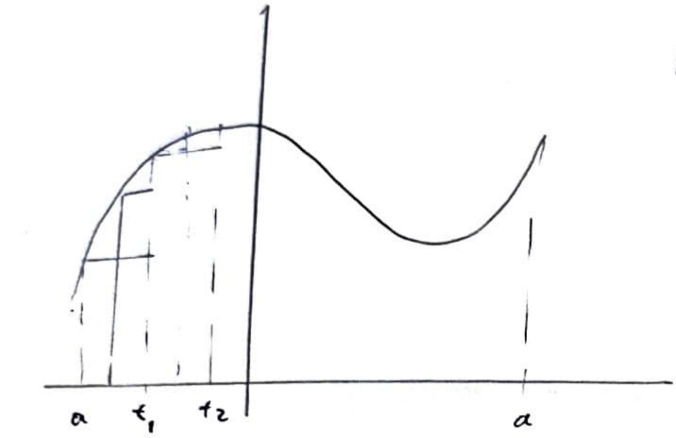
# Probabilities II / Zrovezas

## From Riemann to Lebesgue Integral




Let $f : [a,b] \to \mathbb{R}$, $0 \le f \le M$

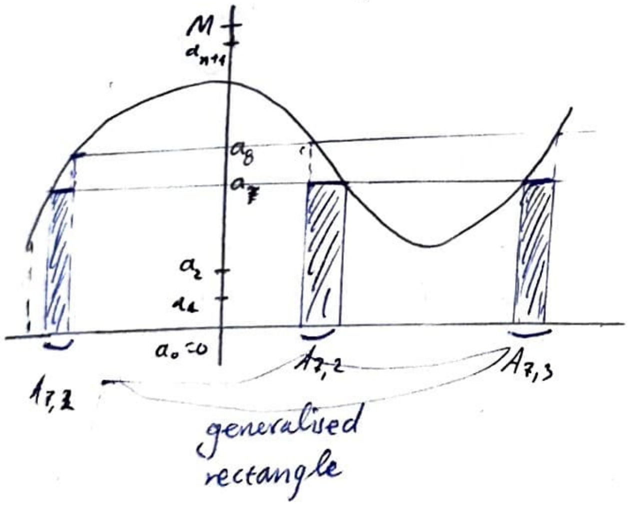
$$\mathcal{P} = \{a < t_0 < t_1 < \dots < t_n < t_{n+1} = b\}$$

$$m_i = \inf\{f(x) : x \in [t_i, t_{i+1}]\}, \quad i = 0, 1, \dots, n$$

For a large and a thin partition

$$\int_a^b f(x)\,dx \cong \sum_{i=0}^{n} m_i \underbrace{(t_{i+1} - t_i)}_{\text{length of the }(i+1)\text{-rectangle}}$$

For the same function




$$\int_a^b f(x)\,dx \cong \sum_{i=0}^{n} a_i \underbrace{\lambda\left(f^{-1}([a_i, a_{i+1}])\right)}_{\text{generalised rectangle}}$$

Lebesgue measure

all the points which $f$ in $[a_i, a_{i+1}]$ took their value.

$$A_7 = f^{-1}([a_i, a_{i+1}]) =$$

$$= A_{7,1} \cup A_{7,2} \cup A_{7,3}$$

measure length of all set of points which we have entered this zone

Lebesgue's thought of integrating a function

Whenever function bounded and has Riemann integral, ints are same
We'll see cases where $\nexists$ Riemann's, but $\exists$ Lebesgue's, bcs Riemann needs continuity (depends strongly on it).

The existense of the Riemann integral, necessitates a "sufficiently" smooth $f$.

<u>Example</u> (The Dirichlet function).

$f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$.

$f$ is not Riemann-integrable.

$\sup\{f(x) : t_i \leq x \leq t_{i+1}\}$

$$U(f,P) = \sum_{i=0}^{n} M_i (t_{i+1} - t_i),$$

$$L(f,P) = \sum_{i=0}^{n} m_i (t_{i+1} - t_i).$$

$$\underline{\int_a^b} f(x)dx = \sup_P L(f,P) = 0$$

$$\overline{\int_a^b} f(x)dx = \inf_P U(f,P) = 1$$

Since these values differ, the R-integral of $f$ does not exist $\Rightarrow$ $f$ is not R-integrable.

<u>$f$ is discontinuous everywhere</u>

However, if

$$Int(f) = 1 \cdot \lambda(f^{-1}(\{1\})) + 0\lambda(f^{-1}(\{0\})) = 1 \cdot \lambda(\mathbb{Q} \cap [0,1]) = 0$$

$$\lambda(\mathbb{Q}) = 0$$

This function behaves more close to zero, except for a set that has Lebesg measure zero.

The sets which have L measure $=0$, are really important, those sets are dominated.

