Probabilities II / Trevezas

Construction of Lebesgue measure via an exterior measure

$$\lambda^*(A) = \inf\left\{ \sum_{n\geq 1} (b_n - a_n) : A \subset \bigcup_n (a_n, b_n),\ a_n, b_n \in \mathbb{R},\ a_n < b_n,\ n \geq 1 \right\}$$

$\downarrow$ $2^{\mathbb{R}}$

exterior measure

$\hookrightarrow$ Καραθεοδωρή theorem $\mathcal{M}_{\lambda^*} \overset{+\text{Propos}}{=} (\mathcal{B}(\mathbb{R}))_\lambda$

$\downarrow$ Lebesgue measurable set

and $\lambda^*|_{\mathcal{M}_{\lambda^*}} \equiv \lambda \rightarrow$ Lebesgue measure

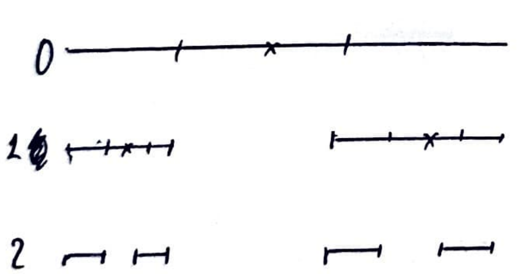
In probability theory it is enough to take $\lambda|_{\mathcal{B}(\mathbb{R})}$ [$\lambda, \nu \rightarrow$ counting measure]

λ-null sets (λ-μηδενική)

$\lambda(\{x\}) = 0$, countable sets A: $\lambda(A) = 0$

$\{x\} = [x, x]$

there are also uncountable, Cantor set $\lambda(C) = 0$



$C_0 = [0, 1]$

$C_1 = [0, 1/3] \cup [2/3, 1]$

...

2^n closed intervals

$C = \bigcap_n C_n \rightarrow$ disjoint union of 2^n closed intervals

$\hookrightarrow$ closed set

closed

$C_{n+1} \subset C_n \Rightarrow (C_n) \downarrow$

$$\lambda(C) = \lim_{n\to\infty} \lambda(C_n) = \lim_{n\to\infty} \frac{2^n}{3^n} = 0$$

$$x \in C \leftrightarrow x = 0, \underset{\substack{\cap \\ \{0,2\}}}{x_1}, \underset{\substack{\cap \\ \{0,2\}}}{x_2}, \dots, x_n$$

$$x = \sum_{n\geq 1} x_n \frac{1}{3^n}$$

$$C = |2^{\mathbb{N}}| = C \quad (\text{cantor set})$$

Questions

1. If $\lambda(A) > 0$, then does it contain an open interval?

↳ if we take (a,b), $a<b \Rightarrow \lambda((a,b)) = b-a > 0$

The answer is no

Example : $\lambda(\mathbb{Q}) = 0 \Rightarrow \lambda(\mathbb{R} \setminus \mathbb{Q}) \overset{\text{disjoint}}{=} \underbrace{\lambda(\mathbb{R})}_{+\infty} - \underbrace{\lambda(\mathbb{Q})}_{0} = +\infty$

↳ it does not contain an open interval

$$\lambda(\underbrace{[0,1] \setminus \mathbb{Q}}_{\substack{\text{no open interval} \\ \text{inside}}}) = 1$$

Definition : Let $(X, \mathcal{A}, \mu)$ be a measurable space. Then we say that μ concertrates (συγκεντρώνεται) on $S \in \mathcal{A}$ if $\mu(X \setminus S) = 0$.

If μ is finite, then μ concertrates on S if $\mu(S) = \mu(X)$

Special case : probability measure $\mu(S) = 1$

$$\downarrow$$
$$P(S)$$

What are different types of probability measures?

