Probabilities II / Trevezas

<u>Definition</u>: let $(X, \mathcal{A})$ be a measurable ~~set~~ space, where X is a topological space with $\mathcal{B}(X) \subset \mathcal{A}$. If μ is a measure on $(X, \mathcal{A})$ the set

$$\text{supp} := \{x \in X : \mu(A) > 0, \ \forall A \text{ open with } x \in A\}$$

↳ support (στήριγμα ή φορέας)

<u>comment</u>: Sometimes the definition demands X to have a countable basis.

$$x \in (\text{supp}(\mu))^c \Leftrightarrow \exists A \text{ open with } x \in A \text{ and } \mu(A) = 0$$

So

$$(\text{supp}(\mu))^c = \bigcup_{\substack{A \text{ open} \\ \text{and } \mu(A)=0}} A \iff \text{supp}(\mu) = \bigcap_{\substack{A \text{ open} \\ \text{and } \mu(A)=0}} A^c = \bigcap_{\substack{F \text{ closed} \\ \text{and } \mu(F^c)=0}} F = \bigcap_{\substack{F \text{ is closed} \\ \text{and } \mu \text{ concentrates} \\ \text{on } F}} F$$

Therefore, $\text{supp}(\mu)$ is the smallest closed set on which μ concentrates.

<u>Remark</u>: So if $\overset{\to \text{ topology}}{\tau}$ on X has a countable basis or X is a seperable metric space, then $\mu(\text{supp}(\mu)^c) = \mu(\text{count. union of open sets}) = 0$

↳ by sub-addition.

<u>Examples</u>: i) Take $P = \sum_{n \geq 0} e^{-\lambda} \frac{\lambda^n}{n!} \delta_n$ → the Poisson measure with $\lambda > 0$.

$\mathbb{N} \subset \text{supp}(\mu)$ since for all $n \in \mathbb{N}$, and an open set with $n \in A$, we have

$$P(\{n\}) \leq P(A) \Rightarrow n \in \text{supp}(P)$$

Here, $\text{supp}(P) = \mathbb{N}$

Take, $x \notin \mathbb{N}$. Then say

$$\varepsilon_x = \min \{ x - \lfloor x \rfloor , \lfloor x \rfloor + 1 - x \}.$$

We have

$$B(x,\varepsilon_x) = (x-\varepsilon_x, x+\varepsilon_x) \subset (\lfloor x \rfloor, \lfloor x \rfloor + 1) \rightarrow P(\underset{\substack{\Psi \\ x}}{\overset{\text{open}}{B(x,\varepsilon_x)}}) = 0.$$

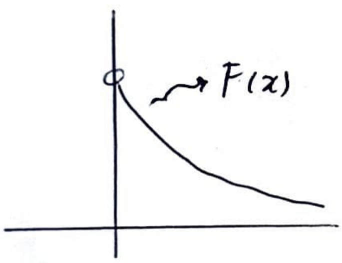
→ δεν περιέχει φυσικό (είνα

ii) $P = \sum_{n \geq 1} p_n \delta_{q_n}$, for $Q = \{q_n\}_{n \geq 1}$ and $p_n > 0$, $\forall n \geq 1$.

We have

$$Q \subset \operatorname{supp}(P) \Rightarrow \underset{\substack{\| \\ \mathbb{R}}}{\bar{Q}} \subseteq \overline{\operatorname{supp}(P)} = \operatorname{supp}(P) \subset \mathbb{R} \Rightarrow \operatorname{supp}(P) = \mathbb{R}$$

iii) $P \to F$. Assume that $\forall x \in \mathbb{R}$, $F(x) = \int_{-\infty}^{x} f(t)\,dt$ where $f(x) = \begin{cases} \overset{\substack{\text{Exp}(1) \\ \downarrow}}{e^{-x}}, & x \geq \\ 0, & \text{else} \end{cases}$



