, as usual:

$$\pi(x) p_{x,x-1} = \pi(x-1) p_{x-1,x},$$

i.e.

$$\pi(x) \left(\frac{x}{5}\right)^2 = \pi(x-1) \left(\frac{5-(x-1)}{5}\right)^2$$

which gives

$$\pi(x) = \left(\frac{6-x}{x}\right)^2 \pi(x-1)$$

We thus have

$$\begin{aligned}
\pi(1) &= \binom{5}{1}^2 \pi(0) = 25\pi(0) \\
\pi(2) &= \binom{4}{2}^2 \pi(1) = \binom{4}{2}^2 \binom{5}{1}^2 \pi(0) = 100\pi(0) \\
\pi(3) &= \binom{3}{3}^2 \pi(2) = \binom{3}{3}^2 \binom{4}{2}^2 \binom{5}{1}^2 \pi(0) = 100\pi(0) \\
\pi(4) &= \binom{2}{4}^2 \pi(3) = \binom{2}{4}^2 \binom{3}{3}^2 \binom{4}{2}^2 \binom{5}{1}^2 \pi(0) = 25\pi(0) \\
\pi(5) &= \binom{1}{5}^2 \pi(4) = \binom{1}{5}^2 \binom{2}{4}^2 \binom{3}{3}^2 \binom{4}{2}^2 \binom{5}{1}^2 \pi(0) = \pi(0).
\end{aligned}$$

We find $\pi(0)$ by normalisation:

$$\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) + \pi(5) = 1$$
$$\Rightarrow \pi(0) = 1/(1 + 25 + 100 + 100 + 25 + 1) = 1/252.$$

Putting everything together, we have

$$\pi(0) = \frac{1}{252}, \quad \pi(1) = \frac{25}{252}, \quad \pi(2) = \frac{100}{252}, \quad \pi(3) = \frac{100}{252}, \quad \pi(4) = \frac{25}{252}, \quad \pi(5) = \frac{1}{252}.$$

This is the answer for the stationary distribution.
We are also asked to interpret $\pi(x)$ as

> *From a lot of 10 (= 5 black + 5 white balls) pick 5 at random and place them in the left urn (place the rest in the right urn) and consider the chance that amongst the 5 balls x are white.*

We know how to answer this problem: it is a hypergeometric distribution:

$$\text{Chance that amongst the 5 balls } x \text{ are white} = \frac{\binom{5}{x}\binom{5}{5-x}}{\binom{10}{5}} = \frac{\binom{5}{x}^2}{255}, \quad x = 0, \ldots, 5.$$

This is PRECISELY the distribution obtained above. Hence $\pi(x)$ IS A HYPERGEOMETRIC DISTRIBUTION.

30.

An auto insurance company classifies its customers in three categories: poor, satisfactory and preferred. No one moves from poor to preferred or from preferred to poor in one year. 40% of the customers in the poor category become satisfactory, 30% of those in the satisfactory category moves to preferred, while 10% become poor; 20% of those in the preferred category are downgraded to satisfactory.
(a) Write the transition matrix for the model.
(b) What is the limiting fraction of drivers in each of these categories? (Clearly state which theorem you are applying in order to compute this.)

Solution. (a) The transition probabilities for this Markov chain with three states are as follows:

	POOR	SATISFACTORY	PREFERRED
POOR	0.6	0.4	0
SATISFACTORY	0.1	0.6	0.3
PREFERRED	0	0.2	0.8

,

so that the transition probability matrix is

$$\mathsf{P} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.1 & 0.6 & 0.3 \\ 0 & 0.2 & 0.8 \end{pmatrix}.$$

(b) We will find the limiting fraction of drivers in each of these categories from the components of the stationary distribution vector π, which satisfies the following equation:

$$\pi = \pi \mathsf{P}.$$

The former is equivalent to the following system of linear equations:

$$\begin{aligned} \pi(1) &= 0.6\pi(1) + 0.1\pi(2) \\ \pi(2) &= 0.4\pi(1) + 0.6\pi(2) + 0.2\pi(3) \\ \pi(3) &= 0.3\pi(2) + 0.8\pi(3) \\ 1 &= \pi(1) + \pi(2) + \pi(3). \end{aligned} \tag{7}$$

This has the following solution: $\pi = (\frac{1}{11}, \frac{4}{11}, \frac{6}{11})$.
Thus, the limiting fraction of drivers in the POOR category is $\frac{1}{11}$, in the SATISFACTORY category—$\frac{4}{11}$, and in the PREFERRED category—$\frac{6}{11}$. By the way, the proportions of the drivers in each category in 15 years approximate these numbers with two significant digits (you can check it, calculating P^{15} and looking at its rows).

31.

The President of the United States tells person A his or her intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability a that a person will change the answer from yes to no when transmitting it to the next person and a probability b that he or she will change it from no to yes. We choose as states the message, either yes or no. The transition probabilities are

$$p_{yes,no} = a, \quad p_{no,yes} = b.$$

The initial state represents the President's choice. Suppose $a = 0.5$, $b = 0.75$.
(a) Assume that the President says that he or she will run. Find the expected length of time before the first time the answer is passed on incorrectly.
(b) Find the mean recurrence time for each state. In other words, find the expected amount of time r_i, for $i = yes$ and $i = no$ required to return to that state.
(c) Write down the transition probability matrix P and find $\lim_{n\to\infty} \mathsf{P}^n$.

(d) Repeat (b) for general a and b.
(e) Repeat (c) for general a and b.

Solution. (a) The expected length of time before the first answer is passed on incorrectly, i.e. that the President will **not** run in the next election, equals the mean of the geometrically distributed random variable with parameter $1 - p_{yes,no} = 1 - a = 0.5$. Thus, the expected length of time before the first answer is passed on incorrectly is 2. What is found can be viewed as the *mean first passage time* from the state *yes* to the state *no*. By making the corresponding ergodic Markov chain with transition matrix

$$\mathsf{P} = \begin{pmatrix} 0.5 & 0.5 \\ 0.75 & 0.25 \end{pmatrix} \tag{8}$$

absorbing (with absorbing state being *no*), check that the time until absorption will be 2. This is nothing but the mean first passage time from *yes* to *no* in the original Markov chain.

(b) We use the following result to find mean recurrence time for each state:
for an ergodic Markov chain, the mean recurrence time for state i is

$$r_i = E_i T_i = \frac{1}{\pi(i)},$$

where $\pi(i)$ is the ith component of the stationary distribution for the transition probability matrix.

The transition probability matrix (8) has the following stationary distribution:

$$\pi = \left(.6, .4\right),$$

from which we find the mean recurrence time for the state *yes* is $\frac{5}{3}$ and for the state *no* is $\frac{5}{2}$.

(c) The transition probability matrix is specified in (8)—it has no zero entries and the corresponding chain is irreducible and aperiodic. For such a chain

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \pi(1) & \pi(2) \\ \pi(1) & \pi(2) \end{pmatrix}.$$

Thus,

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{pmatrix}.$$

(d) We apply the same arguments as in (b) and find that the transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

has the following fixed probability vector:

$$\pi = \left(\tfrac{b}{a+b}, \tfrac{a}{a+b}\right),$$

so that the mean recurrence time for the state *yes* is $1 + \frac{a}{b}$ and for the state *no* is $1 + \frac{b}{a}$.

(d) Suppose $a \neq 0$ and $b \neq 0$ to avoid absorbing states and achieve regularity. Then the corresponding Markov chain is regular. Thus,

$$\lim_{n\to+\infty} \mathsf{P}^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}.$$

32.

A fair die is rolled repeatedly and independently. Show by the results of the Markov chain theory that the mean time between occurrences of a given number is 6.

Solution. We construct a Markov chain with the states $1, 2, \ldots, 6$ and transition probabilities $p_{ij} = \frac{1}{6}$ for each $i, j = 1, 2, \ldots, 6$. Such Markov chain has the transition probability matrix which has all its entries equal to $\frac{1}{6}$. The chain is irreducible and aperiodic and its stationary distribution is nothing but

$$\pi = \left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right).$$

This means that the mean time between occurrences of a given number is 6.

33.

Give an example of a three-state irreducible-aperiodic Markov chain that is not reversible.

Solution.
We will see how to choose transition probabilities in such a way that the chain would not be reversible.
If our three-state chain was a reversible chain, that would meant that the detailed balance equations hold, i.e.

$$\begin{aligned}\pi(1)p_{12} &= \pi(2)p_{21}\\ \pi(1)p_{13} &= \pi(3)p_{31}\\ \pi(2)p_{23} &= \pi(3)p_{32}.\end{aligned}$$

From this it is easy to see that if the detailed balance equations hold, then necessarily $p_{13}p_{32}p_{21} = p_{12}p_{23}p_{31}$. So, choose them in such a way that this does not hold.
For instance, $p_{13} = 0.7$, $p_{32} = 0.2$, $p_{21} = 0.3$, $p_{12} = 0.2$, $p_{23} = 0.2$, $p_{31} = 0.1$. And these specify an ergodic Markov chain which is not reversible.

Another solution is: Consider the Markov chain with three states $\{1, 2, 3\}$ and deterministic transitions: $1 \to 2 \to 3 \to 1$. Clearly, the Markov chain in reverse time moves like $1 \to 3 \to 2 \to 1$ and so its law is not the same. (We can tell the arrow of time by running the film backwards.)

34.

Let P be the transition matrix of an irreducible-aperiodic Markov chain. Let π be its stationary distribution. Suppose the Markov chain starts with $P(X_0 = i) = \pi(i)$, for all $i \in S$.
(a) [Review question] Show that $P(X_n = i) = \pi(i)$ for all $i \in S$ and all n.
(b) Fix $N \geq 1$ and consider the process $X_0^* = X_N, X_1^* = X_{N-1}, \ldots$ Show that it is Markov.
(c) Let P^* be the transition probability matrix of P^* (it is called: the reverse transition matrix). Find its entries $p_{i,j}^*$.
(d) Show that P and P^* they have the same stationary distribution π.

Solution. (a) By definition, $\pi(i)$ satisfies

$$\pi(i) = \sum_j \pi(j)p_{j,i}, \quad i \in S.$$

If $P(X_0 = i) = \pi(i)$, then

$$\begin{aligned} P(X_1 = i) &= \sum_j P(X_0 = j, X_1 = i) \\ &= \sum_j P(X_0 = j, X_1 = i) \\ &= \sum_j \pi(j) p_{j,i} = \pi(i). \end{aligned}$$

Hence $P(X_1 = i) \equiv \pi(i)$. Repeating the process we find $P(X_2 = i) \equiv \pi(i)$, and so on, we have $P(X_n = i) \equiv \pi(i)$ for all n.

(b) Fix n and consider the future of X^* after n. This is $X^*n+1, X^*n+2, \ldots$. Consider also the past of X^* before n. This is $X^*_{n-1}, X^*_{n-2}, \ldots$. But

$$(X^*_{n+1}, X^*_{n+2}, \ldots) = (X_{N-n-1}, X_{N-n-2}, \ldots)$$

is the past of X before time $N - n$. And

$$(X^*_{n-1}, X^*_{n-2}, \ldots) = (X_{N-n+1}, X_{N-n+2}, \ldots)$$

is the future of X after time $N - n$. Since X is Markov, these are independent, conditional on X_{N-n}. But $X_{N-n} = X^*_n$. Hence, given X^*_n, the future of X^* after n is independent of the past of X^* before n, and this is true for all n, and so X^* is also Markov.

(c) Here we assume that $P(X_0 = i) \equiv \pi(i)$. Hence, by (a), $P(X_n = i) \equiv \pi(i)$ for all n. We have

$$\begin{aligned} p^*_{i,j} := P(X^*_{n+1} = j | X^*_n = i) &= P(X_{N-n-1} = j | X_{N-n} = i) \\ &= \frac{P(X_{N-n} = i | X_{N-n-1} = j) P(X_{N-n-1} = j)}{P(X_{N-n} = i)} = \frac{p_{j,i}\pi(j)}{\pi(i)}. \end{aligned}$$

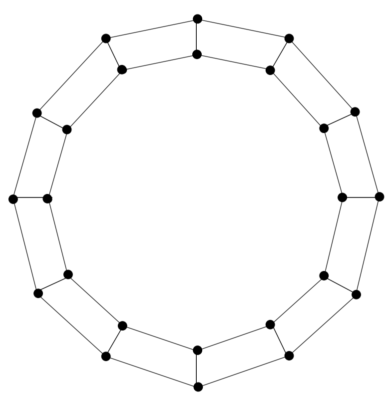
(d) We need to check that, for all $i \in S$,

$$\pi(i) = \sum_k \pi(k) p^*_{k,i}. \tag{9}$$

This is a matter of algebra.

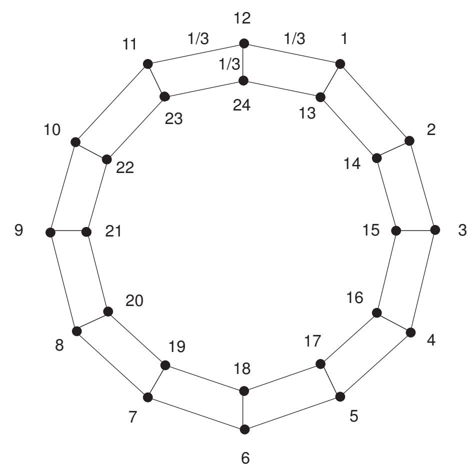
35.

Consider a random walk on the following graph consisting of two nested dodecagons:



(a) Explain why it is reversible (this is true for *any* RWonG).
(b) Find the stationary distribution.
(c) Show that the mean recurrence time (mean time to return) to any state is the same for all states, and compute this time.
(d) Let X_n be the position of the chain at time n (it takes values in a set of 24 elements). Let $Z_n = 1$ if X_n is in the inner dodecagon and $Z_n = 2$ is X_n is at the outer dodecagon. Is (Z_n) Markov?

Solution. (a) Our chain has 24 states. From each of the states we jump to any of three neighbouring states with equal probability $\frac{1}{3}$ (see the figure below: each undirected edge combines two directed edges-arrows). The chain is reversible, i.e. it is possible to move from any state to any other state. This is obviously the case for any random walk on a *connected* graph. Note that the notion of reversibility of the discrete Markov chain is related to the *topology* of the graph on which the chain is being run.



