10/11/2024 Lecture 12

Probabilities II / Trevezas

Properties of Lebesgue Integral

Proposition: Let $f, g : X \to \bar{\mathbb{R}}$ measurable with well-defined integrals, then:

(1) $\int \lambda f \, d\mu = \lambda \int f \, d\mu \quad \forall \lambda \in \mathbb{R}$

(2) $\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu$, whenever the operations are well defined.

(3) $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$

(4) $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$

Application for random variables

If $E(X), E(Y)$ are well defined

(1) $E(\lambda X) = \lambda E(X)$

(2) $E(X+Y) = E(X) + E(Y)$

(3) $X \leq Y \Rightarrow E(X) \leq E(Y)$

(4) $|E(X)| \leq E(|X|)$

Classical Proof technique in measure theory (Τυπική μηχανή)

We want to prove a property for measurable (eventually extended) functions f successively, we prove this property

(S1) for f simple, ≥ 0, measurable (sometimes first only for indicators and then simple)

(S2) for $f \geq 0$, measurable

(S3) for f measurable

Reminder:

$|f| = f^+ + f^-$, $\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu$

f L-integrable ~~w~~ with respect to $\mu \Leftrightarrow \int f^+ d\mu, \int f^- d\mu < +\infty \Leftrightarrow$

$\Leftrightarrow \int |f| d\mu < +\infty$

The expectation of X exists $\Leftrightarrow E|X| < +\infty$

Lebesgue Integral on measurable subsets

Let $(X, \mathcal{A}, \mu)$ be a measurable space and $A \in \mathcal{A}$. We define

$$\int_A f d\mu := \int_X f 1_A d\mu \quad \text{(if it is well defined)}$$

Similarly, for a random variable X on $(\Omega, \mathcal{A}, P)$

$E(X; A) := E(X 1_A)$ [its expectation on A]

Comments

(1) $\displaystyle\int_X f\,d\mu = \int_X f \overset{1}{\overset{\|}{\mathbb{1}_X}}\,d\mu = \int_X f\,d\mu$ συμπίπτει καινούργιος ορισμός με ολοκλ. που είχαμε ορίσει πριν σε όλο το χώρο

$E(X;\Omega) = E(X)$

(2) If $f \geq 0$ and $A, B \in \mathcal{A}$ with $A \subset B$, then:

$$\int_A f\,d\mu \leq \int_B f\,d\mu \quad \Big(\int \underbrace{f \mathbb{1}_A}_{\leq f \mathbb{1}_B}\,d\mu \leq \int f \mathbb{1}_B\,d\mu \Big)$$

Similarly: $E(X;A) \leq E(X;B)$

(3) If $A, B \in \overset{\text{μετρήσιμα}}{\mathcal{A}}$ with $AB = \emptyset$ and integrals are well defined

$$\int_{A\cup B} f\,d\mu = \int f \mathbb{1}_{A\cup B}\,d\mu \overset{AB=\emptyset}{=} \int f(\mathbb{1}_A + \mathbb{1}_B)\,d\mu \overset{\text{prop}}{=} \int f \mathbb{1}_A\,d\mu + \int f \mathbb{1}_B\,d\mu =$$

$$= \int_A f\,d\mu + \int_B f\,d\mu$$

<u>Lemma</u>: If $(X, \mathcal{A}, \mu)$ is a measurable space and $f: X \to \overline{\mathbb{R}}$ measurable, $A \in \mathcal{A}: \mu(A) = 0$. Then

$$\int_A f\,d\mu = 0$$

<u>Proof</u>

Let $A \in \mathcal{A}: \mu(A) = 0$.

We will show that $\int_A f d\mu = 0$.

