# Probabilities II / Trevezas

**Exercise**: 1) If $f:\mathbb{R}\to\mathbb{R}$ is differentiable, then $f'$ is measurable

## Solution

Bcs of differentiability,

$$\forall x\in\mathbb{R} \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Take $h=\frac{1}{n}$, and then

$$f'(x)=\lim_{n\to\infty}\frac{f(x+\frac{1}{n})-f(x)}{1/n}=\lim_{n\to\infty}\underbrace{n\big(\overset{\text{cont}}{f(x+\tfrac{1}{n})}-\overset{\text{cont}}{f(x)}\big)}_{f_n(x)}.$$

continuous

measurable

$f_n$ is continuous for all $n\geq 1$. Finally,

$$f_n \xrightarrow{p.w.} f \Rightarrow f \text{ is measurable}$$

↳ Baire 1 functions (Baire-0 ≡ continuous)

2) If $f,g:\Omega\to\overline{\mathbb{R}}$, where $(\Omega,\mathcal{A})$ is ~~an~~ measurable space and $f_n:\Omega\to\overline{\mathbb{R}}$, $n\geq 1$, which are all measurable.

i) show that $\{f=g\}\in\mathcal{A}$ (is measurable)

ii) $\{\exists \lim f_n\}\equiv\{w\in\Omega: \exists \lim f_n(w)\}\in\mathcal{A}$.

# Solution

$\rightarrow$ $f,g$ τυχαίες είναι το ίδιο $\rightarrow$ $g<f$.

i) $\{f=g\} = \left(\{f<g\} \cup \{f>g\}\right)^c$.

It suffices to show that $\{f<g\} \in \mathcal{A}$.

$$\omega \in \{f<g\} \Longleftrightarrow f(\omega) < g(\omega) \Longleftrightarrow \exists q \in \underbrace{\mathbb{Q}}_{\text{countable}} : f(\omega) < q \text{ and } q < g(\omega).$$

So we have expressed $\{f<g\}$, as

$$\{f<g\} = \bigcup_{q \in \mathbb{Q}} \left(\underset{\mathcal{A}}{\{f<q\}} \cap \underset{\mathcal{A}}{\{q<g\}}\right) \in \mathcal{A} \text{ (Countable union of } \mathcal{A}\text{-sets} \rightarrow \text{measurable)}$$

ii) $\{\exists \lim f_n\} = \{\underbrace{\liminf f_n}_{\text{meas.}} = \underbrace{\limsup f_n}_{\text{measurable}}\} \in \mathcal{A}$ (by i))

## σ-algebra generated by functions

Let $f: \Omega \to \overline{\mathbb{R}}$. The σ-algebra

$$\sigma(f) := f^{-1}(\mathcal{B}(\overline{\mathbb{R}})) = \{f^{-1}(B) : B \in \mathcal{B}(\overline{\mathbb{R}})\}$$

is called the σ-algebra generated by $f$ (παραγόμενη σ-άλγεβρα της $f$)

**Reminder:** $f$ is directly $\sigma(f)$-measurable and $\sigma(f)$ is the smallest

$\sigma$-algebra with respect to which $f$ is $\mathcal{A}$-measurable

$$\underbrace{f^{-1}(\mathcal{B}(\overline{\mathbb{R}}))}_{\sigma(f)} \subset \mathcal{A}$$

Θα πάρει τιμή $a_i$ όταν είναι στο $A_i$

<u>Example</u>: If $f = \sum_{i \in I} a_i \mathbb{1}_{A_i}$, $I$ countable and $f$ is in canonical form

then:

$$\sigma(f) = \sigma(\mathcal{C}), \text{ where } \mathcal{C} = \{A_i, i \in I\}, \ A_i = f^{-1}(\{a_i\}), \ i \in I.$$

Indeed, firstly we'll show that $\sigma(\mathcal{C}) \subset \sigma(f)$.

Since $A_i = f^{-1}(\underset{\mathcal{B}(\overline{\mathbb{R}})}{\underset{\cap}{\{a_i\}}}) \in \sigma(f)$, $\forall i \in I$

$$\rightarrow \mathcal{C} \subset \underset{\sigma\text{-algebra}}{\underset{\downarrow}{\sigma(f)}} \Rightarrow \sigma(\mathcal{C}) \subset \sigma(f). \quad (1)$$

Now, we want to show that $\underset{f^{-1}(\mathcal{B}(\overline{\mathbb{R}}))}{\underset{\parallel}{\sigma(f)}} \subset \sigma(\mathcal{C})$.

If $B \in \mathcal{B}(\overline{\mathbb{R}})$, then set

$$I_B = \{i \in I : a_i \in B\}.$$

Then,

$$B = \{a_i\}_{i \in I_B} \cup B_0, \text{ where } B_0 = B \setminus \{a_i\}_{i \in I_B}$$

and then

$$\overset{\mathcal{E}}{\Downarrow}$$

$$f^{-1}(B)=\underset{\substack{\text{countable}\\ \cup\{a_i\}}}{f^{-1}(\{a_i\}_{i\in I_B})}\cup\underbrace{f^{-1}(B_0)}_{\emptyset}=\bigcup_{i\in I_B}f^{-1}(\{a_i\})=\underbrace{\bigcup_{i\in I_B}A_i}_{\text{countable}}\in\sigma(\mathcal{E})$$

Therefore $\sigma(f)\subset\sigma(\mathcal{E})$. (2).

countable

→ it is a partition of $\Omega$

By (1) + (2), we have $\sigma(f)=\sigma(\mathcal{E})$

(χαρακτηρισμός σ-αλγ που παράγονται από διαμέριση αριθ)

## Applications

1) i) $f(\omega)=c$, $\forall\omega\in\Omega$

ii) $f=1_A$, $\emptyset\subsetneq A\subsetneq\Omega$

### Solution

i) $\sigma(f)=\sigma(f^{-1}(\{c\}))=\sigma(\{\Omega\})=\{\emptyset,\Omega\}$ trivial σ-algebra

↕

χαρακτηριστική σταθερών συναρτ.