(b) The stationary distribution exists and because of the symmetry the stationary vector has all components equal, and since the number of the components is 24 the stationary vector is

$$\pi = (\frac{1}{24}, \frac{1}{24}, \dots, \frac{1}{24}) \in \mathbb{R}^{24}.$$

(c) The mean recurrence time for the state i is $1/\pi(i) = 24$, $\forall i = 1, 2, \dots, 24$.
(d) Observe first that

$$P(Z_n = 2 | X_n = i) = 1/3, \text{ as long as } i = 13, \dots, 24,$$

and

$$P(Z_n = 1 | X_n = i) = 1/3, \text{ as long as } i = 1, \dots, 12.$$

We now verify that (Z_n) is Markov. (We shall argue directly. Alternatively, see section on "functions of Markov chains" from my lecture notes.) By the definition of conditional probability,

$$\begin{aligned} &P(Z_{n+1} = 2 | Z_n = 1, Z_{n-1} = w, \dots) \\ &= \sum_{i=13}^{24} P(Z_{n+1} = 2 | X_n = i, Z_n = 1, Z_{n-1} = w, \dots)\ P(X_n = i | Z_n = 1, Z_{n-1} = w, \dots) \end{aligned}$$

Due to the fact that (X_n) is Markov, when we know that $X_n = i$ that the future after n is independent from the past before n. But Z_{n+1} belongs to the future after n, while $Z_{n-1} = w, \ldots$ belongs to the past before n. Hence, for $i = 13, \ldots, 24$,

$$P(Z_{n+1} = 2 | X_n = i, Z_n = 1, Z_{n-1} = w, \ldots) = P(Z_{n+1} = 2 | X_n = i) = 1/3.$$

Hence

$$P(Z_{n+1} = 2 | Z_n = 1, Z_{n-1} = w, \ldots) = \sum_{i=13}^{24} \frac{1}{3} P(X_n = i | Z_n = 1, Z_{n-1} = w, \ldots) = \frac{1}{3},$$

because, obviously,

$$\sum_{i=13}^{24} P(X_n = i | Z_n = 1, Z_{n-1} = w, \ldots) = 1.$$

(If $Z_n = 1$ then X_n is in the inside dodecagon.) Thus,

$$P(Z_{n+1} = 2 | Z_n = 1, Z_{n-1} = w, \ldots) = P(Z_{n+1} = 2 | Z_n = 1).$$

Similarly, we can show

$$P(Z_{n+1} = 1 | Z_n = 2, Z_{n-1} = w, \ldots) = P(Z_{n+1} = 1 | Z_n = 2).$$

Hence, no matter what the value of Z_n is, the future of Z after n is independent of the past of Z before n. Hence Z is Markov as well.

36.

Consider a Markov chain in the set $\{1, 2, 3\}$ with transition probabilities

$$p_{12} = p_{23} = p_{31} = p, \quad p_{13} = p_{32} = p_{21} = q = 1 - p,$$

where $0 < p < 1$. Determine whether the Markov chain is reversible.

Solution. If $p = 1/2$ then the chain is a random walk on a graph; so it is reversible. If $p \neq 1/2$ then Kolmogorov's loop criterion requires that

$$p_{12} p_{23} p_{31} = p_{13} p_{32} p_{21}.$$

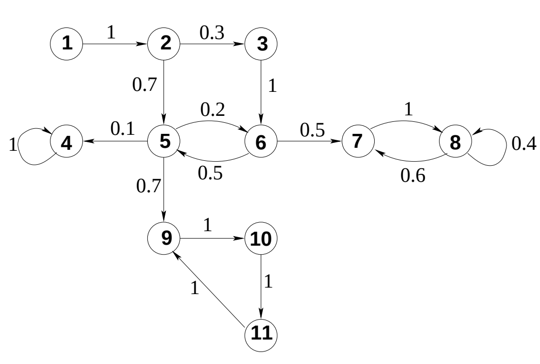
But this is equivalent to

$$p^3 = q^3$$

which is not true (unless $p = 1/2$). Hence the chain is not reversible if $p \neq 1/2$.

37.

Consider a Markov chain whose transition diagram is as below:



(i) Which (if any) states are inessential?
(ii) Which (if any) states are absorbing?
(iii) Find the communication classes.
(iv) Is the chain irreducible?
(v) Find the period of each essential state. Verify that essential states that belong to the same communication class have the same period.
(vi) Are there any aperiodic communication classes?
(vii) Will your answers to the questions (i)–(vi) change if we replace the positive transition probabilities by other positive probabilities and why?

Solution. (i) The inessential states are: 1, 2, 3, 5, 6, because each of them leads to a state from which it is not possible to return.

(ii) 4 is the only absorbing state.

(iii) As usual, let $[i]$ denote the class of state i i.e. $[i] = \{j \in S : \ j \leftrightsquigarrow i\}$. We have:
$[1] = \{1\}$.
$[2] = \{2\}$.
$[3] = \{3\}$.
$[4] = \{4\}$.
$[5] = \{5, 6\}$.
$[6] = \{5, 6\}$.
$[7] = \{7, 8\}$.
$[8] = \{7, 8\}$.
$[9] = \{9, 10, 11\}$
$[10] = \{9, 10, 11\}$
$[11] = \{9, 10, 11\}$
Therefore there are 7 communication classes:

$$\{1\}, \quad \{2\}, \quad \{3\}, \quad \{4\}, \quad \{5, 6\}, \quad \{7, 8\}, \quad \{9, 10, 11\}$$

(iv) No because there are many communication classes.

(v) Recall that for each essential state i, its period $d(i)$ is the gcd of all n such that $p_{i,i}^{(n)} > 0$. So:

$$\begin{aligned} d(4) &= \gcd\{1, 2, 3, \ldots\} = 1 \\ d(7) &= \gcd\{1, 2, 3, \ldots\} = 1 \\ d(8) &= \gcd\{1, 2, 3, \ldots\} = 1 \\ d(9) &= \gcd\{3, 6, 9, \ldots\} = 3 \\ d(10) &= \gcd\{3, 6, 9, \ldots\} = 3 \\ d(11) &= \gcd\{3, 6, 9, \ldots\} = 3 \end{aligned}$$

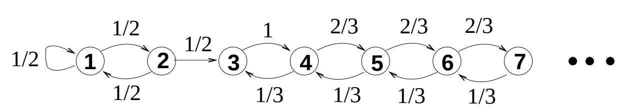
Observe $d(7) = d(8) = 1$, and $d(10) = d(11) = d(9) = 3$.

(vi) Yes: $\{4\}$ and $\{7, 8\}$ are aperiodic communication classes (each has period 1).

(vii) No the answers will not change. These questions depend only on whether, for each i, j, $p_{i,j}$ is positive or zero.

38.

Consider a Markov chain, with state space S the set of all positive integers, whose transition diagram is as follows:



(i) Which states are essential and which inessential?
(ii) Which states are transient and which recurrent?
(iii) Discuss the asymptotic behaviour of the chain, i.e. find the limit, as $n \to \infty$, of $P_i(X_n = j)$ for each i and j.

Solution. (i) The states $3, 4, 5, \ldots$ communicate with one another. So they are all essential. However state 1 leads to 3 but 3 does not lead to 1. Hence 1 is inessential. Likewise, 2 is inessential.

(ii) Every inessential state is transient. Hence both 1 and 2 are transient. On the other hand, the Markov chain will eventually take values only in the set $\{3, 4, 5, \ldots\}$. We observe that the chain on this set is the same type of chain we discussed in gambler's ruin problem with $p = 2/3$, $q = 1/3$. Since $p > q$ the chain is transient. Therefore all states of the given chain are transient.

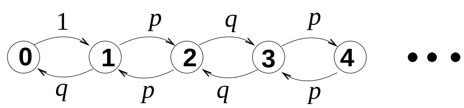
(iii) Since the states are transient, we have that $X_n \to \infty$ as $n \to \infty$, with probability 1. Therefore,

$$P_i(X_n = j) \to 0, \qquad \text{as } n \to \infty,$$

for all i and j.

39.

Consider the following Markov chain, which is motivated by the "umbrellas problem" (see–but it's not necessary–an earlier exercise). Here, $p + q = 1$, $0 < p < 1$.



(i) Is the chain irreducible?
(ii) Does it have a stationary distribution?
Hint: Write the balance equations, together with the normalisation condition and draw your conclusions.
(iii) Find the period $d(i)$ of each state i.
(iv) Decide which states are transient and which recurrent.
Hint: Let τ_j be the first hitting time of state j. Let $N \geq 1$ As in the gambler's ruin problem, let $\varphi(i) := P_i(\tau_N < \tau_0)$. What is $\varphi(0)$? What is $\varphi(N)$? For $1 < i < N$, how does $\varphi(i)$ relate to $\varphi(i-1)$ and $\varphi(i+1)$? Solve the equations you thus obtain to find $\varphi(i)$. Let $N \to \infty$. What do you conclude?

Solution. (i) Yes because all states communicate with one another. (There is just one communication class).

(ii) Let us write balance equations in the form of equating flows (see handout). We have

$$\begin{aligned} \pi(0) &= \pi(1)q \\ \pi(1)p &= \pi(1)p \\ \pi(2)q &= \pi(2)q \\ &\cdots \end{aligned}$$

Let $\pi(1) = c$. Then $\pi(0) = cq$ and

$$\pi(1) = \pi(2) = \pi(3) = \cdots = c.$$

The normalisation condition is $\sum_{i=0}^{\infty} \pi(i) = 1$. This implies that $c = 0$. Hence $\pi(i) = 0$ for all i. This is NOT a probability distribution. Hence there is no stationary distribution.

(iii) We only have to find the period of one state, since all states communicate with one another. Pick state 0. We have $d(0) = \gcd\{2, 4, 6, \dots\} = 2$. Hence $d(i) = 2$ for all i.

(iv) Let $\varphi(i) := P_i(\tau_N < \tau_0)$. We have

$$\varphi(0) = 0, \quad \varphi(N) = 1.$$

Indeed, if $X_0 = 0$ then $\tau_0 = 0$ and so $\varphi(0) = P_0(\tau_N < 0) = 0$. On the other hand, if $X_0 = N$ then $\tau_N = 0$ and $\tau_0 \geq 1$, so $\varphi(N) = P_N(0 < \tau_0) = 1$.

Now, from first-step analysis, for each $i \in [1, N-1]$, we have

$$\varphi(i) = p_{i,i+1}\varphi(i+1) + p_{i,i-1}\varphi(i).$$

But $p_{i,i+1} = p_{i,i-1} = p$ if i is odd and $p_{i,i+1} = p_{i,i-1} = q$ if i is even and positive. So

$$\begin{aligned} p[\varphi(i+1) - \varphi(i)] &= q[\varphi(i) - \varphi(i-1)], \quad && i \text{ odd} \\ q[\varphi(i+1) - \varphi(i)] &= p[\varphi(i) - \varphi(i-1)], \quad && i \text{ even.} \end{aligned}$$

Hence

$$\begin{aligned} \varphi(2) - \varphi(1) &= \frac{q}{p}[\varphi(1) - \varphi(0)] = \frac{q}{p}\varphi(1) \\ \varphi(3) - \varphi(2) &= \frac{p}{q}[\varphi(2) - \varphi(1)] = \varphi(1) \\ \varphi(4) - \varphi(3) &= \frac{q}{p}[\varphi(3) - \varphi(2)] = \frac{q}{p}\varphi(1) \\ \varphi(5) - \varphi(4) &= \frac{p}{q}[\varphi(4) - \varphi(3)] = \varphi(1), \end{aligned}$$

and, in general,

$$\begin{aligned} \varphi(i) - \varphi(i-1) &= \frac{q}{p}\varphi(1) \quad && i \text{ even} \\ \varphi(i) - \varphi(i-1) &= \varphi(1) \quad && i \text{ odd.} \end{aligned}$$

Next, use the "fundamental theorem of (discrete) calculus":

$$\varphi(i) = [\varphi(i) - \varphi(i-1)] + [\varphi(i-1) - \varphi(i-2)] + \cdots + [\varphi(2) - \varphi(1)] + \varphi(1).$$

If i is even then, amongst $1, 2, \dots, i$ there are $i/2$ even numbers and $i/2$ odd numbers.

$$\varphi(i) = \left(\frac{q}{p}\right)^{i/2} \varphi(1) + \frac{i}{2}\varphi(1) \qquad i \text{ even}$$

Suppose N is even. Use $\varphi(N) = 1$ to get that, if both i and N are even,

$$\varphi(i) = \frac{\left(\frac{q}{p}\right)^{i/2} + \frac{i}{2}}{\left(\frac{q}{p}\right)^{N/2} + \frac{N}{2}} = P_i(\tau_N < \tau_0).$$

Taking the limit as $N \to \infty$, we find

$$P_i(\tau_0 = \infty) = 0, \quad i \text{ even.}$$

This implies that $P_i(\tau_0 < \infty) = 1$. The same conclusion holds for i odd. (After all, all states communicate with one another.) Therefore all states are recurrent.

40.

Suppose that $X_1, X_2 \dots$ are i.i.d. random variables with values, say, in $\mathbb{Z}$ and common distribution $p(i) := P(X_1 = i)$, $i \in \mathbb{Z}$.
(i) Explain why the sequence has the Markov property.
(ii) Let A be a subset of the integers such that $\sum_{i \in A} p(i) > 0$. Consider the first hitting time τ_A of A and the random variable $Z := X_{\tau_A}$. Show that the distribution of Z is the conditional distribution of X_1 given that $X_1 \in A$.
Hint: Clearly, $\{Z = i\} = \bigcup_{n=1}^{\infty} \{Z = i, \tau_A = n\}$, *and the events in this union are disjoint; therefore the probability of the union is the sum of the probabilities of the events comprising it.*

Solution. (i) As explained in the beginning of the lectures.

(ii) Since τ_A is the FIRST time that A is hit, it means that

$$\tau_A = n \iff X_1 \notin A, X_2 \notin A, \dots, X_{n-1} \notin A, X_n \in A.$$

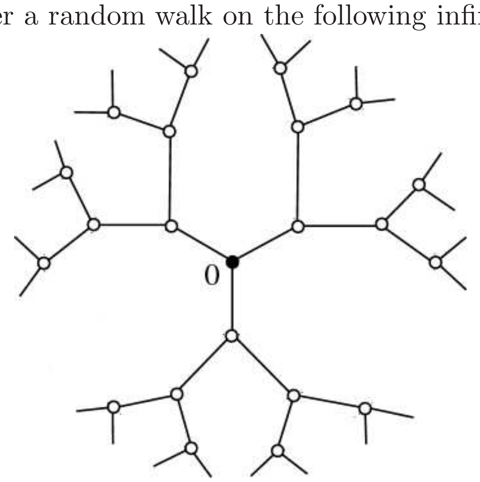
Therefore, with $Z = X_{\tau_A}$, and $i \in A$,

$$\begin{aligned}
P(Z = i) &= \sum_{n=1}^{\infty} P(X_{\tau_A} = i, \tau_A = n) \\
&= \sum_{n=1}^{\infty} P(X_n = i, X_1 \notin A, X_2 \notin A, \dots, X_{n-1} \notin A, X_n \in A) \\
&= \sum_{n=1}^{\infty} P(X_n = i, X_1 \notin A, X_2 \notin A, \dots, X_{n-1} \notin A) \\
&= \sum_{n=1}^{\infty} p(i) P(X_1 \notin A)^{n-1} \qquad \text{[geometric series]} \\
&= p(i) \frac{1}{1 - P(X_1 \notin A)}. \\
&= \frac{p(i)}{P(X_1 \in A)}.
\end{aligned}$$

If $i \notin A$, then, obviously, $P(Z = i) = 0$. So it is clear that $P(Z = i) = P(X_1 = i | X_1 \in A)$, for all i, from the definition of conditional probability.