$\longrightarrow \operatorname{supp}(P) = [0, +\infty)$

Measurable Functions

Definition: Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be 2 measurable spaces. A function $f: X \to Y$ is called <u>$\mathcal{A}/\mathcal{B}$ measurable</u> if:

$$f^{-1}(B) \in \mathcal{A} \quad \forall B \in \underset{\substack{\downarrow \\ \sigma\text{-alg}}}{\mathcal{B}} \quad \text{or equivalently}$$

$$f^{-1}(\mathcal{B}) \subset \mathcal{A} \quad (\text{where } f^{-1}(\mathcal{B}) = \{ f^{-1}(B) : B \in \mathcal{B} \})$$

Comments: 1) Often $\mathcal{B} = \mathcal{B}(\mathbb{R})$ and we refer to Borel measurable function

2) If $X = \Omega$ and $(\Omega, \mathcal{A}, P)$ is a probability space, then F is called a random variable and denoted by X, Y, Z.

$X : \Omega \to \mathbb{R}$ or $\bar{\mathbb{R}}$ or $\mathbb{C}$ or $\mathbb{R}^n$

- $\mathbb{R}$: real random variable
- $\bar{\mathbb{R}}$: extended random variable (εκτεταμένη)

and the B-collection $\mathcal{B}(\mathbb{R})$, $\mathcal{B}(\bar{\mathbb{R}})$, $\mathcal{B}(\mathbb{C})$, $\mathcal{B}(\mathbb{R}^m)$

Any open set in $\bar{\mathbb{R}}$ is characterised as $A \cup B$, A open set on $\mathbb{R}$, B "$\{-\infty, +\infty\}$":

- A open in $\mathbb{R}$
- $A \cup \{-\infty\}$
- $A \cup \{+\infty\}$
- $A \cup \{-\infty, +\infty\}$

Any Borel set in $\bar{\mathbb{R}}$ is $A \cup B$, A Borel in $\mathbb{R}$, $B \subset \{-\infty, +\infty\}$:

- $A \in \mathcal{B}(\mathbb{R})$ $(A \cup \emptyset)$
- $A \cup \{-\infty\}$
- $A \cup \{+\infty\}$
- $A \cup \{-\infty, +\infty\}$

Proposition: Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be measurable spaces, $f : X \to Y$ a function and $\mathcal{E} \subset \mathcal{B} : \sigma(\mathcal{E}) = \mathcal{B}$ (generator). Then:

Then: $f^{-1}(\mathcal{E}) \subset \mathcal{A} \Leftrightarrow f^{-1}(\mathcal{B}) \subset \mathcal{A}$.

Proof

'$\Leftarrow$' Obvious since $\mathcal{E} \subset \mathcal{B}$.

'$\Rightarrow$' We define $D = \{B \in \mathcal{B} : f^{-1}(B) \in \mathcal{A}\}$.

By assumption $f^{-1}(\mathcal{C}) \subset \mathcal{A} \Rightarrow \mathcal{C} \subset D$ (1).

We will prove that <u>D is a σ-algebra</u> (2). Recall that:

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \Rightarrow f^{-1}\Big(\bigcup A_n\Big) = \bigcup_n f^{-1}(A_n)$$

and

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

Also,

$$\left.\begin{aligned} X = f^{-1}(Y) = f^{-1}(B \cup B^c) &= f^{-1}(B) \cup f^{-1}(B^c) \\ f^{-1}(B) \cap f^{-1}(B^c) = f^{-1}(\underbrace{B \cap B^c}_{\emptyset}) &= \emptyset \end{aligned}\right\} \to \boxed{\left(f^{-1}(B)\right)^c = f^{-1}(B^c)}$$

<u>Proof of (2)</u>

i) $\emptyset \in \mathcal{B}$ and $f^{-1}(\emptyset) = \emptyset \in \mathcal{A}$ (σ-algebra) $\to \emptyset \in D$

ii) Take $B \in D \Rightarrow f^{-1}(B) \in \mathcal{A} \xRightarrow{\mathcal{A}\ \sigma\text{-alg}} \left(f^{-1}(B)\right)^c = f^{-1}(\underset{\underset{\mathcal{B}\ \sigma\text{-alg}}{\cap}}{B^c}) \in \mathcal{A} \to B^c \in D$

iii) Take (B_n) in D. So $f^{-1}(B_n) \in \mathcal{A}\ \forall n$. Therefore,

$$f^{-1}\Big(\bigcup_n B_n\Big) = \bigcup_n \underset{\underset{\mathcal{A}}{\cap}}{f^{-1}(B_n)} \in \mathcal{A} \Rightarrow \bigcup_n B_n \in D.$$

By (1)+(2), $\mathcal{B} = \sigma(\mathcal{C}) \subset D \subset \mathcal{B} \Rightarrow D = \mathcal{B}$. So $f^{-1}(\mathcal{B}) \subset \mathcal{A}$.