~~So~~ in any σ-algebra, a constant function is always measurable

ii) $\sigma(f)=\sigma(\{f^{-1}(\{0\}),f^{-1}(\{1\})\})=\sigma(\{A,A^c\})=\{\emptyset,A,A^c,\Omega\}$

2) Let $\Omega$ be the sample space corresponding to a sequence of coin tosses with ~~0 ↔ heads~~ $0\longleftrightarrow K$ (heads) and $1\longleftrightarrow\Gamma$ (tails)

Let also $X_n:\Omega\to\mathbb{R}$, where $X_n(\omega)=\omega_n$ (projection to the n-th coordinate) and $w = (w_1, w_2, \dots, w_n, \dots) \in \{0,1\}^{\mathbb{N}^*}$.

Actually $X_n(w)$ corresponds to the outcome of the $n$-th tossing.

Describe $\sigma(X_n)$.

Since $X_n \in \{0,1\}$, we have that $\sigma(X_n) = \sigma(\{X_n^{-1}(0), X_n^{-1}(1)\})$. Set

$$A_{n,0} = \{0,1\}^{n-1} \times \{0\} \times \{0,1\}^{\mathbb{N}^* \setminus \{1,\dots,n\}}$$

$$A_{n,1} = \{0,1\}^{n-1} \times \{1\} \times \{0,1\}^{\mathbb{N}^* \setminus \{1,\dots,n\}}$$

So we have

$$\sigma(X_n) = \{\emptyset, A_{n,0}, \underbrace{A_{n,1}}_{A_{n,0}^c}, \Omega\}$$

Definition: Let $\Omega$ be a set and $(f_i)_{i\in I}$ be a family of functions $f_i : \Omega \to \overline{\mathbb{R}}$.

The $\sigma$-algebra $\sigma(\{f_i : i \in I\})$

$$\sigma(\{f_i : i\in I\}) = \sigma\Big(\bigcup_{i\in I} \sigma(f_i)\Big)$$

is called the generated $\sigma$-algebra from the family $\{f_i\}_{i\in I}$. It is the smallest $\sigma$-algebra with respect to which all $f_i$ are measurable.

If $I = \{1, 2, \dots, n\}$, we write $\sigma(f_1, f_2, \dots, f_n)$

Exercise: Characterize in the 2nd Application the $\sigma$-algebra:

$\sigma(X_1, X_2, \dots, X_n)$ (κάνω προβολές σε 1 συντεταγμένη, πώς θα εμφανίζονται στην σ-αλγ).

<u>Exercise</u>: Characterize $\sigma(f)$, where $f : \mathbb{R} \to \mathbb{R}$ is

i) $f(x) = x^3$

ii) $f(x) = x^2$.

## <u>Solution</u>

i) $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R})) = f^{-1}(\sigma(\underbrace{\mathcal{T}}_{\substack{\text{usual topology} \\ \text{of } \mathbb{R}}})) = \sigma(f^{-1}(\mathcal{T}))$.

We have that $f(x) = x^3$ is a homeomorphism (1-1 + onto + $f$ cont + $f^{-1}$ cont). This means that $f^{-1}(\mathcal{T}) = \mathcal{T}$, so we have that:

$$\sigma(f) = \sigma(\mathcal{T}) = \mathcal{B}(\mathbb{R})$$

ii) Take $B \in \mathcal{B}(\mathbb{R})$. We have that $f(x) \geq 0$. $B$ can be written as:

$$B = B_+ \cup B_-, \text{ where } B_+ = B \cap [0, +\infty), \; B_- = B \cap (-\infty, 0).$$

So

$$f^{-1}(B) = f^{-1}(B_+) \cup \underbrace{f^{-1}(B_-)}_{\emptyset} = f^{-1}(B_+).$$

$$\mathcal{B}([0, +\infty)) = (\mathcal{B}(\mathbb{R}))_{[0, +\infty)}$$

e.g. $f^{-1}(\{4\}) = \{-2, 2\}$

So

$\sqrt{\ }: [0,+\infty) \to [0,+\infty)$ homeomorphism

$$f^{-1}(B_+) = \underbrace{\sqrt{B_+}}_{\text{symbol}} \cup (-\sqrt{B_+}) = \pm\sqrt{B_+} = \pm \underbrace{B}_{\text{arbitrary}}, \quad B \in \mathcal{B}_{[0,+\infty)}$$

$\{\sqrt{b} : b \in B_+\}$

$f(x) = x^2$ is a homeomorphism from $[0,+\infty) \to [0,+\infty)$

So

$$\sigma(f) = \{\pm B : B \in \mathcal{B}_{[0,+\infty)}\}.$$

e.g. $[-2,2] \in \sigma(f)$

$[-3,2] \notin \sigma(f)$