41.

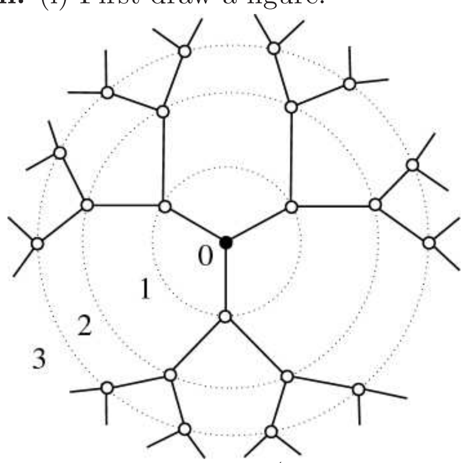
Consider a random walk on the following infinite graph:



The graph continues ad infinitum in the same manner.

Here, each state has exactly 3 neighbouring states (i.e. its degree is 3) and so the probability of moving to one of them is 1/3.
(i) Let 0 be the "central" state. (Actually, a closer look shows that no state deserves to be central, for they are all equivalent. So we just arbitrarily pick one and call it central.) Having done that, let $D(i)$ be the distance of a state i from 0, i.e. the number of "hops" required to reach 0 starting from i. So $D(0) = 0$, each neighbour i of 0 has $D(i) = 1$, etc. Let X_n be the position of the chain at time n. Observe that the process $Z_n = D(X_n)$ has the Markov property. (See lecture notes for criterion!) The question is:
Find its transition probabilities.
(ii) Using the results from the gambler's ruin problem, show that (Z_n) is transient.
(iii) Use (ii) to explain why (X_n) is also transient.

Solution. (i) First draw a figure:



The states with the same distance from 0 *are shown in this figure as belonging to the same circle.*

Next observe that if $Z_n = k$ (i.e. if the distance from 0 is k) then, no matter where X_n is actually located the distance Z_{n+1} of the next state X_{n+1} from 0 will either be $k + 1$ with probability 2/3 or $k - 1$ with probability 1/3. And, of course, if $Z_n = 0$ then $Z_{n+1} = 1$. So

$$P(Z_{n+1} = k + 1|Z_n = k) = 2/3, \quad k \geq 0$$
$$P(Z_{n+1} = k - 1|Z_n = k) = 1/3, \quad k \geq 1$$
$$P(Z_{n+1} = 1|Z_n = 0) = 1.$$

(ii) Since $2/3 > 1/3$, the chain (Z_n) is transient.

(iii) We have that $Z_n \to \infty$ as $n \to \infty$, with probability 1. This means that for any k, there is a time n_0 such that for all $n \geq n_0$ we have $D(X_n) \geq k$, and this happens with probability 1. So, with probability 1, the chain (X_n) will visit states with distance from 0 less than k only finitely many times. This means that the chain (X_n) is transient.

42.

A company requires N employees to function properly. If an employee becomes sick then he or she is replaced by a new one. It takes 1 week for a new employee to be recruited and to start working. Time here is measured in weeks.
(i) If at the beginning of week n there are X_n employees working and Y_n of them get sick during week n then show that at the beginning of week $n + 1$ there will be

$$X_{n+1} = N - Y_n$$

employees working.
(ii) Suppose that each employee becomes sick independently with probability p. Show that

$$P(Y_n = y | X_n = x) = \binom{x}{y} p^y (1-p)^{x-y}, \quad y = 0, 1, \dots, x.$$

(iii) Show that (X_n) is a Markov chain with state space $S = \{0, 1, \dots, N\}$ and derive its transition probabilities.
(vi) Write the balance equation for the stationary distribution π of the chain.
(v) What is the number of employees working in steady state?
Do this without using (vi) by assuming that the X is in steady state [i.e. that X_0 (and therefore each X_n) has distribution π] and by taking expectations on the equation you derived in (i).

Solution. (i) This is elementary: Since every time an employee gets sick he or she is replaced by a new one, but it takes 1 week for the new employee to start working, it means that those employees who got sick during week $n-1$ will be replaced by new ones who will start working sometime during week n and so, by the end of week n, the number of employees will be brought up to N, provided nobody got sick during week n. If the latter happens, then we subtract the Y_n employees who got sick during week n to obtained the desired equation.

(ii) Again, this is easy: If $X_n = x$, at most x employees can get sick. Each one gets sick with probability p, independently of one another, so the total number, Y_n, of sick employees has the Binomial(x, p) distribution.

(iii) We have that Y_n depends only on X_n and not on $X_{n-1}, X_{n-2}, \dots$, and therefore $P(X_{n+1} = j | X_n = i, X_{n-1} = i_1, X_{n-2} = i_2 \dots) = P(X_{n+1} = j | X_n = i)$. Hence X is Markov. We are asked to derive $p_{i,j} = P(X_{n+1} = j | X_n = i)$ for all $i, j \in S$. If $X_n = i$ then $Y_n \le i$ and so $X_{n+1} \ge N - i$, so the only possible values j for which $p_{i,j} > 0$ are $j = N - i, \dots, N$. In fact, $P(X_{n+1} = j | X_n = i) = P(Y_n = N - j | X_n = i)$ and so, using the formulae of (ii),

$$p_{i,j} = \begin{cases} \binom{i}{N-j} p^{N-j} (1-p)^{i-N+j}, & j = N-i, \dots, N \\ 0, & \text{otherwise} \end{cases}, \qquad i = 0, 1, \dots, N.$$

(vi) The balance equations are:

$$\begin{aligned} \pi(j) &= \sum_{i=0}^{N} \pi(i) p_{i,j} \\ &= \sum_{i=N-j}^{N} \pi(i) \binom{i}{N-j} p^{N-j} (1-p)^{i-N+j}. \end{aligned}$$

(v) If X_0 has distribution π then X_n has distribution π for all n. So $EX_n \equiv \mu$ does not depend on n. Now, if $X_n = x$, Y_n is Binomial(x, p) and therefore $E(Y_n | X_n = x) = px$. So

$$EY_n = \sum_{x=0}^{N} px P(X_n = x) = pEX_n = p\mu.$$

Since $EX_{n+1} = N - EY_n$ we have

$$\mu = N - p\mu,$$

whence

$$\mu = \frac{N}{1+p}.$$

This is the mean number of employees in steady state. So, for example, if $p = 10\%$, then $\mu \approx 0.91N$.

43.

(i) Let X be the number of heads in n i.i.d. coin tosses where the probability of heads is p. Find the generating function $\varphi(z) := Ez^X$ of X.
(ii) Let Y be a random variable with $P(Y = k) = (1-p)^{k-1}p$, $k = 1, 2, \dots$ Find the generating function of Y.

Solution. (i) The random variable X, which is defined as the number of heads in n i.i.d. coin tosses where the probability of heads is p, is binomially distributed:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Thus,

$$\begin{aligned}
\varphi(z) := Ez^X &= \sum_{k=0}^{n} P(X = k) z^k \\
&= \sum_{k=0}^{n} \binom{n}{k} (1-p)^{n-k} (pz)^k \\
&= ((1-p) + zp)^n = (q + zp)^n, \text{ where } q = 1 - p.
\end{aligned}$$

(ii) The random variable Y, defined by

$$P(Y = k) = (1-p)^{k-1}p, \ k = 1, 2, \dots$$

has the following generating function:

$$\begin{aligned}
\varphi(z) := Ez^Y &= \sum_{k=1}^{\infty} P(Y = k) z^k \\
&= \sum_{k=1}^{\infty} (1-p)^{k-1} p z^k \\
&= \frac{p}{1-p} \sum_{k=1}^{\infty} [(1-p)z]^k \\
&= \frac{p}{1-p} \left[\frac{1}{1 - z(1-p)} - 1 \right] \\
&= \frac{pz}{1 - zq}, \text{ where } q = 1 - p.
\end{aligned}$$

44.

A random variable X with values in $\{1, 2, \dots, \} \cup \{\infty\}$ has generating function $\varphi(z) = Ez^X$.

(i) Express $P(X=0)$ in terms of φ.
(ii) Express $P(X=\infty)$ in terms of φ.
(iii) Express EX and $varX$ in terms of φ.

Solution. (i) $\varphi(0) = \sum\limits_{k=0}^{\infty} P(X=k)z^k|_{z=0} = P(X=0)$, thus, $P(X=0) = \varphi(0)$.

(ii) The following must hold: $\sum\limits_{k=0}^{\infty} P(X=k) + P(X=\infty) = 1$. This may be rewritten as follows: $\varphi(1) + P(X=\infty) = 1$, from which we get

$$P(X=\infty) = 1 - \varphi(1).$$

(iii) By definition of the expected value of a discrete random variable

$$EX = \sum_{k=0}^{\infty} kP(X=k).$$

Now note, that

$$\varphi'(z) = \sum_{k=0}^{\infty} kP(X=k)z^{k-1},$$

so that $\varphi'(1)$ should give nothing but EX. We conclude that

$$EX = \varphi'(1).$$

Let $p_k := P(X=k)$. Now we take the second derivative of $\varphi(z)$:

$$\varphi''(z) = \sum_{k=2}^{\infty} k(k-1)p_k z^{k-2},$$

so that

$$\begin{aligned}\varphi''(1) = \sum_{k=2}^{\infty}(k^2 p_k - kp_k) &= \sum_{k=2}^{\infty} k^2 p_k - \sum_{k=2}^{\infty} kp_k \\ &= \sum_{k=0}^{\infty} k^2 p_k - \sum_{k=0}^{\infty} kp_k \\ &= EX^2 - EX = EX^2 - \varphi'(1),\end{aligned}$$

from which we get that $EX^2 = \varphi'(1) + \varphi''(1)$. But this is enough for $\operatorname{var} X$, since

$$\operatorname{var} X = EX^2 - (EX)^2 = \varphi'(1) + \varphi''(1) - \left[\varphi'(1)\right]^2.$$

45.

A random variable X with values in $\{1, 2, \dots, \} \cup \{\infty\}$ has generating function

$$\varphi(z) = \frac{1 - \sqrt{1 - 4pqz^2}}{2qz},$$

where $p, q \geq 0$ and $p + q = 1$.
(i) Compute $P(X = \infty)$. (Consider all possible values of p).
(ii) For those values of p for which $P(X = \infty) = 0$ compute EX.

Solution.
(i) As it was found above, $P(X = \infty) = 1 - \varphi(1)$, and particularly

$$P(X = \infty) = 1 - \varphi(1) = 1 - \frac{1 - \sqrt{1 - 4pq}}{2q} = 1 - \frac{1 - |p - q|}{2q} = \begin{cases} 1 - \frac{p}{q}, \ p < q \\ 0, \ p \geq q \end{cases}$$

(ii) It follows that $P(X = \infty) = 0$ for $p \geq \frac{1}{2}$. The expected value of X is given by

$$EX = \varphi'(1) = \begin{cases} \frac{1}{p-q}, \ p > \frac{1}{2} \\ \infty, \ p = \frac{1}{2} \end{cases}$$

and we are done.

46.

You can go up the stair by climbing 1 or 2 steps at a time. There are n steps in total. In how many ways can you climb all steps?
Hint 1: If $n = 3$, you can reach the 3d step by climbing 1 at a time, or 2 first and 1 next, or 1 first and 2 next, i.e. there are 3 ways.
Hint 2: if w_m is the number of ways to climb m steps, how is w_m related to w_{m-1} and w_{m-2}?
Hint 3: Consider the generating function $\sum_m z^m w_m$.

Solution. Just before being at step m you are either at step $m - 1$ or at step $m - 2$. Hence

$$w_m = w_{m-1} + w_{m-2}, \quad m \geq 2. \tag{10}$$

Here, step 0 means being at the bottom of the stairs. So

$$w_0 = 1, \quad w_1 = 1.$$

So

$$\begin{aligned} w_2 &= w_1 + w_0 = 2 \\ w_3 &= w_2 + w_1 = 3 \\ w_4 &= w_3 + w_2 = 5 \\ w_5 &= w_4 + w_3 = 8 \\ w_6 &= w_5 + w_4 = 13 \\ w_7 &= w_6 + w_5 = 21 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

How do we find a formula for w_n? Here is where *generating functions come to rescue.* Let

$$W(s) = \sum_{m \geq 0} w_m s^m$$

be the generating of $(w_m, m \geq 0)$. Then the generating function of $(w_{m+1}, m \geq 0)$ is

$$\sum_{m \geq 0} w_{m+1} s^m = s^{-1}(W(s) - w_0)$$

and the generating function of $(w_{m+2}, m \geq 0)$ is

$$\sum_{m \geq 0} w_{m+2} s^m = s^{-2}(W(s) - w_0 - sw_1).$$

From the recursion

$$w_{m+2} = w_{m+1} + w_m, \quad m \geq 0$$

(obtained from (10) by replacing m by $m+2$) we have (*and this is were linearity is used*) that the generating function of $(w_{m+2}, m \geq 0)$ equals the sum of the generating functions of $(w_{m+1}, m \geq 0)$ and $(w_m, m \geq 0)$, namely,

$$s^{-2}(W(s) - w_0 - sw_1) = s^{-1}(W(s) - w_0) + W(s). \tag{11}$$

Since $w_0 = w_1 = 1$, we can solve for $W(s)$ and find

$$W(s) = \frac{-1}{s^2 + s - 1}.$$

Essentially, what generating functions have done for us is to transform the LINEAR recursion (10) into the ALGEBRAIC equation (11). This is something you have learnt in your introductory Mathematics courses. The tools and recipes associated with LINEARITY are indispensable for anyone who does anything of value. Thus, keep them always in your bag of tricks.

The question we ask is:
Which sequence $(w_n, n \geq 0)$ has generating function $W(s)$?
We start by noting that the polynomial $s^2 + s - 1$ has two roots:

$$a = (\sqrt{5} - 1)/2, \quad b = (-1 - \sqrt{5})/2.$$

Hence $s^2 + s - 1 = (s-a)(s-b)$, and so, by simple algebra,

$$W(s) = \frac{1}{b-a}\left(\frac{1}{s-a} - \frac{1}{s-b}\right).$$

Write this as

$$W(s) = \frac{1}{b-a}\left(\frac{b}{bs-ab} - \frac{a}{as-ab}\right).$$

Noting that $ab = -1$, we further have

$$W(s) = \frac{b}{b-a}\,\frac{1}{1+bs} - \frac{a}{b-a}\,\frac{1}{1+as}.$$

But $\frac{1}{1+bs} = \sum_{n=0}^{\infty}(-bs)^n$, $\frac{1}{1+as} = \sum_{n=0}^{\infty}(-as)^n$, and so $W(s)$ is the generating function of

$$w_n = \frac{b}{b-a}\,(-b)^n - \frac{a}{b-a}\,(-a)^n, \quad n \geq 0.$$

This can be written also as

$$w_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

which is always an integer (why?)

47.