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Definition: In general, a <u>measure is discrete</u>, if it concentrates on a countable set. In particular, for a probability measure there exist a countable $S \in \mathcal{A}$:

$$p(S) = 1 \quad (\text{διακριτό μέτρο πιθανότητας})$$

Example

$\Omega \to$ sample space (finite / countable)

e.g. $\Omega = \{0,1\}$, $(\{0,1\}, P(\Omega))$

$\hookrightarrow$ Bernoulli(p), $0<p<1$, $\underset{\underset{1-p}{|||}}{p_0} = P(\{0\})$, $\underset{\underset{p}{|||}}{p_1} = P(\{1\})$ $\to$ probability of success

$$P(A) = \sum_{w \in A} p_w$$

Binomial(n,p), $\Omega = \{0,1,\dots,n\}$

$$p_k = P(\{k\}) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0,1,\dots,n$$

<u>Theorem</u> (Characterisation of discrete probability measures): Let $(\Omega, \mathcal{A})$ be a measurable space with $\{w\} \in \mathcal{A}$, $\forall w \in \Omega$. Then

i) P is a discrete probability measure $\iff \exists S$ countable and $(p_w)_{w \in S}$ with $p_w \geq 0$, $\forall w \in S$ such that $\sum_{w \in S} p_w = 1 \ \wedge \ P = \sum p_w \delta_w$

$\hookrightarrow$ dirac measure at w

ii) Any discrete probability measure is determined completely by $P(\{w\})$.

<u>Proof</u>

"$\Rightarrow$" Assume that P is a discrete probability measure. Then

$\exists S \in \mathcal{A}$ countable : $p(S) = 1$.

Take $p_w = P(\{w\})$ $\forall w \in S$. Let $A \in \mathcal{A}$.

$$P(A) = P(A \cap S) + \underbrace{P(A \cap S^c)}_{\substack{\subset S \\ 0}} = P(A \cap S) = \sum_{w \in S \cap A} P(\{w\})$$

$$= \sum_{w \in S} P(\{w\}) \cdot 1_A\{w\} = \sum_{w \in S} p_w \cdot \delta_w(A) \Rightarrow$$

$$\Rightarrow p = \sum_{w \in S} p_w \delta_w$$

"$\Leftarrow$" (a) measure, (b) probability measure, (c) discrete

(a) if $\lambda_i \geq 0$ and μ_i measure $\forall i \in I$ (countable), then $\mu = \sum_{i \in I} \lambda_i \mu_i$ is a measure.

$$\cdot\ \mu(\emptyset) = \sum_{i \in I} \underbrace{\lambda_i \mu_i(0)}_{0} = 0$$

$\cdot$ Take (A_n) + in $\mathcal{A}$.

$$\mu(\bigcup_n A_n) = \sum_{i \in I} \lambda_i \underbrace{\mu_i(\bigcup_n A_n)}_{\sigma\text{-addit}}$$

$$= \sum_{i \in I} \lambda_i \sum_n \mu_i(A_n) = \sum_n \sum_i \lambda_i \mu_i(A_n) = \sum_n \mu(A_n)$$

So P is a measure, $P = \sum_{w \in S} p_w \delta_w$ (λ_w, μ_w)

(b) Is it a probability measure?

$$P(\Omega) = \sum_{w \in S'} p_w \delta_w(\Omega) = \sum_{w} p_w = 1$$

(c) just take S' countable

$$P(S') = \sum_{w \in S'} p_w \underbrace{\delta_w(s)}_{1} = \sum_{w \in S'} p_w = 1$$

So P is a discrete probability measure.

Discrete probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Bernoulli(p), Bin(n,p), $Geo^*(p)$ $\to 1,2,3,\dots$, $Geo(p)$ $\to 0,1,2,\dots$, $P(\lambda)$, $\lambda > 0$

They are all considered probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$Ber(p) \longrightarrow P = (1-p)\delta_0 + p\,\delta_1$ (combination of Diracs measures)

$Bin(n,p) \longrightarrow P = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k$

