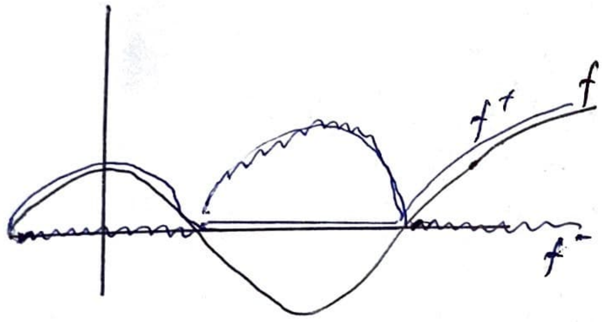
Probabilities II / Trevezas

Proposition: Let $f, g : \Omega \to \bar{\mathbb{R}}$ be measure functions on $(\Omega, \mathcal{A})$ and $\lambda \in \mathbb{R}$. Then

$$\lambda f,\ |f|,\ f+g,\ f \cdot g,\ f/g,\ \min\{f,g\},\ \max\{f,g\},\ \underset{\max\{0,f\}}{\underset{\shortparallel}{f^+}},\ \underset{\max(0,-f)}{\underset{\shortparallel}{f^-}}$$

are measurable, provided they are well defined.

Reminder: Problems arise when we have $\frac{0}{0}$, $+\infty-\infty$ and other interminute ~~m~~expressions.




$f = f^+ - f^-$, $f^+, f^- \geq 0$

$|f| = f^+ + f^-$

Proposition: Let $(f_n)$ be a sequence of measurable functions on $(\Omega, \mathcal{A})$, $f_n : \Omega \to \bar{\mathbb{R}}$ then

i) $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$, $\limsup_n f_n$ are measurable

ii) if $f_n \xrightarrow[\text{pointwise}]{p.w} f$ (κατά σημείο), then $f = \lim f_n$ is measurable (continuity is not preserved by pointwise limits, however!)

Reminder: In contrast to continuity measurability is transferred to the limit function

## Proof

$f_n : \Omega \to \overline{\mathbb{R}}$ are measurable, $\forall n \geq 1$. We want to show that $\inf\limits_n f_n$ is also measurable.

generator intervals of rational numbers, so $x \in \mathbb{Q}$ can

For all $x \in \mathbb{R}$ (we take preferable generator)

$$\{\inf f_n < x\} = \bigcup_{\substack{q \in \mathbb{Q} \\ q < x}} \underbrace{\{f_n < q\}}_{\in \mathcal{A}} \in \mathcal{A}$$

$\inf f_n < x$

$\inf f_n(\omega) < x \iff \exists n \in \mathbb{N} : f_n(\omega) < x$

$\bigcup\limits_n \{f_n < x\}$

αρκεί να $\exists$ ένας ρητός ανάμεσα από πλήθυσμα

$\{\omega \in \Omega : \inf\limits_n f_n(\omega) < x\}$

$\Rightarrow \inf f_n$ is measurable.

- $\sup f_n = -\inf(-f_n) \Rightarrow \sup f_n$ also measurable

- $\liminf f_n = \underbrace{\sup_n \underbrace{\underbrace{\inf_{k \geq n} f_k}_{\text{measur}}}_{h_n}}_{\text{measurable}}$ also measurable

<u>Proposition</u>: If $f : X \to Y$, $g : Y \to Z$, $(X, \mathcal{A})$, $(Y, \mathcal{B})$, $(Z, \mathcal{C})$ are measurable spaces, the

$$f \circ g : X \to Z \text{ is } \mathcal{A}/\mathcal{C} \text{ measurable}$$

$f \circ g(x) = f(g(x))$

## Proof

$\cancel{(f \circ g)^{-1}}$

Let $C \in \mathcal{C}$

$$(f \circ g)^{-1}(C) = f^{-1}(\underbrace{g^{-1}(C)}_{\in \mathcal{B}}) \in \mathcal{A} \quad \text{bcs } f \text{ is measurable.}$$

Therefore, $f \circ g$ is $\mathcal{A}/\mathcal{C}$ measurable

<u>Definition</u>: A function $f : \Omega \to \bar{\mathbb{R}}$ is called <u>simple</u> (anlı̇) if $f(\Omega)$ is a finite set.

e.g. $f = 1_A$, $A \subset \Omega$    or    $f(w) = c$

$\{0,1\}$      ↳ a simple value.