Consider a branching process starting with $Z_0 = 1$ and branching mechanism

$$p_1 = 1 - p, \quad p_2 = p.$$

(Each individual gives birth to 1 or 2 children with probability $1-p$ or p, respectively.) Let Z_n be the size of the n-th generation. Compute the probabilities $P(Z_n = k)$ for all possible values of k, the generating function $\varphi_n(z) = Ez^{Z_n}$, and the mean size of the n-th generation $m_n = EZ_n$. Do the computations in whichever order is convenient for you.

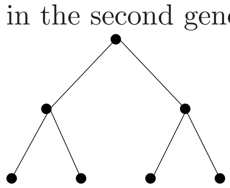
Solution. The mean number of offspring of a typical individual is

$$m := (1-p) + 2p = 1 + p.$$

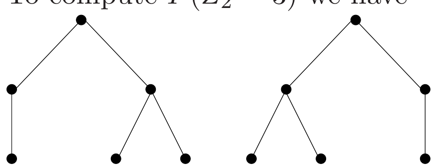
Therefore

$$EZ_n = m^n = (1+p)^n.$$

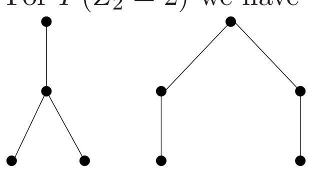
Let $q = 1 - p$. To compute $P(Z_2 = 4)$, we consider all possibilities to have 4 children in the second generation. There is only one possibility:



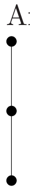
Therefore $P(Z_2 = 4) = p^2$.
To compute $P(Z_2 = 3)$ we have



and so $P(Z_2 = 3) = pqp + ppq$.
For $P(Z_2 = 2)$ we have



and so $P(Z_2 = 2) = qp + pq^2$
And for $P(Z_2 = 1)$ there is only one possibility,



and so $P(Z_2 = 2) = q^2$.

You can continue in this manner to compute $P(Z_3 = k)$, etc.

The generating function of the branching mechanism is

$$\varphi(z) = p_1 z + p_1 z^2 = qz + pz^2.$$

So $\varphi_1(z) = Ez^{Z_1} = \varphi(z)$. Next, we have $\varphi_2(z) = \varphi_1(\varphi(z))$ and so

$$\begin{aligned}\varphi_2(z) = \varphi(\varphi(z)) &= q\varphi(z) + p\varphi(z)^2 \\ &= p^3 z^4 + 2\,p^2 q z^3 + \left(qp + pq^2\right) z^2 + q^2 z.\end{aligned}$$

Similarly, $\varphi_3(z) = \varphi_2(\varphi(z))$ and so

$$\begin{aligned}\varphi_3(z) &= p^3\varphi(z)^4 + 2\,p^2 q\varphi(z)^3 + \left(qp + pq^2\right)\varphi(z)^2 + q^2\varphi(z) \\ &= p^7 z^8 + 4\,p^6 q z^7 + p(2\,(qp + pq^2)p^3 + 4\,p^4 q^2)z^6 + p(2\,q^2 p^3 + 4\,(qp + pq^2)p^2 q)z^5 \\ &\quad + (qp^3 + p(4\,q^3 p^2 + (qp + pq^2)^2))z^4 + (2\,q^2 p^2 + 2\,pq^2(qp + pq^2))z^3 \\ &\quad + (q(qp + pq^2) + pq^4)z^2 + q^3 z.\end{aligned}$$

48.

Consider a branching process with $Z_0 = 1$ and branching mechanism

$$p_0 = \frac{1}{10}, \quad p_1 = \frac{7}{10}, \quad p_2 = \frac{2}{10}.$$

(i) Compute probability of ultimate extinction.
(ii) Compute the mean size of the n-th generation.
(iii) Compute the standard deviation of the size of the n-th generation.

Solution. (i) The generating function of the branching mechanism is

$$\varphi(z) = \frac{1}{10}z^0 + \frac{7}{10}z^1 + \frac{2}{10}z^2 = \frac{1}{10}(1 + 7z + 2z^2).$$

The probability ε of ultimate extinction is the smallest positive z such that

$$\varphi(z) = z.$$

We have to solve

$$1 + 7z + 2z^2 = 10z.$$

Its solutions are $1, 1/2$. Therefore,

$$\varepsilon = 1/2.$$

(ii) The mean number of offspring of an individual is

$$m = \frac{2}{10} + \frac{1}{10} \times 2 = \frac{11}{10}.$$

Therefore the mean size of the n-th generation is

$$EZ_n = m^n = (11/10)^n.$$

(iii) As in Exercise 2 above, we have that

$$\varphi'(1) = EX, \quad \varphi''(1) = EX^2 - EX, \quad \operatorname{var} X = EX^2 - (EX)^2 = \varphi''(1) + EX - (EX)^2.$$

Since $\varphi_n(z) = \varphi_{n-1}(\varphi(z))$, we have

$$\begin{aligned}\varphi_n'(z) &= \varphi_{n-1}'(\varphi(z))\ \varphi'(z) \\ \varphi_n''(z) &= \varphi_{n-1}''(\varphi(z))\ \varphi'(z)^2 + \varphi_{n-1}'(\varphi(z))\ \varphi''(z).\end{aligned}$$

Setting $z = 1$ and using that $\varphi(1) = 1$ we have

$$\varphi_n''(1) = \varphi_{n-1}''(1)\ \varphi'(1)^2 + \varphi_{n-1}'(1)\ \varphi''(1).$$

But $\varphi'(1) = m$, $\varphi_{n-1}'(1) = m^{n-1}$ and so

$$\varphi_n''(1) = \varphi_{n-1}''(1)\ m^2 + m^{n-1}\ \varphi''(1).$$

Iterating this we find

$$\varphi_n''(1) = \varphi''(1) \sum_{k=n-1}^{2n-2} m^k.$$

We here have $m = 11/10$, $\varphi''(1) = 4/10$. But then

$$\begin{aligned}
\sigma_n^2 = \operatorname{var} Z_n = \varphi_n''(1) + EZ_n - (EZ_n)^2 &= \varphi''(1) \sum_{k=n-1}^{2n-2} m^k + m^n - m^{2n} \\
&= \varphi''(1) m^{n-1} \frac{m^n - 1}{m - 1} + m^n - m^{2n} \\
&= \frac{4}{10} m^{n-1} \frac{m^n - 1}{1/10} - m^n(m^n - 1) \\
&= 4m^{n-1}(m^n - 1) - m^n(m^n - 1) \\
&= (4 - m)m^{n-1}(m^n - 1) \\
&= \frac{29}{10} \left(\frac{11}{10}\right)^{n-1} \left(\left(\frac{11}{10}\right)^n - 1\right)
\end{aligned}$$

Of course, the standard deviation is the square root of this number.

49.

Consider the same branching process as above, but now start with $Z_0 = m$, an arbitrary positive integer. Answer the same questions.

Solution. (i) The process behaves as the superposition of N i.i.d. copies of the previous process. This becomes extinct if and only if each of the N copies becomes extinct and so, by independence, the extinction probability is

$$\varepsilon^N = (1/2)^N.$$

(ii) The n-th generation of the new process is the sum of the populations of the n-th generations of each of the N constituent processes. Therefore the mean size of the n-th generation is

$$Nm^n = N(11/10)^n.$$

(iii) For the same reason, the standard deviation of the size of the n-th generation is

$$\sqrt{N}\sigma_n.$$

50.

Show that a branching process cannot have a stationary distribution π with $\pi(i) > 0$ for some $i > 0$.

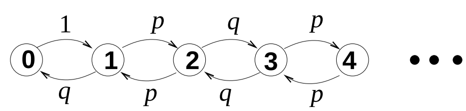
Solution. If the mean number m of offspring is ≤ 1 then we know that the process will become extinct for sure, i.e. it will be absorbed by state 0. Hence the *only* stationary distribution satisfies

$$\pi(0) = 1, \quad \pi(i) = 0, \quad i \geq 1.$$

If the mean number m of offspring is > 1 then we know that the probability that it will become extinct is $\varepsilon < 1$, i.e. $P_1(\tau_0 = \infty) = 1 - \varepsilon > 0$. But we showed in Part (i) of Problem 8 above that $P_i(\tau_0 = \infty) = 1 - \varepsilon^i > 0$ for all i. Hence the process is transient. And so there is NO stationary distribution at all.

51.

Consider the following Markov chain, which is motivated from the "umbrellas problem" (see earlier exercise). Here, $p + q = 1$, $0 < p < 1$.



Is it positive recurrent?

Solution. We showed in another problem that the chain is irreducible and recurrent. Let us now see if it is positive recurrent. In other words, let us see if $E_i T_i < \infty$ for some (and thus all) i.

As we said in the lectures, this is equivalent to having $\pi(i) > 0$ for all i where π is solution to the balance equations. We solved the balance equations in the past and found that $\pi(i) = c$ for all i, where c is a constant. But there is no $c > 0$ for which $\sum_{i=0}^{\infty} \pi(i) = 1$. And so the chain is not positive recurrent; it is null recurrent.

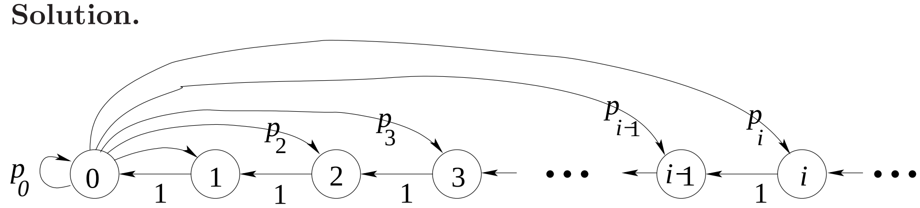
52.

Consider a Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probabilities

$$p_{i,i-1} = 1, \quad i = 1, 2, 3, \ldots$$
$$p_{0,i} = p_i, \quad i = 0, 1, 2, 3, \ldots$$

where $p_i > 0$ for all i and $\sum_{i \geq 0} p_i = 1$.
(i) Is the chain irreducible?
(ii) What is the period of state 0?
(iii) What is the period of state i, for all values of i?
(iv) Under what condition is the chain positive recurrent?
(v) If the chain is positive recurrent, what is the mean number of steps required for it to return to state i if it starts from i?

Solution.



(i) Yes it is. It is possible to move from any state to any other state.

(ii) It is 1.

(iii) Same.

(iv) We write balance equations:

$$\pi(i) = \pi(i+1) + \pi(0)p_i, \quad i \geq 0.$$

Solving this we find

$$\pi(i) = \pi(0)(1 - p_0 - \cdots - p_{i-1}), \quad i \geq 1.$$

The normalising condition gives

$$1 = \sum_{i=0}^{\infty} \pi(i) = \pi(0) \sum_{i=0}^{\infty} (1 - p_0 - \cdots - p_{i-1}).$$

This can be satisfied if and only if

$$\sum_{i=0}^{\infty} (1 - p_0 - \cdots - p_{i-1}) < \infty$$

This is the condition for positive recurrence.

Note that, since $p_0 + \cdots + p_{i-1} = P_0(X_1 \leq i-1)$, the condition can be written as

$$\sum_{i=0}^{\infty} P_0(X_1 \geq i) < \infty$$

But

$$\sum_{i=0}^{\infty} P_0(X_1 \geq i) == \sum_{i=0}^{\infty} E_0(X_1 \geq i) = E_0 \sum_{i=0}^{\infty} \mathbf{1}(X_1 \geq i) = E_0 \sum_{i=0}^{X} 1 = E_0(X+1)$$

so the condition is equivalent to

$$E_0 X_1 < \infty.$$

(v)

$$E_i T_i = \frac{1}{\pi(i)}.$$

53.

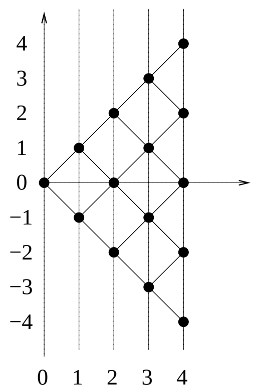
Consider a SIMPLE SYMMETRIC RANDOM WALK $S_n = \xi_1 + \cdots + \xi_n$, started from $S_0 = 0$. Find the following probabilities:
(i) $P(S_4 = k)$, for all possible values of k.
(ii) $P(S_n \geq 0 \ \ \forall n = 1, 2, 3, 4)$.
(iii) $P(S_n \neq 0 \ \ \forall n = 1, 2, 3, 4)$.
(iv) $P(S_n \leq 2 \ \ \forall n = 1, 2, 3, 4)$.
(v) $P(|S_n| \leq 2 \ \ \forall n = 1, 2, 3, 4)$.

Solution. (i) We have

$$P(S_4 = k) = \binom{4}{\frac{4+k}{2}} 2^{-4}, \quad k = -4, -2, 0, 2, 4,$$

and so

$$P(S_4=-4)=P(S_4=4)=1/16, \quad P(S_4=-2)=P(S_4=2)=4/16, \quad P(S_4=0)=6/16.$$



(ii) Since the random walk is symmetric, all paths of the same length are equally likely. There are 6 paths comprising the event $\{(S_n \geq 0 \ \ \forall n = 1,2,3,4\}$ and so $P((S_n \geq 0 \ \ \forall n = 1,2,3,4) = 6/16$.
(iii) There are just 2 paths comprising the event $\{S_n > 0 \ \ \forall n = 1,2,3,4\}$. Hence $P(S_n \neq 0 \ \ \forall n = 1,2,3,4) = 4/16$.
(iv) There are 2 paths violating the condition $\{S_n \leq 2 \ \ \forall n = 1,2,3,4\}$. Hence $P(S_n \leq 2 \ \ \forall n = 1,2,3,4) = (16-2)/16 = 14/16$.
(v) There are 4 paths violating the condition $\{|S_n| \leq 2 \ \ \forall n = 1,2,3,4\}$. Hence $P(|S_n| \leq 2 \ \ \forall n = 1,2,3,4) = (16-4)/16 = 12/16$.

54.

Consider a SIMPLE RANDOM WALK $S_n = \xi_1 + \cdots + \xi_n$, started from $S_0 = 0$, with $P(\xi_1 = 1) = p$, $P(\xi_1 = -1) = q$, $p + q = 1$.
(i) Show that

$$E(S_m \mid S_n) = \begin{cases} \frac{m}{n} S_n, & \text{if } m \leq n \\ S_n, & \text{if } m > n \end{cases}.$$

(ii) Are you surprised by the fact that the answer does not depend on p?