$Geo^*(p) \longrightarrow P = \sum_{n \geq 1} (1-p)p^{n-1} \delta_n$

$Geo(p) \to P = \sum_{n \geq 0} (1-p) p^n \delta_n$

$P(\lambda) \to P = \sum_{n \geq 0} e^{-\lambda} \frac{\lambda^n}{n!} \delta_n$

P probability measure $\longleftrightarrow$ F distribution function

i) $\nearrow$ ii) right continuous iii) $F(-\infty) = 0$, $F(+\infty) = 1$

$P(\{x\}) > 0 \leftrightarrow F(x) - F(x^-) > 0 \Leftrightarrow F$ is discontinuous at x

$P((-\infty, x] \setminus (-\infty, x))$

$D_F = \{x \in R : F(x) - F(x^-) > 0\}$

P is discrete probability measure (dpm) $\xleftrightarrow{P \to F_P} P(D_{F_P}) = 1$

<u>Definition</u>: P is a continuous probability measure $\longleftrightarrow F_P$ is continuous $\Longleftrightarrow$

$P(\{x\}) = 0 \quad \forall x \in R$

$P \to$ discrete

$P \to$ continuous

continuous $\to$ absolutely continuous (απόλυτα συνεχείς) $\to$ have a probability density function

continuous $\to$ singular continuous (ιδιάζουσα συνεχείς)

<u>Absolutely continuous</u>

$\exists F \geq 0, \; F(y) - F(x) = \int_x^y f(t)\,dt \;\longrightarrow\; F(x) = \int_{-\infty}^x f(t)\,dt$

<u>Singular continuous</u>

$\exists S \in \mathcal{B}(\mathbb{R})$ with $\lambda(S) = 0$, for which $P(S) = 1$ (F is continuous)

$0, x_1, x_2, \dots x_n \dots$

$x_1 \in \{0,2\}, \; x_2 \in \{0,2\}$

$\longrightarrow$

F_d, F_{ac}, F_{sc}

F_d: discrete

F_{ac}: absolutely continuous

F_{sc}: singular continuous

if F is a discrete function

$F = p_1 F_d + p_2 F_{ac} + p_3 F_{sc}$, $p_1, p_2, p_3 \geq 0$ and $p_1 + p_2 + p_3 = 1$

$0, x_1, x_2, x_3$

$\vert\vert\vert$

$X = \sum_n x_n \frac{1}{2^n}$

$\{0,1\}$

If $X_n \sim Ber\left(\frac{1}{2}\right) \Rightarrow X \sim Uniform\ (0,1)$

$0 < p < \frac{1}{2}$, $\frac{1}{2} < p < 1$

$X_n \sim Ber(p)$, $p \neq 1/2$, then X will have singular continuous distribution

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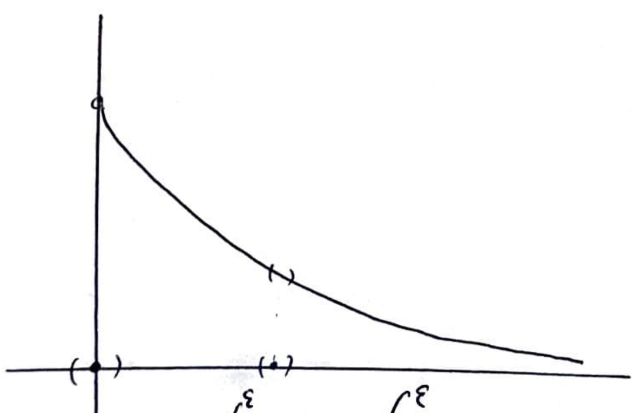
Support of a measure (στήριγμα/φορέας μέτρου), $(X, \mathcal{A}, \mu)$

$supp(\mu) = \{x \in X : \text{if } A \text{ is open and } x \in A, \text{ then } \mu(A) > 0\}$

$\hookrightarrow$ closed set

e.g. $Exp(1)$



$[0, +\infty)$ $\quad \int_{-\varepsilon}^{\varepsilon} f(t)dt = \int_0^{\varepsilon} f(t)dt > 0.$

$(supp(\mu))^c$