<u>Canonical form of a simple function</u>

If $a_1, a_2, \dots, a_n$ are the distinct values of $f$, then

$$f = \sum_{i=1}^{n} \bar{a}_i 1_{A_i}, \text{ where } A_i = f^{-1}(\{a_i\}), \ 1 \le i \le n$$ (if sth missing, it will be 0)

Be careful if 0 is indicated as a value:

e.g. $f = 1_{[0,1]} = 1 \cdot 1_{[0,1]} + 0 \cdot 1_{[0,1]^c}$

$\{A_i\}_{1 \le i \le n}$ form a partition of $\Omega$

<u>Proposition</u>: Let $(\Omega, \mathcal{A})$ be a measurable space. Then

i) if $A \subset \Omega$, $1_A$ is measurable $\Leftrightarrow A \in \mathcal{A}$

ii) a simple function in canonical form

$$f = \sum_{i=1}^{n} a_i 1_{A_i} \Longleftrightarrow A_1, A_2, \dots, A_n \in \mathcal{A} \text{ (they are measurable)}$$

<u>Proof</u>

let $x \in \mathbb{R}$

$$1_A^{-1}([-\infty, x)) = \begin{cases} \emptyset, & x < 0 \\ A^c, & 0 \leq x < 1 \\ \Omega, & x \geq 1 \end{cases}$$

So $1_A$ is measurable $\Leftrightarrow A^c \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}$.

closed, belongs to Borel

ii) "$\Longrightarrow$" $f^{-1}(\{a_i\}) \in \mathcal{A} \ \forall 1 \leq i \leq n$

$\{a_i\} \in \mathcal{B}(\bar{\mathbb{R}})$, $f^{-1}(\{a_i\}) = A_i$

"$\Longleftarrow$" $A_i \in \mathcal{A} \ \forall 1 \leq i \leq n \overset{i)}{\Longrightarrow} 1_{A_i}$ is measurable $1 \leq i \leq n$

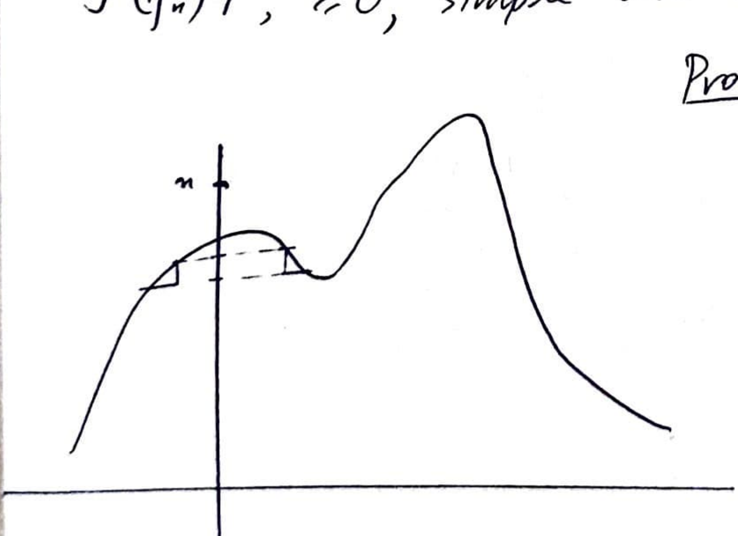
$\Rightarrow a_i \cdot 1_{A_i}$ is measurable $\forall 1 \leq i \leq n \xrightarrow[\text{sum is finite}]{} \displaystyle\sum_{i=1}^{n} a_i 1_{A_i}$ is measurable

$\displaystyle\sum_{i=1}^{n} a_i 1_{A_i} = f$

<u>Proposition</u>: Let $f: \Omega \to [0, +\infty]$ be a measurable function. Then

$\exists (f_n) \uparrow, \geq 0$, simple and measurable functions : $f_n \xrightarrow{p.w.} f$

<u>Proof</u>



$$D_n = \left\{ \frac{k}{2^n}, 0 \leq k \leq n \cdot 2^n \right\} \subset D_{n+1}, \ \forall n \geq 1$$

συνέχεια υποδιαιρ 2 και περιλαμβ προηγούμενα, θα γίνεται συνέχεια λεπτότερη e.g. $D_2 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, \frac{8}{4} = 2\}$

$D_3 = \{0, \frac{1}{8}, \frac{2}{4}, \frac{3}{8}, \frac{2}{4}, \dots, \frac{8}{4}, \frac{9}{4}, \frac{10}{4}, \frac{11}{4}, \frac{12}{4}, 3\}$

$$f_n(\omega) = \left(\sum_{k=0}^{n\cdot 2^n - 1} \frac{k}{2^n} \mathbb{1}_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(f(\omega))\right) + \mathbb{1}_{[n, +\infty)} f(\omega)$$

προσέγγιση $f$,
όταν είναι $f(\omega)$ σε αυτό
το διάστ, δώσε τιμή αυτή –

$$= \sum_{k=0}^{n 2^n - 1} \frac{k}{2^n} \mathbb{1}_{f^{-1}([\frac{k}{2^n}, \frac{k+1}{n}))}(\omega) + \mathbb{1}_{f^{-1}([n, +\infty))}(\omega) .$$

Set $A_{k,n} = f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{n}\right)\right), \ 0 \le k \le n 2^n;$

$A_{n2^n, n} = f^{-1}([n, +\infty))$.