Solution. (i) If $m > n$ then $S_m = S_n + (S_m - S_n)$, so

$$E(S_m \mid S_n) = S_n + E(S_m - S_n \mid S_n).$$

But $S_m - S_n = \xi_{n+1} + \cdots + \xi_m$. Since $\xi_{n+1}, \ldots, \xi_m$ are independent of S_n, we have

$$E(S_m - S_n \mid S_n) = E(S_m - S_n) = \sum_{k=n+1}^{m} E\xi_k = (m-n)(p-q).$$

Thus,

$$E(S_m - S_n \mid S_n) = S_n + (p-q)(m-n), \quad \text{if } m > n.$$

If $m \leq n$, then

$$E(S_m \mid S_n) = E\left(\sum_{k=1}^{m} \xi_k \mid S_n\right) = \sum_{k=1}^{m} E(\xi_k \mid S_n)$$

Now notice that for all $k = 1, \ldots, n$,

$$E(\xi_k \mid S_n) = E(\xi_1 \mid S_n),$$

because the random variables $\xi_1, \ldots, \xi_n$ are i.i.d. and S_n is a symmetric function of them (interchanging two does not change the sum). Hence for all $m = 1, \ldots, n$,

$$E(S_m \mid S_n) = mE(\xi_1 \mid S_n).$$

This is true even for $m = n$. But, in this case, $E(S_m \mid S_n) = E(S_n \mid S_n) = S_n$, so that $E(\xi_1 \mid S_n) = S_n/n$. Thus,

$$E(S_m - S_n \mid S_n) = \frac{m}{n} S_n, \qquad \text{if } m \le n.$$

(ii) At first sight, yes, you should be surprised. But look (think) again...

55.

Consider a SIMPLE RANDOM WALK S_n again, *which does not necessarily start from* 0, and define the processes the processes:

$$\begin{aligned} X_n &= S_{2n}, \quad n \ge 0 \\ Y_n &= S_{2n+1}, \quad n \ge 0 \\ Z_n &= e^{S_n}, \quad n \ge 0 \end{aligned}$$

(i) Show that each of them is Markov and identify their state spaces.
(ii) Compute their transition probabilities.

Solution. (i) The first two are Markov because they are a subsequence of a Markov chain. The third is Markov because $x \mapsto e^x$ is a bijection from $\mathbb{R}$ into $(0, \infty)$. The state space of the first two is $\mathbb{Z}$. The state space of the third is the set $S = \{e^k : k \in \mathbb{Z}\} = \{\dots, e^{-2}, e^{-1}, 1, e, e^2, e^3, \dots\}$.

(ii) For the first one we have

$$P(X_{n+1} = j | X_n = i) = P(S_{2n} = j | S_{2n} = i) = P(i + \xi_{2n+1} + \xi_{2n} = j) = P(\xi_1 + \xi_2 = j - i).$$

Hence, given i, the only possible values of j are $i-2, i, i+2$. For all other values of j, the transition probability is zero. We have

$$\begin{aligned} P(X_{n+1} = i+2 | X_n = i) &= P(\xi_1 = \xi_2 = 1) = p^2 \\ P(X_{n+1} = i-2 | X_n = i) &= P(\xi_1 = \xi_2 = -1) = q^2 \\ P(X_{n+1} = i | X_n = i) &= P(\xi_1 = 1, \xi_2 = -1 \text{ or } \xi_1 = -1, \xi_2 = 1) = 2pq \end{aligned}$$

The second process has the same transition probabilities.

For the third process we have

$$\begin{aligned} P(Z_{n+1} = e^{k+1} | Z_n e^k) &= P(S_{n+1} = k+1 | S_n = k) = p \\ P(Z_{n+1} = e^{k-1} | Z_n e^k) &= P(S_{n+1} = k-1 | S_n = k) = q \end{aligned}$$

56.

Consider a SIMPLE RANDOM WALK S_n again, and suppose it starts from 0. As usual, $P(\xi_1 = 1) = p$, $P(\xi_1 = -1) = q = 1 - p$. Compute $Ee^{\alpha S_n}$ for $\alpha \in \mathbb{R}$.

Solution. We have $S_n = \xi_1 + \cdots + \xi_n$. By independence

$$Ee^{\alpha S_n} = E\left[e^{\alpha\xi_1} \cdots e^{\alpha\xi_n}\right] = E\left[e^{\alpha\xi_1}\right] \cdots E\left[e^{\alpha\xi_n}\right] = \left(E\left[e^{\alpha\xi_1}\right]\right)^n = (pe^{\alpha} + qe^{-\alpha})^n.$$

57.

(i) Explain why $P(\lim_{n\to\infty} S_n = \infty) = 1$ is $p > q$ and, similarly, $P(\lim_{n\to\infty} S_n = -\infty) = 1$ if $p < q$.
(ii) What can you say about the asymptotic behaviour of S_n as $n \to \infty$ when $p = q$?

Solution. (i) The Strong Law of Large Numbers (SLLN) says that

$$P(\lim_{n\to\infty} S_n/n = p - q) = 1,$$

because $E\xi_1 = p - q$. If $p > q$, then SLLN implies that

$$P(\lim_{n\to\infty} S_n/n > 0) = 1.$$

But

$$\{\lim_{n\to\infty} S_n/n > 0\} \subset \{\lim_{n\to\infty} S_n = \infty\}.$$

Since the event on the left has probability 1, so does the event on the right, i.e.

$$P(\lim_{n\to\infty} S_n = \infty) = 1, \quad \text{if } p > q.$$

If, on the other hand, $p < q$, then $p - q < 0$, and so SLLN implies that

$$P(\lim_{n\to\infty} S_n/n < 0) = 1.$$

But

$$\{\lim_{n\to\infty} S_n/n < 0\} \subset \{\lim_{n\to\infty} S_n = -\infty\}.$$

Since the event on the left has probability 1, so does the event on the right, i.e.

$$P(\lim_{n\to\infty} S_n = -\infty) = 1, \quad \text{if } p < q.$$

(ii) If $p = q$, then $p - q = 0$, and the fact that S_n/n converges to 0 cannot be used to say something about the sequence S_n other than that the sequence S_n has no limit. So, we may conclude that

$$P(S_n \text{ has no limit as } n \to \infty) = 1, \quad \text{if } p = q.$$

Stronger conclusions are possible, as we saw in the lectures.

58.

For a SIMPLE SYMMETRIC RANDOM WALK let f_n be the probability of first return to 0 at time n. Compute f_n for $n = 1, \dots, 6$ first by applying the general formula and then by path counting (i.e. by considering the possible paths that contribute to the event).

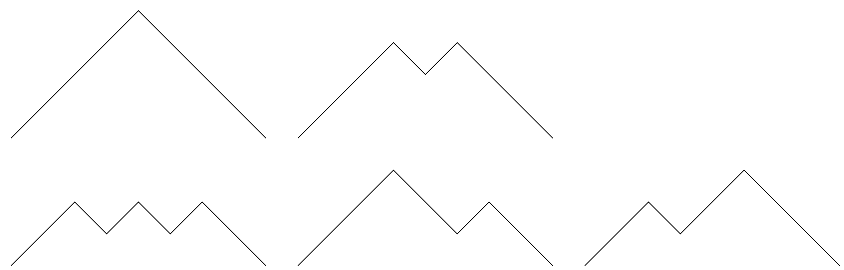
Solution. Obviously, $f_n = 0$ if n is odd. Recall the formula

$$f_{2k} = \binom{1/2}{k}(-1)^{k-1}, \quad k \in \mathbb{N}.$$

With $k = 1, 2, 3$, we have

$$f_2 = \binom{1/2}{1} = \frac{1}{2}$$
$$f_4 = -\binom{1/2}{2} = -\frac{(1/2)(1/2-1)}{2} = \frac{1}{8}$$
$$f_6 = \binom{1/2}{3} = -\frac{(1/2)(1/2-1)(1/2-2)}{6} = \frac{3}{16}$$
$$f_8 = -\binom{1/2}{4} = -\frac{(1/2)(1/2-1)(1/2-2)(1/2-3)}{24} = \frac{5}{128}.$$

To do path counting, we consider, e.g. the last case. The possible paths contributing to the event $\{T_0' = 8\}$ are the ones in the figure below as well as their reflections:



Each path consists of 8 segments, so it has probability 2^{-8}. There are 5 paths, so $f_8 = 10/2^8 = 5/128$.

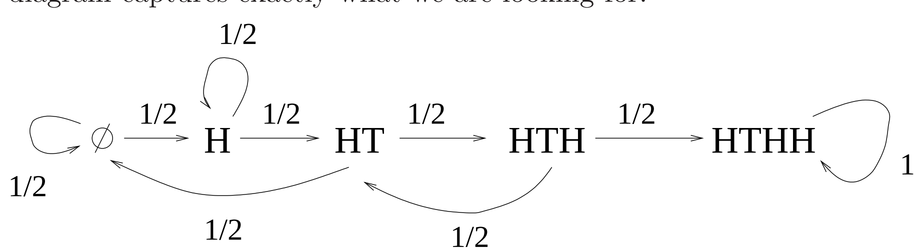
59.

Consider a SIMPLE SYMMETRIC RANDOM WALK starting from 0. Equalisation at time n means that $S_n = 0$, and its probability is denoted by u_n.
(i) Show that for $m \geq 1$, $f_{2m} = u_{2m-2} - u_{2m}$.
(ii) Using part (i), find a closed-form expression for the sum $f_2 + f_4 + \cdots + f_{2m}$.
(iii) Using part (i), show that $\sum_{k=1}^{\infty} f_{2k} = 1$. (One can also obtain this statement from the fact that $F(x) = 1 - (1-x)^{1/2}$.)
(iv) Show that the probability of no equalisation in the first $2m$ steps equals the probability of equalisation at $2m$.

60.

A fair coin is tossed repeatedly and independently. Find the expected number of tosses required until the patter HTHH appears.

Solution. It's easy to see that the Markov chain described by the following transition diagram captures exactly what we are looking for.



Rename the states $\emptyset, H, HT, HTH, HTHH$ as $0, 1, 2, 3, 4$, respectively, and let ψ_i be the average number of steps required for the state 4 to be reached if the starting state is i. Writing first-step (backwards) equations we have

$$\begin{aligned}
\psi_0 &= 1 + \tfrac{1}{2}\psi_0 + \tfrac{1}{2}\psi_1 \\
\psi_1 &= 1 + \tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_2 \\
\psi_2 &= 1 + \tfrac{1}{2}\psi_0 + \tfrac{1}{2}\psi_3 \\
\psi_3 &= 1 + \tfrac{1}{2}\psi_3 + \tfrac{1}{2}\psi_4.
\end{aligned}$$

Also, obviously, $\psi_4 = 0$. Solving, we find

$$\psi_3 = 8, \quad \psi_2 = 14, \quad \psi_1 = 16, \quad \psi_0 = 18.$$

So the answer is: "it takes, on the average, 18 coin tosses to see the pattern HTHH for the first time".

61.

Show that the stationary distribution for the Ehrenfest chain is Binomial.

Solution. The Ehrenfest chain has state space

$$S = \{0, 1, \dots, n\}$$

and transition probabilities

$$p_{i,i+1} = 1 - \frac{i}{n}, \quad p_{i,i-1} = \frac{i}{n}, \quad i = 0, \dots, n.$$

From the transition diagram we immediately deduce that *detailed balance equations* must hold, so, if π denotes the stationary distribution,

$$\pi(i)p_{i,i-1} = \pi(i-1)p_{i-1,i}, \quad 1 \le i \le n,$$

or

$$\pi(i) = \frac{n-i+1}{i}\pi(i-1), \quad 1 \le i \le n,$$

iterating of which gives

$$\pi(i) = \frac{n-i+1}{i} \, \frac{n-i+2}{i-1} \, \cdots \, \frac{n-1}{2} \, \frac{n}{1} \, \pi(0) = \frac{n!}{(n-i)!i!}\pi(0),$$

which is immediately recognisable as Binomial distribution.

62.

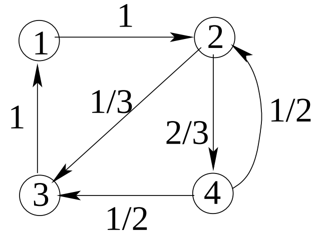
A Markov chain has transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Draw the transition diagram.
Are there any absorbing states?
Which are the communicating classes?
Can you find a stationary distribution?

What are the periods of the states?
Are there any inessential states?
Which states are recurrent?
Which states are transient?
Which states are positive recurrent?

Solution.



• There are no absorbing states because there is no state i for which $p_{i,i} = 1$.

• All states communicate with one another. Therefore there is only one communicating class, $\{1, 2, 3, 4\}$, the whole state space. (We refer to this by saying that the chain is irreducible.)

• Yes, of course we can. We can ALWAYS find a stationary distribution if the state space is FINITE. It can be found by solving the system of equations (known as balance equations)

$$\pi \mathsf{P} = \pi,$$

which, in explicit form, yield

$$\begin{aligned}
\pi(1) &= \pi(3) \\
\pi(2) &= \pi(1) + \tfrac{1}{2}\pi(4) \\
\pi(3) &= \tfrac{1}{3}\pi(2) + \tfrac{1}{2}\pi(4) \\
\pi(4) &= \tfrac{2}{3}\pi(2)
\end{aligned}$$

Solving these, along with the normalisation condition $\pi(1) + \pi(2) + \pi(3) + \pi(4) = 1$, we find

$$\pi(1) = \pi(4) = \pi(3) = 9/2, \quad \pi(2) = 27/4.$$

• Since the chain is irreducible the periods of all the states are the same. So let take a particular state, say state 4 and consider the set

$$\{n \geq 1 : \ p_{4,4}^{(n)} > 0\}.$$

We see that the first few elements of this set are

$$\{2, 5, 6, \ldots\}.$$

We immediately deduce that the greatest common divisor of the set is 1. Therefore the period of state 4 is 1. And so each state has period 1. (We refer to this by saying that the chain is aperiodic.)

• Since all states communicate with one another there are no inessential states.

• Since $\pi(i) > 0$ for all i, all states are recurrent.

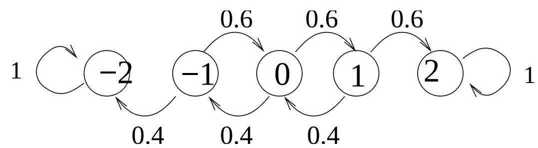
• Since all states are recurrent there are no transient states.

• Since $\pi(i) > 0$ for all i, all states are positive recurrent.

63.

In tennis the winner of a game is the first player to win four points, unless the score is 4–3, in which case the game must continue until one player wins by two points. Suppose that the game has reached the point where one player is trying to get two points ahead to win and that the server will independently win the point with probability 0.6. What is the probability the server will win the game if the score is tied 3-3? if she is ahead by one point? Behind by one point?

Solution. Say that a score x-y means that the server has x points and the other player y. If the current score is 3-3 the next score is either 4-3 or 3-4. In either case, the game must continue until one of the players is ahead by 2 points. So let us say that i represents the difference $x - y$. We model the situation by a Markov chain as follows:



Let φ_i be the probability that the server wins, i.e. that state 2 is reached before state -2. First-step equations yield:

$$\varphi_i = 0.6\varphi_{i+1} + 0.4\varphi_{i-1}, \quad -1 \le i \le 1.$$

In other words,

$$\begin{aligned} \varphi_{-1} &= 0.6\varphi_0 + 0.4\varphi_{-2} \\ \varphi_0 &= 0.6\varphi_1 + 0.4\varphi_{-1} \\ \varphi_1 &= 0.6\varphi_2 + 0.4\varphi_0 \end{aligned}$$

Of course,

$$\varphi_{-2} = 0, \quad \varphi_2 = 1.$$

Solving, we find

$$\varphi_0 = \frac{0.6^2}{1 - 2 \times 0.6 \times 0.4} \approx 0.69, \quad \varphi_1 \approx 0.88, \quad \varphi_{-1} \approx 0.42.$$

64.