Step 1: f simple, ≥ 0, measurable in canonical form

$$f = \sum_{i=1}^{m} a_i 1_{A_i}$$

$$\int_A f d\mu = \int \left(\sum_{i=1}^{m} a_i 1_{A_i}\right) 1_A d\mu = \int \sum_{i=1}^{n} a_i 1_{A_i} 1_A d\mu = \int \sum_{i=1}^{n} a_i 1_{A_i A} d\mu =$$

$$= \sum a_i \mu(A_i A) \overset{A_i A \subset A}{\underset{\mu(A)=0}{=\!=}} \sum_{i=1}^{n} a_i \cdot 0 = 0$$

Step 2: $f \geq 0$, measurable

$$\int_A f d\mu = \int f 1_A d\mu = \sup \{ \underbrace{\int s d\mu : s \text{ simple}, \geq 0, \text{measurable}, 0 \leq s \leq f 1_A}_{S} \}$$

Let $s \in S$.

$$\int s d\mu = \int_A s d\mu + \underbrace{\int_{A^c} s d\mu}_{=0} = \int_A s d\mu \overset{\mu(A)=0}{\underset{\text{STEP 1}}{=\!=}} 0 \Rightarrow \int_A f d\mu = 0.$$

$x \in A^c$ $s(x)=0$ $(0 \leq s \leq f 1_A)$

Step 3: f measurable

$f = f^+ - f^-$

$$\int_A f^+ d\mu \overset{SZ}{\underset{\mu(A)=0}{=}} \int_A f^- d\mu = 0 \Rightarrow \int_A f\, d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = 0.$$

<u>Proposition</u>: Let $(X, \mathcal{A}, \mu)$ a measurable space and $f, g : X \to [0, +\infty]$ measurable functions. Then

(i) $\int f\, d\mu = 0 \iff f = 0$, μ-a.e (almost everywhere)

(ii) $f = g$ μ-a.e. $\Rightarrow \int f\, d\mu = \int g\, d\mu$ [also holds for $f, g : X \to \overline{\mathbb{R}}$]

(iii) $\int f\, d\mu < +\infty \Rightarrow f < +\infty$, μ-a.e.

<u>Proof</u>

(ii)
$$\int f\, d\mu = \int_{\{f=g\}} f\, d\mu + \overbrace{\int_{\{f\neq g\}} f\, d\mu}^{\overset{0}{\shortparallel}} = \int_{\{f=g\}} f\, d\mu = \int_{\{f=g\}} g\, d\mu = \int_{\{f=g\}} g\, d\mu + \overbrace{\int_{\{f\neq g\}} g\, d\mu}^{\overset{0}{\shortparallel}} = \int g\, d\mu$$

Due to the fact that $f = g$ μ-a.e, $\mu(\{f \neq g\}) = 0 \underset{\text{lemma}}{\overset{\text{Prev}}{\Rightarrow}} \int_{\{f\neq g\}} f\, d\mu = 0$

Έχοντας ότι $\{f=g\}$ και $\{f\neq g\}$ είναι μετρήσιμα σύνολα

(i) "$\Leftarrow$" $f = 0$ μ-a.e. $\Rightarrow \int f\, d\mu = 0$ ✓ (from ii)

"$\Rightarrow$" Let $a > 0$. If $\mu(f > a) = 0$ $\forall a > 0 \underset{\substack{a = 1/n \\ \forall n \geq 1}}{\Rightarrow} \mu(f > 1/n) = 0$ $\forall n \geq 1$.

So,

$$\mu(f > 0) = \mu\Big(\bigcup_{n=1}^{\infty} \underbrace{\{f > 1/n\}}_{\nearrow}\Big) = \lim_n \underbrace{\mu(f > 1/n)}_{\overset{\shortparallel}{0}}$$

Consequently

$$\mu(f \neq 0) \underset{f \geq 0}{=} \mu(f > 0) \overset{\text{by Assumption}}{=} 0 \Rightarrow f = 0 \quad \mu\text{-a.e.}$$

Αρκεί νδο για τυχαίο a.