Now, if

$$g(x) = \mathbb{1}_{(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]} \Rightarrow Int(g)=1$$

<u>Definition</u>: Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that ~~the~~ a property "P" holds $\mu$- almost everywhere $\mu$-a.e ($\mu$-σ.π. σχεδόν παντού), $\forall x \in X$ if

negation

if $\bar{P} := \{x \in X : P(x) \text{ is false}\} \subset A \in \mathcal{A}$ with $\mu(A)=0$

άρνηση P

($\bar{P}$ is $\mu$-null set)

<u>Application</u>

Let $f, g$ be $\mathcal{A}$-measurable. We say that $f \overset{a.e}{\underset{(\mu\text{-}a.e)}{=}} g$ if $\mu(f \neq g) = 0$

(αν $f, g$ μετρήσιμες $f \neq g$ θα είναι σύνολο μετρήσιμο)

For random variables $X, Y$, we write

$X \overset{a.s}{\underset{σ.β}{=}} Y$ (almost surely) (σχεδόν βεβαίως) υπονοείται ∃ μέτρο πιθ εδώ

$\mathbb{P}$ - a.s. ($\mathbb{P}$ - σ.β.)

∃ probability measure

$\leftrightarrow \mathbb{P}[X=Y]=1$

same probability measure.

e.g. i) $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$

$f \overset{a.e.}{=} 0$ στα ολοκλήρ θα έχει συμπεριφορά μηδενικής συν.

ii) $f = \mathbb{1}_{\mathbb{Q}} \to f \overset{a.e}{=} 0$

iii) $g = \mathbb{1}_{(\mathbb{R} \setminus \mathbb{Q}) \cap [0,1]} \to g \overset{a.e}{=} 1.$

**Definition** (Formal Definition of Lebesgue Integral)

3 Steps

S1 : simple, $\geq 0$, measurable $f$ (1)

S2 : $\geq 0$, measurable $f$ (2)

S3 : measurable $f$ (3)

Take a measure space $(X, \mathcal{A}, \mu)$ and $f : X \to \overline{\mathbb{R}}$.

S1 : $f$ as in (1) in canonical form $\left(f = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i},\ A_i = f^{-1}(\{a_i\})\right)$

multiplies with length of sets
th where $f$ takes these values $a_i$

Define

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$$

S2 : $f$ as in (2)

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu : s \text{ as in } (1), \text{ with } 0 \leq s \leq f \right\}$$

Comments :

(1) $\int f \, d\mu$ in S2 extends that of S1

(2) if we take the sequence of simple, measurable, $\geq 0$ functions $(f_n)\uparrow$ as in the Proposition

with $f_n \xrightarrow{pw} f$, then

$$\int f \, d\mu = \lim_n \int f_n \, d\mu \qquad \text{(sometimes given as the definition)}$$

(S3) $f$ as in (3), so only measurable

$$f = f^+ - f^- \quad \text{and} \quad \int f \, d\mu = \int f^+ d\mu - \int f^- d\mu,$$

whenever this is well defined.

$$\left(\int f^+ d\mu < +\infty \text{ or } \int f^- d\mu < +\infty\right)$$

<u>Comments</u>

(1) $\int f \, d\mu \in \overline{\mathbb{R}}$, when it is well defined

(2) If both are finite $\left(\int f^+ d\mu, \int f^- d\mu < +\infty\right)$ then $\int f \, d\mu \in \mathbb{R}$ and we say that $f$ is L-integrable (with respect to $\mu$)

(3) If $f = \sum_{i=1}^{n} a_i 1_{A_i}$ or $f = \sum_{n=1}^{\infty} a_n 1_{A_n}$, with $a_n \geq 0$, then always (not necessarily in canonical form)

$$\int f \, d\mu = \sum_{i=1}^{n} a_i \mu(A_i) \quad \text{or} \quad \int f \, d\mu = \sum_{n=1}^{+\infty} a_n \mu(A_n)$$

respectively.

## Application to Random Variables

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \overline{\mathbb{R}}$ a random variable.

(i) $\int X^+ dP$, $\int X^- dP < +\infty$, then $X$ is L-integrable with respect to P, but we say that its expectation exists / or it has expectation and $E(X) := \int X dP = \int X^+ dP - \int X^- dP$ → αναμενόμενη τιμή

(ii) $E(X^+) = +\infty$, $E(X^-) < +\infty$. Then, $E(X) = +\infty$

(iii) $E(X^+) < +\infty$, $E(X^-) = +\infty$. Then, $E(X) = -\infty$

(iv) $E(X^+) = +\infty$, $E(X^-) = +\infty$, Then $E(X)$ <u>is not defined</u> Cauchy random variable.