Consider a simple random walk with $p = 0.7$, starting from zero. Find the probability that state 2 is reached before state -3. Compute the mean number of steps until the random walk reaches state 2 or state 3 for the first time.

Solution. Let φ_i be the probability that state 2 is reached before state -3, starting from state i. By writing first-step equations we have:

$$\varphi_i = p\varphi_{i+1} + q\varphi_{i-1}, \quad -3 < i < 2.$$

In other words,

$$\begin{aligned} \varphi_{-2} &= 0.7\varphi_{-1} + 0.3\varphi_{-3} \\ \varphi_{-1} &= 0.7\varphi_0 + 0.3\varphi_{-2} \\ \varphi_0 &= 0.7\varphi_1 + 0.3\varphi_{-1} \\ \varphi_1 &= 0.7\varphi_2 + 0.3\varphi_0 \end{aligned}$$

We also have, of course,

$$\varphi_{-3} = 0, \quad \varphi_2 = 1.$$

By solving these equations we find:

$$\varphi_i = \frac{(q/p)^{i+3} - 1}{(q/p)^5 - 1}, \quad -3 \le i \le 2.$$

Therefore

$$\varphi_0 = \frac{(3/7)^3 - 1}{(3/7)^5 - 1} \approx 0.93.$$

Next, let t_i be the mean number of steps until the random walk reaches state 2 or state 3 for the first time, starting from state i. By writing first-step equations we have:

$$t_i = 1 + pt_{i+1} + qt_{i-1}, \quad -3 < i < 2.$$

In other words,

$$\begin{aligned}
t_{-2} &= 1 + 0.7t_{-1} + 0.3t_{-3} \\
t_{-1} &= 1 + 0.7t_0 + 0.3t_{-2} \\
t_0 &= 1 + 0.7t_1 + 0.3t_{-1} \\
t_1 &= 1 + 0.7t_2 + 0.3t_0
\end{aligned}$$

We also have, of course,

$$t_2 = 0, \quad t_{-3} = 0.$$

By solving these equations we find:

$$t_i = \frac{5}{p-q}\frac{(q/p)^{i+3} - 1}{(q/p)^5 - 1} - \frac{i+3}{p-q}, \quad -3 \le i \le 2.$$

Therefore

$$t_0 = \frac{5}{0.4}\frac{(3/7)^3 - 1}{(3/7)^5 - 1} - \frac{3}{0.4} \approx 4.18.$$

65.

A gambler has £9 and has the opportunity of playing a game in which the probability is 0.4 that he wins an amount equal to his stake, and probability 0.6 that he loses his stake. He is allowed to decide how much to stake at each game (in multiple of 10p). How should he choose the stakes to maximise his chances of increasing his capital to £10?

66.

Let $\xi_1, \xi_2, \dots$ be i.i.d. r.v.'s with values in, say, $\mathbb{Z}$ and $P(\xi_1 = x) = p(x), x \in \mathbb{Z}$. Let $A \subseteq \mathbb{Z}$ such that $P(\xi_1 \in A) > 0$. Let $T_A = \inf\{n \ge 1 : \xi_n \in A\}$. Show that $P(\xi_{T_A} = x) = p(x)/\sum_{a \in A} p(a)$, $x \in A$.

Solution. Let $x \in A$.

$$P(\xi_{T_A} = x) = \sum_{n=1}^{\infty} P(\xi_n = x, \xi_1 \notin A, \dots, \xi_{n-1} \notin A)$$

$$= \sum_{n=1}^{\infty} p(x) P(\xi_1 \notin A)^{n-1} = p(x) \frac{1}{P(\xi_1 \notin A)} = \frac{p(x)}{\sum_{a \in A} p(a)}.$$

$\square$

67.

For a SIMPLE SYMMETRIC RANDOM WALK starting from 0, compute ES_n^4.

Solution. We have that $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \ldots, \xi_n$ are i.i.d. with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. When we expand the fourth power of the sum we have

$$\begin{aligned} S_n^4 =& \xi_1^4 + \cdots + \xi_n^4 \\ &+ \xi_1^2\xi_2^2 + \cdots + \xi_{n-1}^2\xi_n^2 \\ &+ \xi_1^3\xi_2 + \cdots + \xi_{n-1}^3\xi_n \\ &+ \xi_1^2\xi_2\xi_3 + \cdots + \xi_{n-2}^2\xi_{n-1}\xi_n \\ &+ \xi_1\xi_2\xi_3\xi_4 + \cdots + \xi_{n-3}\xi_{n-2}\xi_{n-1}\xi_n. \end{aligned}$$

After taking expectation, we see that the expectation of each term in the last three rows is zero, because $E\xi_i = 0$ and because of independence. There are n terms in the first row and $3(n^2 - n)$ terms in the second one. Hence

$$ES_n^4 = nE\xi_1^4 + 3(n^2 - n)E\xi_1^2 E\xi_2^2.$$

But $\xi_1^2 = \xi_1^4 = 1$. So the answer is:

$$ES_n^4 = n + 3(n^2 - n) = 3n^2 - 2n.$$

68.

For a SIMPLE RANDOM WALK, compute $E(S_n - ES_n)^4$ and observe that this is less than Cn^2 for some constant C.

Solution. Write

$$S_n - ES_n = \widehat{S}_n = \sum_{k=1}^n \widehat{\xi}_k,$$

where

$$\widehat{\xi}_k = \xi_k - E\xi_k = \xi_k - (p - q).$$

Notice that

$$E\widehat{\xi}_k = 0,$$

and repeat the computation of $\widehat{S}_n^4$ as above but with $\widehat{\xi}_k$ in place of ξ_k:

$$\begin{aligned} \widehat{S}_n^4 =& \widehat{\xi}_1^4 + \cdots + \widehat{\xi}_n^4 \\ &+ \widehat{\xi}_1^2\widehat{\xi}_2^2 + \cdots + \widehat{\xi}_{n-1}^2\widehat{\xi}_n^2 \\ &+ \widehat{\xi}_1^3\widehat{\xi}_2 + \cdots + \widehat{\xi}_{n-1}^3\widehat{\xi}_n \\ &+ \widehat{\xi}_1^2\widehat{\xi}_2\widehat{\xi}_3 + \cdots + \widehat{\xi}_{n-2}^2\widehat{\xi}_{n-1}\widehat{\xi}_n \\ &+ \widehat{\xi}_1\widehat{\xi}_2\widehat{\xi}_3\widehat{\xi}_4 + \cdots + \widehat{\xi}_{n-3}\widehat{\xi}_{n-2}\widehat{\xi}_{n-1}\widehat{\xi}_n. \end{aligned}$$

After taking expectation, we see that the expectation of each term in the last three rows is zero. Hence

$$E\widehat{S}_n^4 = nE\widehat{\xi}_1^4 + 3(n^2 - n)E\widehat{\xi}_1^2 E\widehat{\xi}_2^2.$$

This is of the form $c_1 n^2 + c_2 n$. And clearly this is less than Cn^2 where $C = c_1 + |c_2|$.

69.

For a simple symmetric random walk let T_0' be the time of first return to 0. Compute $P(T_0' = n)$ for $n = 1, \dots, 6$ first by applying the general formula and then by path counting.

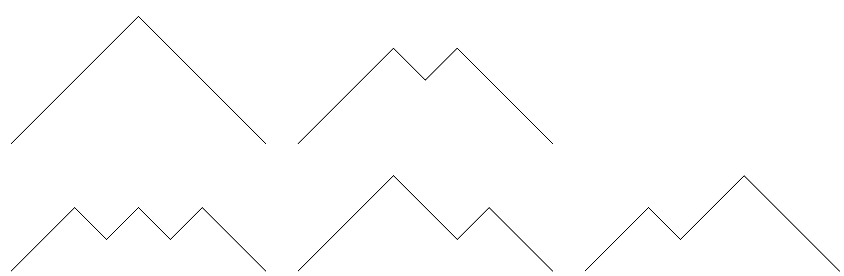
Solution. Obviously, $P(T_0' = n) = 0$ if n is odd. Recall the formula

$$P(T_0' = 2k) = \binom{1/2}{k}(-1)^{k-1}, \quad k \in \mathbb{N}.$$

With $k = 1, 2, 3$, we have

$$\begin{aligned}
P(T_0' = 2) &= \binom{1/2}{1} = \frac{1}{2} \\
P(T_0' = 4) &= -\binom{1/2}{2} = -\frac{(1/2)(1/2-1)}{2} = \frac{1}{8} \\
P(T_0' = 6) &= \binom{1/2}{3} = -\frac{(1/2)(1/2-1)(1/2-2)}{6} = \frac{3}{16} \\
P(T_0' = 8) &= -\binom{1/2}{4} = -\frac{(1/2)(1/2-1)(1/2-2)(1/2-3)}{24} = \frac{5}{128}.
\end{aligned}$$

To do path counting, we consider, e.g. the last case. The possible paths contributing to the event $\{T_0' = 8\}$ are the ones in the figure below as well as their reflections: Each path consists of 8 segments, so it has probability 2^{-8}. There are 5 paths, so



$P(T_0' = 8) = 10/2^8 = 5/128$.

70.

Show that the formula $P(M_n \geq x) = P(|S_n| \geq x) - \frac{1}{2}P(|S_n| = x)$ can also be derived by summing up over y the formula $P(M_n < x, S_n = y) = P(S_n = y) - P(S_n = 2x - y)$, $x > y$.

Solution.

71.

How would you modify the formula we derived for Es^{T_a} for a SIMPLE RANDOM WALK starting from 0 in order to make it valid for all a, positive or negative? Here T_a is the first hitting time of a.

Solution. For a simple random walk $S_n = \xi_1 + \cdots + \xi_n$, with $P(\xi_i = +1) = p$, $P(\xi_i = -1) = q$, we found

$$Es^{T_1} = \frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \equiv \psi(p, s),$$

where T_1 is the first time that the RW hits 1 (starting from 0), and we use the notation $\psi(p,s)$ just to denote this function as a function of p and s. We argued that when $a>0$, the random variable T_a is the sum of a i.i.d. copies of T_1 and so

$$Es^{T_a} = \psi(p,s)^a.$$

Let us now look at T_{-1}. Since the distribution of T_{-1} is the same as that of T_1 but *for a RW where the p and q are interchanged* we have

$$Es^{T_{-1}} = \psi(q,s).$$

Now, if $-a<0$, the random variable T_{-a} is the sum of a i.i.d. copies of T_{-1}. Hence

$$Es^{T_{-a}} = \psi(q,s)^a.$$

72.

Consider a SIMPLE SYMMETRIC RANDOM WALK starting from 0 and let T_a be the first time that state a will be visited. Find a formula for $P(T_a=n)$, $n\in\mathbb{Z}$.

Solution. Let us consider the case $a>0$, the other being similar. We have

$$Es^{T_a} = \left(\frac{1-\sqrt{1-4pqs^2}}{2qs}\right)^a,$$

whence

$$(2q)^a E(s^{a+T_a}) = \big(1-\sqrt{1-4pqs^2}\big)^a.$$

We write power series for the right and left hand sides separately.

$$RHS = \sum_{r=0}^{a}\binom{a}{r}\big(-\sqrt{1-4pqs^2}\big)^r = \sum_{r=0}^{a}\binom{a}{r}(-1)^r\sum_{n=0}^{\infty}\binom{r/2}{n}(-4pqs^2)^n$$

$$= \sum_{n=0}^{\infty}\left[\sum_{r=0}^{a}\binom{a}{r}\binom{r/2}{n}(-1)^{n+r}(4pq)^n\right]s^{2n} = \sum_{n=0}^{\infty}\left[\sum_{r=1}^{a}\binom{a}{r}\binom{r/2}{n}(-1)^{n+r}(4pq)^n\right]s^{2n}$$

$$LHS = (2q)^a\sum_{m=0}^{\infty}P(T_a=m)s^{a+m}.$$

Equating powers of s in both sides, we see that we need $a+m=2n$, i.e. $m=2n-a$.

$$LHS = (2q)^a\sum_{n\ge a}P(T_a=2n-a)s^{2n}.$$

We conclude:

$$P(T_a=2n-a) = \frac{1}{(2q)^a}\left[\sum_{r=1}^{a}\binom{a}{r}\binom{r/2}{n}(-1)^{n+r}(4pq)^n\right], \quad n\ge a.$$

73.

Consider a SIMPLE SYMMETRIC RANDOM WALK starting from 0 and let T_a be the first time that state a will be visited. Derive the formulæ for $P(T_a < \infty)$ in detail.

Solution. We have
$$P(T_a < \infty) = \lim_{s \uparrow 1} E s^{T_a}.$$
First consider the case $a > 0$. We have that, for all values of p,
$$E s^{T_a} = (E s^{T_1})^a = \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \right)^a, \quad |s| < 1.$$
So
$$P(T_a < \infty) = \lim_{s \uparrow 1} \left(\frac{1 - \sqrt{1 - 4pqs^2}}{2qs} \right)^a = \left(\frac{1 - \sqrt{1 - 4pq}}{2q} \right)^a = \left(\frac{1 - |p - q|}{2q} \right)^a.$$
If $p \geq q$ then $|p - q| = p - q$ and simple algebra gives $P(T_a < \infty) = 1$. If $p < q$ then $|p - q| = q - p$ and simple algebra gives $P(T_a < \infty) = (q/p)^a$. Next consider the case $a < 0$. By interchanging the roles of p and q we have
$$P(T_a < \infty) = \left(\frac{1 - |q - p|}{2p} \right)^a.$$
If $p \geq q$ then $|q - p| = p - q$ and simple algebra gives $P(T_a < \infty) = (p/q)^a$. If $p > q$ then $|q - p| = q - p$ and simple algebra gives $P(T_a < \infty) = 1$.

74.

Show that for a symmetric simple random walk any state is visited infinitely many times with probability 1.

Solution.

75.

Derive the expectation of the running maximum M_n for a SSRW starting from 0:
$$E M_n = E|S_n| + \binom{n}{n/2} 2^{-n-1} - \frac{1}{2}.$$
Conclude that $E M_n / E|S_n| \to 1$, as $n \to \infty$.

Solution. This follows from the formula
$$P(M_n \geq x) = P(|S_n| \geq x) - \frac{1}{2} P(|S_n| = x).$$
We have $E M_n = \sum_{x=1}^{\infty} P(M_n \geq x)$, $E|S_n| = \sum_{x=1}^{\infty} P(|S_n| \geq x)$, so:
$$E M_n = E|S_n| - \frac{1}{2} \sum_{x=1}^{\infty} P(|S_n| = x).$$
The last sum equals $1 - P(S_n = 0) = 1 - \binom{n}{n/2} 2^{-n}$.