<u>Comments</u>

1) The measurability condition can be now checked on a smaller

family being a generator of $\mathcal{B}$.

2) If $F: \Omega \to \mathbb{R}$ where $(\Omega, \mathcal{A}, P)$ is a probability space then we can check that $\{f \le x\} \in \mathcal{A}, \forall x \in \mathbb{R}$ in order to obtain measureability of F, so F will be a real random variable.

$$\{f \le x\} = F^{-1}((-\infty, x])$$

Indeed, the family

$$\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\} \qquad (\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}))$$

is a generator of $\mathcal{B}(\mathbb{R})$

Corollary (Πόρισμα):

i) if $f: (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then f is a measurable function or a real random variable $\Leftrightarrow \{f \le x\} \in \mathcal{A}, \forall x \in \mathbb{R}$

ii) if $f: (\Omega, \mathcal{A}) \to (\overline{\mathbb{R}}, \mathcal{B}(\mathbb{R}))$, then f is a measurable or equivalently an extended real random variable $\Leftrightarrow \{f \le x\} = F^{-1}([-\infty, x]) \in \mathcal{A}, \forall x \in \mathbb{R}$

Exercise

Show that $\mathcal{C} = \{[-\infty, x] : x \in \underset{\text{or } \overline{\mathbb{R}}}{\mathbb{R}}\}$ is a generator of $\mathcal{B}(\overline{\mathbb{R}})$

Theorem: A function $F: \overline{\mathbb{R}} \to [0,1]$ is a distribution function of a probability measure on $\overline{\mathbb{R}} \to \mathcal{B}(\overline{\mathbb{R}}) \Leftrightarrow$

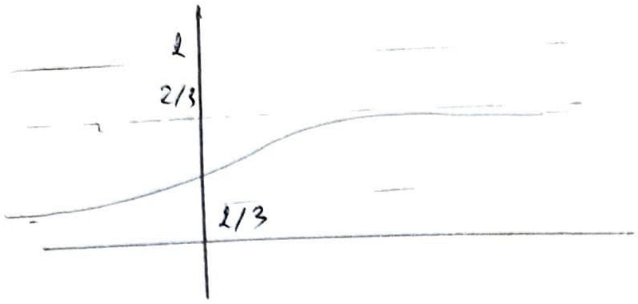
i) $F \uparrow$

ii) F is right continuous

iii) $0 \le F(-\infty) \le F(+\infty) \le 1$

Comments

- $F(-\infty) = P(\{-\infty\})$
- $F(+\infty) = 1 - P(\{-\infty\})$



If $F(-\infty) > 0$ or $F(+\infty) < 1$, then we say that the distribution function is <u>improper</u>/difective (μη γνησια) and if $F(-\infty) = 0$ and $F(+\infty) = 1$, it is called <u>proper</u> (γνησια)

Corollary: Let (X, d_X) and (Y, d_Y) be metric spaces. If $f: X \to Y$ is continuous, then f is $\mathcal{B}(X)/\mathcal{B}(Y)$ measurable:

<u>Proof</u>

f is continuous $\Rightarrow f^{-1}(\mathcal{T}_{d_Y}) \subset \mathcal{T}_{d_X}$ (every open set returns an open set).

However:

$$\left.\begin{array}{l} \mathcal{T}_{d_X} \text{ is a generator of } \mathcal{B}(X) \\ \mathcal{T}_{d_X} \subset \mathcal{B}(X) \\ \text{and} \\ f^{-1}(\mathcal{T}_{d_Y}) \subset \mathcal{B}(X) \\ \sigma(\mathcal{T}_{d_Y}) = \mathcal{B}(Y) \end{array}\right\} \Rightarrow f \text{ is } \mathcal{B}(X)/\mathcal{B}(Y) \text{ measurable}$$