Indeed,

$$0 = \int f\,d\mu \underset{f \geq 0}{\geq} \int_{\{f>a\}} f\,d\mu \geq \int_{\{f>a\}} a\,d\mu = a\,\mu(f>a) \geq 0$$

$$\rightarrow \mu(f>a) = 0$$

iii) $$\int f\,d\mu = \int_{\{f<+\infty\}} f\,d\mu + \int_{\{f=+\infty\}} f\,d\mu = \underbrace{\int_{\{f<+\infty\}} f\,d\mu}_{\substack{<+\infty \\ \text{by assumption}}} + (+\infty)\cdot\mu(\{f=+\infty\}).$$

If $\mu(\{f=+\infty\}) > 0$, then $\int f\,d\mu = +\infty$.

$$\Rightarrow \mu(\{f=+\infty\}) = 0 \Rightarrow f < +\infty \quad \mu\text{-a.e.}$$

Applications to Random Variables

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X, Y : \Omega \to [0, +\infty]$. Then

i) $E(X) = 0 \iff X \underset{\sigma.\beta.}{\overset{a.s}{=}} 0$ (almost surely/with probability 1)

Μέση τιμή τυχαίων μεταβλητών σαν ολοκληρώματα Lebesgue ως προς μέτρο P

ii) $X \overset{a.s}{=} Y \Rightarrow E(X) = E(Y)$ (it also holds for $X, Y \in \bar{R}$)

iii) $E(X) < +\infty \Rightarrow X \overset{a.s.}{<} +\infty$

<u>Definition</u>: $X \in \bar{R}$ with $E|X| < +\infty$ $(E(X) \in R)$. Define $V(X) =$
$= E(X - E(X))^2$, the variance of X.

<u>Properties</u>

(i) $V(X) \in [0, +\infty]$

(ii) $V(X) < +\infty \Leftrightarrow E(X^2) < +\infty$

(iii) $V(\alpha X + \beta) = \alpha^2 V(X)$, $\forall \alpha, \beta \in R$

(iv) $V(X) = 0 \leftrightarrow X \overset{a.s}{=} c$, for some $c \in R$

<u>Proof</u>

everything is immediate.

iv) $\underset{}{\overset{1}{\Rightarrow}}$ $V(X) = 0 \Rightarrow E\underbrace{(X-\mu)^2}_{\geq 0} = 0 \xrightarrow{\text{Propos. ii}} (X-\mu)^2 \underset{a.s.}{=} 0 \Rightarrow$

$\Rightarrow |X - \mu| \underset{a.s}{=} 0 \Rightarrow X \overset{a.s.}{=} \mu$, where μ constant $\in R$.

[$X \overset{a.s.}{=} Y \rightarrow g(X) \overset{a.s.}{=} g(Y)$

$P(X=Y)=1 \Rightarrow P(g(X) = g(Y)) = 1$

$\{X=Y\} \subset \{g(X) = g(Y)\}$]

Basic Inequalities

Proposition: Let $(\Omega, \mathcal{A}, P)$ be a probability space. Then, the following hold true:

i) Markov inequality:

If $X: \Omega \to [0, +\infty]$ a random variable and $a > 0$, then

$$P(X \geq a) \leq \frac{E(X)}{a}$$

(Generalized Markov inequality)

If $X: \Omega \to \overline{\mathbb{R}}$ a random variable and $h \geq 0$, as well as increasing then,:

$$h(a) > 0 \Rightarrow P(X \geq a) \leq \frac{E[h(X)]}{h(a)}$$

ii) Chebyshev inequality

If $X: \Omega \to \overline{\mathbb{R}}$ random variable with $E|X| < +\infty$, and any $a > 0$ we have that

$$P(|X - E(X)| \geq a) \leq \frac{V(X)}{a^2}$$

iii) Jensen inequality

Let $X: \Omega \to \overline{\mathbb{R}}$ random variable with $E(X) < +\infty$, $g: I \to \mathbb{R}$ a convex (κυρτή) function on an interval, $I \subset \mathbb{R}$ and $P(X \in I) = 1$, $E[|g(X)|] < +\infty$, then:

(concave ≡ κοίλη)

$$E[g(X)] \geq g(E(X))$$

Proof

i) $X \geq a \mathbb{1}_{\{X \geq a\}} \Rightarrow E(X) \geq a E(\mathbb{1}_{\{X \geq a\}}) = a \cdot P(X \geq a) \Rightarrow$

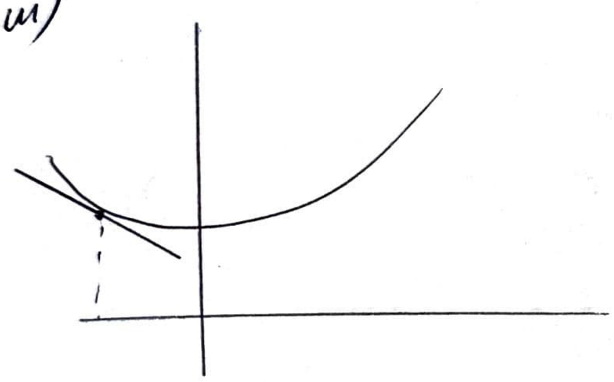
$\Rightarrow \boxed{P(X \geq a) \leq \frac{E(X)}{a}}$

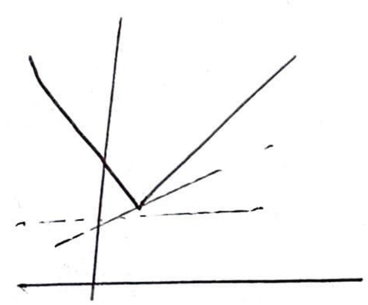
Generalized:

$$\{X \geq a\} \underset{h\uparrow}{\subset} \{h(X) \geq h(a)\} \Rightarrow P(\{X \geq a\}) \overset{h\uparrow}{\leq} P(h(X) \geq \underbrace{h(a)}_{>0}) \overset{\text{Markov}}{\leq} \frac{E[h(X)]}{h(a)}$$

ii) $P(|X - E(X)| \geq a) \underset{(=)}{\leq} P((X - E(X))^2 \geq a^2) \underset{\text{Markov}}{\leq} \frac{E[(X - E(X))^2]}{a^2} = \frac{V(X)}{a^2}$

iii)





convexity $\Rightarrow \forall x \in I^0 \rightarrow$ interior of the interval I, $\exists \lambda(x_0) \equiv \lambda_0$:

$g(x) \geq g(x_0) + \lambda_0 (x - x_0)$, $\forall x \in I$.

We want to show that $E[g(X)] \geq g[E(X)]$. For x_0 we choose $E(X)$ (with no loss of generality $E(X)$ not an endpoint of I). So, for :

$\omega : X(\omega) \in I$, we have $\exists \lambda_0 = \lambda(x_0)$:

$$g(X(\omega)) \geq g(E(X)) + \lambda_0 (X(\omega) - E(X)).$$

So with probability 1, (a.s.)

$$g(X) \geq g(E(X)) + \lambda_0 (X - E(X)) \Rightarrow$$

$$\Rightarrow E[g(X)] \geq g(E(X)) + \lambda_0 \underbrace{(E(X) - E(X))}_{0} \Rightarrow \boxed{E[g(X)] \geq g(E(X))}$$

* if E(X) is an endpoint, for example $I = [a,b]$ and $E(X) = a$,

then $X \overset{a.s.}{\geq} a \Rightarrow X - a \overset{a.s.}{\geq} 0 \Rightarrow E(X-a) = E(X) - a = 0 \Rightarrow X \overset{a.s.}{=} a$ and

$$E[g(X)] = g(E(X)) = g(a).$$