76.

Using the ballot theorem, show that, for a SSRW starting from 0,

$$P(S_1 > 0, \ldots, S_n > 0) = \frac{ES_n^+}{n},$$

where $S_n^+ = \max(S_n, 0)$.

Solution. The ballot theorem says

$$P(S_1 > 0, \ldots, S_n > 0 \mid S_n = x) = x/n, \quad x > 0.$$

Hence

$$\begin{aligned} P(S_1 > 0, \ldots, S_n > 0) &= \sum_{x=1}^{\infty} P(S_1 > 0, \ldots, S_n > 0 \mid S_n = x) P(S_n = x) \\ &= \sum_{x=1}^{\infty} \frac{x}{n} P(S_n = x) = \frac{1}{n} ES_n^+. \end{aligned}$$

77.

For a SIMPLE RANDOM WALK with $p < q$ show that $EM_\infty = \frac{p}{q-p}$.

Solution. We have

$$EM_\infty = \sum_{x=1}^{\infty} P(M_\infty \geq x) = \sum_{x=1}^{\infty} P(T_x < \infty) = \sum_{x=1}^{\infty} (p/q)^x = \frac{p/q}{1-(p/q)} = \frac{p}{q-p}.$$

78.

Consider a SSRW, starting from some positive integer x, and let T_0 be the first n such that $S_n = 0$. Let $M = \max\{S_n : 0 \leq n \leq T_0\}$. Show that M has the same distribution as the integer part of (i.e. the largest integer not exceeding) x/U, where U is a uniform random variable between 0 and 1.

Solution. Let T_a be the first time that the random walk reaches level $a \geq x$. Then

$$P(M \geq a) = P(T_a < T_0) = x/a.$$

On the other hand, if $[y]$ denotes the largest integer not exceeding the real number y, we have, for all $a \geq x$,

$$P([x/U] \geq a) = P(x/U \geq a) = P(U \leq x/a) = x/a.$$

Hence $P([x/U] \geq a) = P(M \geq a)$, for all $x \geq a$ (while both probabilities are 1 for $x < a$). Hence M has the same distribution as $[x/U]$.

79.

Show that $(X_n, n \in Z_+)$ is Markov if and only if for all intervals $I = [M, N] \subseteq \mathbb{Z}_+$ $(X_n, n \in I)$ (the process inside) is independent of $(X_n, n \notin I)$ (the process outside), conditional on the pair (X_M, X_N) (the process on the boundary).

80.

A deck of cards has 3 Red and 3 Blue cards. At each stage, a card is selected at random. If it is Red, it is removed from the deck. If it is Blue then the card is not removed and we move to the next stage. Find the average number of steps till the process ends.

Solution. The problem can be solved by writing first step equations for the Markov chain representing the number of red cards remaining: The transition probabilities are:

$$p(3,2) = 3/6, \quad p(2,1) = 2/5, \quad p(1,0) = 1/4,$$
$$p(i,i) = 1 - p(i,i-1), \quad i = 3,2,1, \qquad p(0,0) = 1.$$

Let $\psi(i)$ be the average number of steps till the process ends if the initial state is i. Then

$$\psi(3) = 1 + (3/6)\psi(3) + (3/6)\psi(2),$$
$$\psi(2) = 1 + (3/5)\psi(2) + (2/5)\psi(1),$$
$$\psi(1) = 1 + (3/4)\psi(1).$$

Alternatively, observe that, to go from state i to $i-1$, we basically toss a coin with probability of success equal to $p(i,i-1)$. Hence the expected number of tosses till success is $1/p(i,i-1)$. Adding these up we have the answer:

$$\psi(3) = (6/3) + (5/2) + (2/1).$$

81.

There are two decks of cards. Deck 1 contains 50 Red cards, 30 Blue cards, and 20 Jokers. Deck 2 contains 10, 80, 10, respectively. At each stage we select a card from a deck. If we select a Red card then, we select a card of the other deck at the next stage. If we select a Blue card then we select a card from the same deck at the next stage. If, at any stage, a Joker is selected, then the game ends. Cards are always replaced in the decks. Set up a Markov chain and find, if we first pick up a card at random from a deck at random, how many steps it takes on the average for the game to end.

Solution. The obvious Markov chain has three states: 1 (you take a card from Deck 1), 2 (you take a card from Deck 2), and J (you selected a Joker). We have:

$$p(1,2) = 30/100, \quad p(1,J) = 20/100,$$
$$p(2,1) = 80/100, \quad p(2,J) = 10/100,$$
$$p(J,J) = 1.$$

Let $\psi(i)$ be the average number of steps for the game to end. when we start from deck i Then

$$\psi(1) = 1 + (50/100)\psi(1) + (30/100)\psi(2),$$
$$\psi(2) = 1 + (10/100)\psi(2) + (80/100)\psi(1).$$

Solve for $\psi(1), \psi(2)$. Since the initial deck is selected at random, the answer is $(\psi(1) + \psi(1))/2$.

82.

Give an example of a Markov chain with a small number of states that
(i) is irreducible
(ii) has exactly two communication classes
(iii) has exactly two inessential and two essential states
(iv) is irreducible and aperiodic
(v) is irreducible and has period 3
(vi) has exactly one stationary distribution but is not irreducible
(vii) has more than one stationary distributions
(viii) has one state with period 3, one with period 2 and one with period 1, as well as a number of other states
(ix) is irreducible and detailed balance equations are satisfied
(x) has one absorbing state, one inessential state and two other states which form a closed class
(xi) has exactly 2 transient and 2 recurrent states
(xii) has exactly 3 states, all recurrent, and exactly 2 communication classes
(xiii) has exactly 3 states and 3 communication classes
(xiv) has 3 states, one of which is visited at most finitely many times and the other two are visited infinitely many times, with probability one
(xv) its stationary distribution is unique and uniform

83.

Give an example of a Markov chain with infinitely many states that
(i) is irreducible and positive recurrent
(ii) is irreducible and null recurrent
(iii) is irreducible and transient
(iv) forms a random walk
(v) has an infinite number of inessential states and an infinite number of essential states which are all positive recurrent

84.

A drunkard starts from the pub (site 0) and moves one step to the right with probability 1. If, at some stage, he is at site k he moves one step to the right with probability p^k, one step to the left with probability q^k, or stays where he is with the remaining probability. Suppose $p + q = 1$, $0 < p < q$. Show that the drunkard will visit 0 infinitely many times with probability 1.

Solution. Write down balance equations for the stationary distribution. Observe that, since $\sum_{k=1}^{\infty}(p/q)^{k(k-1)/2} < \infty$, we have that $\pi(k) > 0$ for all k. Hence, not only 0 will be visited infinitely many times, but also the expected time, starting from 0, till the first return to 0 is $1/\pi(0) < \infty$.

85.

A Markov chain takes values $1, 2, 3, 4, 5$. From i it can move to any $j > i$ with equal probability. State 5 is absorbing. Starting from 1, how many steps in the average will it take till it reaches 5?

Solution. 2.08

86.

There are N individuals, some infected by a disease (say the disease of curiosity) and some not. At each stage, exactly one uninfected individual is placed in contact with the infected ones. An infected individual infects with probability p. So an uninfected individual becomes infected if he or she gets infected by at least one of the infected individuals. Assume that, to start with, there is only one infected person. Build a Markov chain with states $1, 2, \ldots, N$ and argue that $p(k, k+1) = 1 - (1-p)^k$. Show that, on the average, it will take $N + q(1-q^N)/(1-q)$ for everyone to become infected.

Solution. When there are k infected individuals and one uninfected is brought in contact with them, the chance that the latter is not infected is $(1-p)^k$. So

$$p(k,k) = (1-p)^k, \quad p(k,k+1) = 1-(1-p)^k, \quad , k = 1, \ldots, N-1.$$

Of course, $p(N,N) = 1$. The average number of steps to go from k to $k+1$ is $1/p(k,k+1)$. Hence, starting with 1 infected individual it takes

$$\sum_{k=1}^{N-1} \frac{1}{1-(1-p)^k}$$

for everyone to become infected.

87. ---

Assume, in addition, that exactly one infected individual is selected for treatment and he or she becomes well with probability $\alpha > 0$, and this occurs independently of everything else. (i) What is the state space and the transition probabilities? (ii) How many absorbing states are there? (iii) What kind of question would you like to ask here and how would you answer it?

Solution. (i) Since $p(1,0)$ is equal to α which is positive, we now need to include 0 among the states. So the state space is

$$S = \{0, 1, 2, \ldots, N\}.$$

The transition probabilities now become

$$p(k,k-1) = \alpha q^k, \quad p(k,k) = (1-\alpha)q^k, \quad p(k,k+1) = (1-\alpha)(1-q^k), \quad k = 1, \ldots N-1$$
$$p(N,N-1) = \alpha q^N, \quad p(N,N) = 1-\alpha q^N, \quad p(0,0) = 1.$$

where $q = 1-p$. (ii) There is only one absorbing state: the state 0. (iii) The question here is: How long will it take for the chain to be absorbed at 0? Letting $g(k)$ be the mean time to absorption starting from k, we have

$$g(k) = 1 + (1-\alpha q^k)g(k) + \alpha q^k g(k-1) + (1-\alpha)(1-q^k)g(k+1), \quad 1 \le k \le N$$
$$g(N) = 1 + (1-\alpha q^N)g(N) + \alpha q^N g(N-1),$$
$$g(0) = 0.$$

There is a unique solution.

88. ---

Prove that for an irreducible Markov chain with N states it is possible to go from any state to any other state in at most $N-1$ steps.

Solution. For any two distinct states i, j there is a path that takes you from i to j. Cut out any loops from this path and you still have a path that takes you from i to j. But this path has distinct states and distinct arrows. There are at most $N-1$ such arrows.

89.

Consider the general 2-state chain, where $p(1,2) = a$, $p(2,1) = b$. Give necessary and sufficient conditions for the chain (i) to be aperiodic, (ii) to possess an absorbing state, (iii) to have at least one stationary distribution, (iv) to have exactly one stationary distribution.

Solution. (i) There must be at least one self-loop. The condition is:

$$a < 1 \text{ or } b < 1.$$

(ii)

$$a = 0 \text{ or } b = 0.$$

(iii) It always does because it has a finite number of states.
(iv) There must exist exactly one communication class:

$$a > 0 \text{ or } b > 0.$$

90.

Show that the stationary distribution on any (undirected) graph whose vertices have all the same degree is uniform.

Solution. We know that

$$\pi(i) = cd(i),$$

where $d(i)$ is the degree of i, and c some constant. Indeed, the detailed balance equations

$$\pi(i)p(i,j) = \pi(j)p(j,i), \quad i \neq j$$

are trivially satisfied because $p(i,j) = 1/d(i)$, $p(j,i) = 1/d(i)$, by definition.
So when $d(i) = d =$ constant, the distribution π is uniform.

91.

Consider the random walk on the graph

$$\boxed{1} \text{ — } \boxed{2} \text{ — } \boxed{3} \text{ — } \boxed{4} \text{ — } \cdots \text{ — } \boxed{\text{N-1}} \text{ — } \boxed{\text{N}}$$

(i) Find its stationary distribution. (ii) Find the average number of steps to return to state 2, starting from 2. (iii) Repeat for 1. (iv) Find the average number of steps for it to go from i to N. (v) Find the average number of steps to go from i to either 1 or N. (vi) Find the average number of steps it takes to visit all states at least once.

Solution. Hint for (vi): The time to visit all states at least once is the time to hit the boundary plus the time to hit the other end of the boundary.

92.

When a bus arrives at the HW campus, the next bus arrives in $1, 2, \ldots, 20$ minutes with equal probability. You arrive at the bus stop without checking the schedule, at some fixed time. How long, on the average, should you wait till the next bus arrives? What is the standard deviation of this time?

Solution. This is based on one of the examples we discussed: Let X_n be the time elapsed from time n till the arrival of the next bus. Then X_n is a Markov chain with transition probabilities

$$p(k, k-1) = 1, \quad k > 0,$$
$$p(0, k) = p_k, \quad k > 0,$$

where $p_k = (1/20)\mathbf{1}(1 \le k \le 20)$. We find that the stationary distribution is

$$\pi(k) = c \sum_{j \ge k} p_k = c \sum_{j=k}^{20} \frac{1}{20} = \frac{c(21-k)}{20}, \quad 0 \le k \le 20.$$

where c a constant determined by normalisation:

$$1 = \sum_{k=0}^{20} \pi(k) = \frac{c}{20} \sum_{k=0}^{20} (21-k) = \frac{c}{20} \frac{21 \times 22}{2}, \qquad c = \frac{20}{231}.$$

Hence

$$\pi(k) = \frac{21-k}{231}, \quad 0 \le k \le 20,$$

and so the average waiting time is

$$\sum_{k=0}^{20} k\pi(k) = \sum_{k=0}^{20} k \frac{21-k}{231} = 20/3 = 6'40''$$

The standard deviation is

$$\sqrt{\sum_{k=0}^{20} k^2 \pi(k) - (20/3)^2} = \sqrt{230}/3 \approx 5'3''$$

Note: To do the sums without too much work, use the formulae

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

93.

Build a Markov chain as follows: When in state k ($k = 1, 2, 3, 4, 5, 6$), roll a die k times, take the largest value and move to that state. (i) Compute the transition probabilities and write down the transition probability matrix. (ii) Is the chain aperiodic? (iii) Does it have a unique stationary distribution? (iv) Can you find which state will be visited more frequently on the average?

Solution. (i) Let M_k be the maximum of k independent rolls. Then

$$P(M_k \le \ell) = (\ell/6)^k, \quad k, \ell = 1, \ldots, 6.$$

The transition probability from state k to state ℓ is

$$p(k, \ell) = P(M_k = \ell) = (\ell/6)^k - ((\ell-1)/6)^k, \quad k, \ell = 1, \ldots, 6.$$

The transition probability matrix is

$$\begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/36 & 3/36 & 5/36 & 7/36 & 9/36 & 11/36 \\ 1/216 & 7/216 & 19/216 & 37/216 & 61/216 & 91/216 \\ 1/1296 & 15/1296 & 65/1296 & 175/1296 & 369/1296 & 671/1296 \\ 1/7776 & 31/7776 & 211/7776 & 781/7776 & 2101/7776 & 4651/7776 \\ 1/46656 & 63/46656 & 665/46656 & 3367/46656 & 11529/46656 & 31031/46656 \end{pmatrix}$$

$$\approx \begin{pmatrix} 0.167 & 0.167 & 0.167 & 0.167 & 0.167 & 0.167 \\ 0.0278 & 0.0833 & 0.139 & 0.194 & 0.250 & 0.306 \\ 0.00463 & 0.0324 & 0.0880 & 0.171 & 0.282 & 0.421 \\ 0.000772 & 0.0116 & 0.0502 & 0.135 & 0.285 & 0.518 \\ 0.000129 & 0.00399 & 0.0271 & 0.100 & 0.270 & 0.598 \\ 0.0000214 & 0.00135 & 0.0143 & 0.0722 & 0.247 & 0.665 \end{pmatrix}$$

(ii) The chain is obviously aperiodic because it has at least one self-loop.
(iii) Yes it does because it is finite and irreducible.
(iv) Intuitively, this should be state 6.