<u>Examples</u>

1) counting measure (αριθμητικό μέτρο)

$(X, \mathcal{A}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$, $\nu(A) = |A|$

if $f : \mathbb{N} \to [0, +\infty]$, the $Int(f) = \int f d\nu = \sum_{n \geq 0} f(n)$

$a \equiv (a_n)_{n \in \mathbb{N}}$ : $\int a \, d\nu = \sum_{n=0}^{+\infty} a_n$

<u>Proof</u>

$f = \sum_{n \geq 0} f(n) \cdot \mathbb{1}_{\{n\}}$

$$\int f\,dv = \sup\left\{ \int s\,dv : s \text{ is simple, measurable, } 0 \le s \le f \right\}$$

function satisfying ↑ will be called $\mathcal{S}$

Let $s \in \mathcal{S}$. Then, obviously

$$s = \sum_{n \in I} s(n) \mathbb{1}_{\{n\}} .$$

↳ finite $\subset \mathbb{N}$

$$\int s\,dv = \sum_{n \in I} s(n) \underbrace{v(\{n\})}_{= 1} \overset{s \in \mathcal{S}}{\le} \sum_{n \in I} f(n) \overset{f \text{ non-}}{\underset{\text{negative}}{\le}} \sum_{n \ge 0} f(n)$$

↓

it's upper bound of integral of $f$:

So, $\int f\,dv \le \sum_{n \ge 0} f(n)$ (1)

Take $s_m = \sum_{n=0}^{m} f(n) \mathbb{1}_{\{n\}}$, $s_m \ge 0$, simple and measurable, $0 \le s_m \le f$.

So $s_m \in \mathcal{S}$ $\forall m \ge 1$,

$$\int s_m\,dv = \sum_{n=0}^{m} f(n).$$

Obviously,

$$\sup_m \left\{ \int s_m\,dv \right\} \le \int f\,dv.$$

But,

$$\sup_m \int s_m\,dv = \sup_m \sum_{n=0}^{m} f(n) = \sum_{n \ge 0} f(n) \Rightarrow$$

$\sum_n f(n) \le \int f \, d\nu$ (2).

By (1)+(2) we have equality.

<u>Remark</u>: The same holds true for $f$: $\sum_n |f(n)| < +\infty$ (absolutely convergent sequences). So,

$$\int f \, d\nu = \sum_n f(n)$$

<u>Lebesgue measure on bounded intervals</u>

<u>Theorem</u>: If $f : [a,b] \to \mathbb{R}$ is a bounded (φραγμένη) function and call $D(f)$ the set of discontinuity points of $f$, then we have that

(i) $f$ is Riemann-integrable $\leftrightarrow$ $\lambda(\underbrace{D(f)}) = 0$ $\leftrightarrow$
$F_\sigma$ set (countable union of closed sets)

$\leftrightarrow$ $f$ is continuous $\lambda$-almost everywhere

(ii) $f$ is Riemann-integrable $\Rightarrow$ $f$ L-integrable and

$$\int_{[a,b]} f \, d\lambda = \int_a^b f(x)\,dx$$

$\lambda(D_f) = 0$ ✓
$> 0$ ✗

<u>Remark</u>: $f$ is continuous $\lambda$-almost everywhere $\not\Rightarrow$ $\not\Leftarrow$ $f = g$ $\lambda$-a.e with $g$ continuous

[~~modify f in a λ-null set~~ of discontinuity points "excluding the $\lambda$-null set, $f$ becomes continuous on the restriction"]

[modify $f$ in a $\lambda$-null set to become continuous]

Take $f$ to be: $\qquad$ $f$ is discontinuous everywhere

$$f = \mathbb{1}_{\mathbb{Q}\cap[0,1]} . \qquad \qquad \lambda(D_f) = 1$$

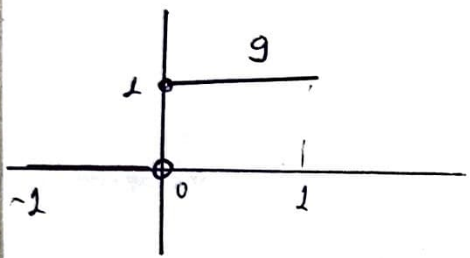
$\Updownarrow$

This $f$ is everywhere discontinuous.

$$f \overset{\lambda\text{-a.e}}{=} 0 \ , \ \lambda(\mathbb{Q}\cap[0,1]) = 0$$

bcs $\lambda([f \neq 0]) = 0$

modification: $f(q) = 0 , q \in \mathbb{Q}$




$$D_g = \{0\} \mid \lambda(D_g) = 0$$

↳ is continuous $\lambda$-a.e.

<u>However</u>, $g \neq$ continuum function on $[-1,1]$ $\lambda$-a.e.

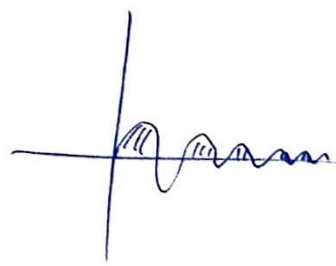
$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \lim_{M\to\infty} \int_0^{M} \frac{\sin x}{x}\, dx$$

δε συγκλίνει απόλυτα ολοκλήρωμα

συγκλίνει κατά συνθήκη

↳ <u>This is not L-integrable</u>:



positive and negative parts have $+\infty$

Είναι αν συγκλ απλά, όχι αναδιάταξη και να καν θα βρω το ίδιο το πάνω ολοκλήρωμα όπως σαν αυτό. Τα μαθηματικά έχουν συγκεκριμένο τρόπο να τα προσεγγίζω

Τα L-int δεν είναι γνήσια επέκταση όλων των ολοκληρωμάτων (για γενικευμένα ολοκληρώματα δεν ισχύει).