Therefore,

$$f_n = \sum_{k=0}^{n 2^n} a_{k,n} \mathbb{1}_{A_{k,n}} \quad , \text{where} \quad a_{k,n} = \frac{k}{2^n}, \ 0 \le k \le n \cdot 2^n$$

i) $f_n \geq 0$

ii) $f_n$ are simple

iii) $f_n$ are measurable

iv) $f_n \uparrow$ (bcs it is appr(downwards))

$f_n \xrightarrow{p.w.} f$

Indeed,

i) $\omega$: $f_n(\omega) = +\infty$ , $f_n(\omega) = n \xrightarrow[n\to\infty]{} +\infty$

ii) $\omega$: $0 \leq f_n(\omega) < +\infty$ , $\forall n$: $f(\omega) < n$ such that

$0 \leq |f(\omega) - f_n(\omega)| \leq \frac{1}{2^n} \longrightarrow 0$

↳ δεν το έγραψα [illegible] μου έχω παρα

$\xrightarrow{i) + ii)} f_n(\omega) \to f(\omega) \quad \forall \omega \in \Omega.$

Conclusion

$f(\omega) \in [0, +\infty]$ : $\exists (f_n)$: $f_n \xrightarrow{pw} f$ , with $f_n$ as (simple) friendly as possible

this will help in integration later on

$f_n \geq 0$, simple & measurable, $(f_n \uparrow)$

Corollary: If $f: \Omega \to \overline{\mathbb{R}}$ is measurable, then $\exists (f_n)$ simple & measurable, with $(|f_n|)$, such that $f_n \xrightarrow{pw} f$.

Proof

$f = f^+ - f^-, f^+, f^- \in [0, +\infty]$

$\exists (s_n)\uparrow, (r_n)\uparrow$ simple, measurable, $\geq 0$:

$f^+ = \lim s_n$ and $f^- = \lim r_n$

So, take $f_n = s_n - r_n$ and then we have

$f_n$ is simple & measurable and $\lim\limits_n f_n = \lim\limits_n s_n - \lim\limits_n r_n =$

$= f^+ - f^- = f$

$$\left.\begin{array}{l} |f_n| = s_n + r_n \\ (s_n)\uparrow, (r_n)\uparrow \end{array}\right\} \to (|f_n|)\uparrow$$

<u>Exercise</u>: Show that every monotone $f: \mathbb{R} \to \mathbb{R}$ is Borel measurable

<u>Proof</u>

Let $\beta \in \mathbb{R}$ and with no loss of generality take $f$ to be increasing

$$\{f \leq \beta\} = \begin{cases} \emptyset, & f(x) > \beta \ \forall x \in \mathbb{R} \\ \mathbb{R}, & f(x) \leq \beta, \forall x \in \mathbb{R} \\ \emptyset \neq A \neq \mathbb{R}, & \text{otherwise} \end{cases}$$

In the third case,

$\{f \leq \beta\} \neq \emptyset$ and $\{f > \beta\} \neq \emptyset$,

but $\mathbb{R} = \underbrace{\{f \leq \beta\} \cup \{f > \beta\}}_{\text{disjoint}}$

Take $x \in \{f \leq \beta\}$ and $y \in \{f > \beta\}$, so

$$f(x) \leq \beta < f(y) \xrightarrow{f\uparrow} x < y$$

So

$\{f \leq \beta\} = (-\infty, x^*)$ or $(-\infty, x^*] \in \mathcal{B}(\mathbb{R})$

Therefore, $f$ is Borel measurable.

<u>Exercise</u>: let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be 2 measurable spaces $f: X \to Y$, $\mathcal{E} \subset \mathcal{B}$. Show that,

$$f^{-1}(\sigma(\mathcal{E})) = \sigma(f^{-1}(\mathcal{E}))$$

<u>Proof</u>

We've already proved that for any $\sigma$-algebra $\mathcal{B}$, $f^{-1}(\mathcal{B})$ is a $\sigma$-algebra on $X$, so $f^{-1}(\sigma(\mathcal{E}))$ is a $\sigma$-algebra on $X$.

Notice that, $f^{-1}(\mathcal{E}) \subset f^{-1}(\sigma(\mathcal{E}))$ (bcs $\mathcal{E} \subset \sigma(\mathcal{E})$) $\Rightarrow$

$$\Rightarrow \boxed{\sigma(f^{-1}(\mathcal{E})) \subset f^{-1}(\sigma(\mathcal{E}))}$$

Let's recall that for any $\mathcal{B}$ $\sigma$-algebra where $\mathcal{B} = \sigma(\underset{\text{generator}}{\mathcal{E}})$, a function $f$ is $\mathcal{A}/\mathcal{B}$ measurable, if

$$f^{-1}(\sigma(\mathcal{E})) \subset \mathcal{A}.$$

Take

$$\mathcal{A} = \sigma(f^{-1}(\mathcal{E}))$$

and $f$ will be $\sigma(f^{-1}(\mathcal{E}))/\sigma(\mathcal{E})$ measurable if

$$f^{-1}(\sigma(\mathcal{E})) \subset \sigma(f^{-1}(\mathcal{E}))$$

(2)

Exercise: If $f:\mathbb{R}\to\mathbb{R}$ is differentiable, then $f'$ is measurable

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$