94.

Simple queueing system: Someone arrives at a bank at time n with probability α. He or she waits in a queue (if any) which is served by one bank clerk in a FCFS fashion. When at the head of the queue, the person requires a service which is distributed like a random variable S with values in $\mathbb{N}$: $P(S = k) = p_k$, $k = 1, 2, \ldots$. Different people require services which are independent random variables. Consider the quantity W_n which is the total waiting time at time n: if I take a look at the queue at time n then W_n represents the time I have to wait in line till I finish my service. (i) Show that W_n obeys the recursion

$$W_{n+1} = (W_n + S_n \xi_n - 1)^+,$$

where the S_n are i.i.d. random variables distributed like S, independent of the ξ_n. The latter are also i.i.d. with $P(\xi_n = 1) = \alpha$, $P(\xi_n = 0) = 1 - \alpha$. Thus $\xi_n = 1$ indicates that there is an arrival at time n. (ii) Show that W_n is a Markov chain and compute its transition probabilities $p(k, \ell)$, $k, \ell = 0, 1, 2, \ldots$, in terms of the parameters α and p_k. (iii) Suppose that $p_1 = 1 - \beta$, $p_2 = \beta$. Find conditions on α and β so that the stationary distribution exists. (iv) Give a physical interpretation of this condition. (v) Find the stationary distribution. (vi) Find the average waiting time in steady-state. (vii) If $\alpha = 4/5$ (4 customers arrive every 5 units of time on the average–heavy traffic), what is the maximum value of β so that a stationary distribution exists? What is the average waiting time when $\beta = 0.24$?

Solution. (i) If, at time n the waiting time W_n is nonzero and nobody arrives then $W_{n+1} = W_n - 1$, because, in 1 unit of time the waiting time decreases by 1 unit. If, at time n, somebody arrives and has service time S_n then, immediately the waiting time becomes $W_n + S_n$ and so, in 1 unit of time this decreases by 1 so that $W_{n+1} = W_n + S_n - 1$. Putting things together we arrive at the announced equation. Notice that the superscript $+$ means maximum with 0, because, if $W_n = 0$ and nobody arrives, then $W_{n+1} = W_n = 0$.

(ii) That the W_n form a Markov chain with values in $\mathbb{Z}_+$ follows from the previous exercise. To find the transition probabilities we argue as follows: Let $p(k,\ell) = P(W_{n+1} = \ell \mid W_n = k)$. First, observe that, for a given k, the ℓ cannot be less than $k-1$. In fact, $p(k,k-1) = 1-\alpha$ (the probability that nobody arrives). Next, for ℓ to be equal to k we need that somebody arrives and brings work equal to 1 unit: $p(k,k) = \alpha p_1$. Finally, for general $\ell > k$ we need to have an arrival which brings work equal to $\ell - k + 1$: $p(k,\ell) = \alpha p_{\ell-k+1}$.
(iii) Here we have

$$p(k,k-1) = 1-\alpha, \quad p(k,k+1) = \alpha\beta, \quad p(k,k) = \alpha(1-\beta).$$

To compute the stationary distribution we write balance equations:

$$\pi(k)(1-\alpha) = \pi(k-1)\alpha\beta, \quad k \geq 0.$$

Iterating this we get

$$\pi(k) = \left(\frac{\alpha\beta}{1-\alpha}\right)^k \pi(0).$$

We need to be able to normalise:

$$\sum_{k=0}^{\infty} \left(\frac{\alpha\beta}{1-\alpha}\right)^k \pi(0) = 1.$$

We can do this if and only if the geometric series converges. This happens if and only if

$$\frac{\alpha\beta}{1-\alpha} < 1,$$

(iv) The condition can also be written as

$$1 + \beta < 1/\alpha.$$

The left side is the average service time $(1 \times (1-\beta) + 2 \times \beta)$. The right side is the average time between two successive arrivals. So the condition reads:
Average service time < average time between two successive arrivals.
(v) From the normalisation condition, $\pi(0) = (1 - \alpha(1+\beta))/(1-\alpha)$. This follows because

$$\sum_{k=0}^{\infty} \left(\frac{\alpha\beta}{1-\alpha}\right)^k = \frac{1}{1 - \frac{\alpha\beta}{1-\alpha}} = \frac{1-\alpha}{1-\alpha(1+\beta)}.$$

Hence

$$\pi(k) = \frac{1-\alpha(1+\beta)}{1-\alpha} \left(\frac{\alpha\beta}{1-\alpha}\right)^k, \quad k \geq 0.$$

This is of the form $\pi(0) = (1-\rho)\rho^k$, where $\rho = \alpha\beta/(1-\alpha)$, hence a geometric distribution.
(vi) The average waiting time in steady-state is

$$\sum_{k=0}^{\infty} k\pi(k) = \sum_{k=0}^{\infty} k\rho^k(1-\rho) = \frac{\rho}{1-\rho} = \frac{\alpha\beta}{1-\alpha(1+\beta)}.$$

(vii) If $\alpha = 4/5$ then $\beta < (5/4) - 1 = 1/4$. So the service time must be such that $P(S=2) = \beta < 1/4$, and $P(S=1) = 1-\beta > 3/4$. When $\beta = 0.24$ we are OK since $0.2 < 0.25$. In this case, the average waiting time is equal to 24, which is quite large compared to the maximum value of S.

95.

Let S_n be a simple symmetric random walk with $S_0 = 0$. Show that $|S_n|$ is Markov.

Solution. If we know the value of S_n, we know two things: its absolute value $|S_n|$ and its sign. But to determine the value of $|S_{n+1}|$, knowledge of the sign is irrelevant, since, by symmetry $|S_{n+1}| = |S_n| \pm 1$ with probability 1/2 if $S_n \neq 0$, while $|S_{n+1}| = 1$ is $S_n = 0$. Hence $P(|S_{n+1}| = j \mid S_n = i)$ depends only on $|i|$: if $|i| > 0$ it is 1/2 for $j = i \pm 1$, and if $|i| = 0$ it is 1 for $j = 1$.

96.

Let S_n be a simple but not symmetric random walk. Show that $|S_n|$ is not Markov.

Solution. In contrast to the above, $P(|S_{n+1}| = j \mid S_n = i)$ is not a function of $|i|$.

97.

Consider a modification of a simple symmetric random walk that takes 1 or 2 steps up with probability 1/4 or a step down with probability 1/2. Let Z_n be the position at time n. Show that $P(Z_n \to \infty) = 1$.

Solution. From the Strong Law of Large Numbers, Z_n/n converges, as $n \to \infty$, with probability 1, to the expected value of the step:

$$(-1) \times 0.5 + (+1) \times 0.25 + (+2) \times 0.25 = 0.25.$$

Since this a positive number it follows that Z_n must converge to infinity with probability 1.

98.

There are N coloured items. There are c possible colours. Pick an items at random and change its colour to one of the other $c - 1$ colours at random. Keep doing this. What is the Markov chain describing this experiment? Find its stationary distribution. (Hint: When $c = 2$ it is the Ehrenfest chain.)

Solution. The Markov chain has states

$$x = (x_1, \ldots, x_c),$$

where x_i is the number of items having colour i. Of course, $x_1 + \cdots + x_c = N$. If we let e_i be the vector with 1 in the i-th position and 0 everywhere else, then we see that from state x only a transition to a state of the form $x - e_i + e_j$ is possible if $x_i > 0$. The transition probability is

$$p(x, x - e_i + e_j) = \frac{x_i}{N} \frac{1}{c-1},$$

because x_i/N is the probability that you pick an item with colour i, and $1/(c-1)$ is the probability that its new colour will be j. To find the stationary distribution $\pi(x)$ we just try to see if *detailed balance equations* hold. If they do then we are happy and know that we have found it. If they don't, well, we don't give up and try to see how to satisfy the *(full) balance equations*. Recall:

• Full balance equations: $\pi(x) = \sum_y \pi(y)p(y, x)$, for all x.

If the chain is finite (and here it is), we can always find a probability distribution π that satisfies the full balance equations.

• Detailed balance equations: $\pi(x)p(x, y) = \pi(y)p(y, x)$, for all x, y.

Even if the chain is finite, it is NOT always the case that detailed balance hold. If they do, then we should feel lucky!
Since, for $c = 2$ (the Ehrenfest chain) the stationary distribution is the binomial distribution, we may GUESS that the stationary distribution here is multinomial:

$$\text{GUESS:} \qquad \pi(x) = \pi(x_1, \ldots, x_c) = \binom{N}{x_1, \ldots, x_c} c^{-N} = \frac{N!}{x_1! \cdots x_c!} c^{-N}.$$

Now

$$\text{CHECK WHETHER:} \qquad \pi(x)p(x, y) = \pi(y)p(y, x) \qquad \text{HOLD FOR ALl } x, y.$$

If y, x are not related by $y = x - e_i + e_j$ for some distinct colours i, j, then $p(x, y) = p(y, x) = 0$, and so the equations hold trivially. Suppose then that $y = x - e_i + e_j$ for some distinct colours i, j, then $p(x, y) = p(y, x) = 0$, and so the equations hold trivially. Suppose then that $y = x - e_i + e_j$ for some distinct colours i, j. We have

$$\pi(x)p(x, x - e_i + e_j) = \frac{N!c^{-N}}{x_1! \cdots x_i! \cdots x_j! \cdots x_c!} \frac{x_i}{N} \frac{1}{c-1}$$

and

$$\pi(x, x - e_i + e_j)p(x - e_i + e_j, x) = \frac{N!c^{-N}}{x_1! \cdots (x_i - 1)! \cdots (x_j + 1)! \cdots x_c!} \frac{x_j + 1}{N} \frac{1}{c-1}$$

The two quantities are obviously the same. Hence detailed balance equations are satisfied. Hence the multinomial distribution IS THE stationary distribution.

99.

In the previous problem: If there are 9 balls and 3 colours (Red, Green, Blue) and we initially start with 3 balls of each colour, how long will it take on the average till we see again the same configuration? (Suppose that 1 step = 1 minute.) If we start we all balls coloured Red, how long will it take on the average till we see the same again?

Solution.

$$\pi(3, 3, 3) = \frac{9!}{3!3!3!} 3^{-9} = 560/6561.$$

Hence the average number of steps between two successive occurrences of the state $(3, 3, 3)$ is $6561/560 \approx$ **11.72 minutes**. Next,

$$\pi(9, 0, 0) = \frac{9!}{9!0!0!} 3^{-9} = 1/19683.$$

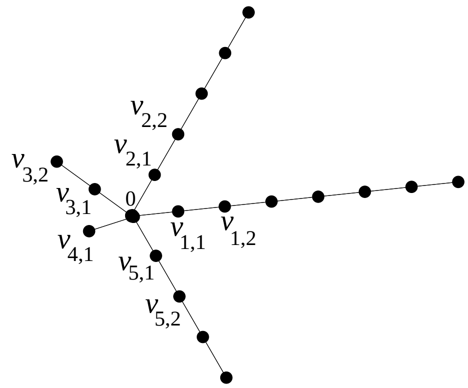
Hence the average number of steps between two successive occurrences of the state $(9, 0, 0)$ is 19683 minutes = 328.05 hours $\approx$ **13 and a half days**.

100.

Consider a random walk on a star-graph that has one centre vertex 0 and N legs emanating from 0. Leg i contains ℓ_i vertices (in addition to 0) labelled

$$v_{i,1}, v_{i,2}, \ldots, v_{i,\ell_i}.$$

The vertices are in sequence: 0 is connected to $v_{i,1}$ which is connected to $v_{i,2}$, etc. till the end vertex v_{i,ℓ_i}. (i) A particle starts at 0. Find the probability that it reaches the



end of leg i before reaching the end of any other leg. (ii) Suppose $N = 3$, $\ell_1 = 2, \ell_2 = 3, \ell_3 = 100$. Play a game as follows: start from 0. If end of leg i is reached you win ℓ_i pounds. Find how much money you are willing to pay to participate in this game.

Solution. Let $\varphi_i(x)$ be the probability that end of leg i is reached before reaching the end of any other leg. Clearly,

$$\varphi_i(v_{i,\ell_i}) = 1, \quad \varphi_i(v_{k,\ell_k}) = 0, \; k \neq i.$$

Now, if $v_{i,r}$ is an interior vertex of leg i (i.e. neither 0 nor the end vertex), then

$$\varphi_i(v_{k,r}) = \frac{1}{2}\varphi_i(v_{k,r-1}) + \frac{1}{2}\varphi_i(v_{k,r+1}).$$

This means that the function

$$r \mapsto \varphi_i(v_{k,r})$$

must be linear for each k (for the same reason that the probability of hitting the left boundary of an interval before hitting the right one is linear for a simple symmetric random walk). Hence

$$\varphi_i(v_{k,r}) = a_{i,k}r + b_{i,k},$$

where $a_{i,k}, b_{i,k}$ are constants. For any leg we determine the constants in terms of the values of φ_i at the centre 0 and the end vertex. Thus,

$$\begin{aligned} \varphi_i(v_{k,r}) &= \frac{\varphi_i(0)}{\ell_k}(\ell_k - r), \quad k \neq i, \\ \varphi_i(v_{i,r}) &= \frac{r}{\ell_i} + \frac{\varphi_i(0)}{\ell_i}(\ell_i - r). \end{aligned}$$

Now, for vertex 0 we have

$$\begin{aligned} \varphi_i(0) = \frac{1}{N}\sum_{k=1}^{N}\varphi_i(v_{k,1}) &= \frac{1}{N}\left[\frac{1}{\ell_i} + \frac{\varphi_i(0)}{\ell_i}(\ell_i - 1) + \sum_{k\neq i}\frac{\varphi_i(0)}{\ell_k}(\ell_k - 1)\right] \\ &= \frac{1}{N}\frac{1}{\ell_i} + \varphi_i(0)\frac{1}{N}\sum_{k=1}^{N}\left(1 - \frac{1}{\ell_k}\right) \end{aligned}$$

whence

$$\varphi_i(0) = \frac{\frac{1}{\ell_i}}{\sum_{k=1}^{N}\frac{1}{\ell_k}}.$$

(ii)

$$\varphi_3(0) = \frac{1/600}{1/2 + 1/3 + 1/600}, \ \varphi_2(0) = \frac{1/3}{1/2 + 1/3 + 1/600}, \ \varphi_1(0) = \frac{1/2}{1/2 + 1/3 + 1/600}.$$

The average winnings are:

$$600\varphi_3(0) + 3\varphi_2(0) + 2\varphi_1(0) \approx 1.2 \text{ pounds.}$$