Probabilities II / Treterzas

Mid-Term Solutions

Question 1: $X = \{1,2,3,4,5\}$, $\mathcal{C} = \{\underset{A}{\{1,2\}}, \underset{B}{\{2,3\}}\}$. Find:

i) the generated Dynkin class $\delta(\mathcal{C})$.

ii) the generated σ-algebra $\sigma(\mathcal{C})$

Solution

i) By the properties of a Dynkin class, we have that

$$\mathcal{D} = \{\emptyset, X, \underset{A}{\{1,2\}}, \underset{B}{\{2,3\}}, \underset{A^c}{\{3,4,5\}}, \underset{B^c}{\{2,3\}}\} \subseteq \delta(\mathcal{C}).$$

So we ~~have~~ will prove that $\mathcal{D}$ is a Dynkin class and since $\mathcal{C} \subset \mathcal{D}$, we ~~will~~ have necessarily $\mathcal{D} = \delta(\mathcal{C})$

Dynkin class $\equiv$ $\mathcal{D}$-class : (a) $X \in \mathcal{D}$ (b) closed under completion

(c) closed under disjoint unions.

The only possibilities are $(\emptyset, X), (A, A^c), (B, B^c)$ other than the empty set) so (c) is satisfied and

$$\delta(\mathcal{C}) = \mathcal{D}.$$

ii) $\{1\} = A \setminus B$, $\{2\} = A \cap B$, $\{3\} = B \setminus A$: $\{4,5\} = (A \cup B)^c$

So

$$\mathcal{A} = \sigma(\{\{1\},\{2\},\{3\},\{4,5\}\}) \subset \sigma(\mathcal{C})$$

Obviously, $\mathcal{C} \subset \mathcal{A}$ and $\mathcal{A}$ is a σ-algebra, so

$$\sigma(\mathcal{C}) = \mathcal{A}$$

and in particular we ~~also~~ have that $\mathcal{A}$ includes all the unions of the collection $\{\{1\},\{2\},\{3\},\{4,5\}\}$ which is a partition of X.

(We have $2^4 = 16$ elements

<u>Question 2</u>: Let $(X,\mathcal{A})$ be a measurable space and (A_n), (B_n) two sequences of $\mathcal{A}$-measurable sets for which we assume that $A_n \to A$, $B_n \to B$. Show that:

i) $A, B \in \mathcal{A}$

ii) $A_n \cup B_n \to A \cup B$

<u>Solution</u>

i) $A_n \to A \Rightarrow A = \liminf A_n = \underbrace{\bigcup_{n\geq 1} \underbrace{\bigcap_{k\geq n} A_k}_{\in \mathcal{A}}}_{\in \mathcal{A}} \in \mathcal{A}$

σ-algebra is closed under countable unions and intersections

Similarly $B \in \mathcal{A}$.

ii) <u>a' solution</u>

We need to prove that $\liminf(A_n \cup B_n) = \limsup(A_n \cup B_n) = A \cup B$

$$A \cup B \xlongequal[B_n \to B]{A_n \to A} (\liminf A_n) \cup (\liminf B_n) \underset{B_n \subset A_n \cup B_n}{\overset{A_n \subset A_n \cup B_n}{\subset}} \liminf (A_n \cup B_n) \underset{\text{always}}{\subset}$$

$$\subset \limsup (A_n \cup B_n) \underset{(4)^*}{=\!=} \limsup A_n \cup \limsup B_n \xlongequal[B_n \to B]{A_n \to A} A \cup B \Rightarrow$$

$$\Rightarrow \liminf A_n \cup B_n = \limsup A_n \cup B_n = A \cup B \Leftrightarrow A_n \cup B_n \to A \cup B$$

$(4)^*$ $\omega \in \limsup (A_n \cup B_n) \Leftrightarrow \omega \in A_n \cup B_n$ for infinite n

$\Leftrightarrow \omega \in A_n$ for infinite n or $\omega \in B_n$ for infinite n

$\Leftrightarrow \omega \in \limsup A_n$ or $\omega \in \limsup B_n$

$\Leftrightarrow \omega \in \limsup A_n \cup \limsup B_n$

So $\limsup (A_n \cup B_n) \underset{\text{ral}}{\overset{\text{in gene}}{=\!=}} \limsup A_n \cup \limsup B_n$.

Attenzione! $\liminf (A_n \cup B_n) \underset{\uparrow \atop \text{not always}}{=\!=} \liminf A_n \cup \liminf B_n$.

So in particular, $\liminf A_n \cup \liminf B_n \overset{\text{could be}}{\subsetneq} \liminf A_n \cup B_n$

Take

$$A_n = \begin{cases} \emptyset, & n = 1, 3, 5, \dots \\ X, & n = 2, 4, 6, \dots \end{cases}, \quad B_n = \begin{cases} X, & n = 1, 3, 5, \dots \\ \emptyset, & n = 2, 4, 6, \dots \end{cases}$$

Obviously: $\liminf A_n = \liminf B_n = \emptyset$

But

$$\liminf (A_n \cup B_n) = \liminf (\{X, X, \ldots\}) = X$$

and strict inequality holds.

Be cautious also here:

$$\liminf (A_n \cup B_n) = \bigcup_{n \geq 1} \bigcap_{k \geq n} (A_k \cup B_k)$$

$$\bigcap_{k \geq n} (A_k \cup B_k) \overset{\text{in general}}{\neq} \Big(\bigcap_{k \geq n} A_k\Big) \cup \Big(\bigcap_{k \geq n} B_k\Big)$$

$$\bigcup_{k \geq n} (A_k \cap B_k) \overset{\text{in general}}{\neq} \Big(\bigcup_{k \geq n} A_k\Big) \cap \Big(\bigcup_{k \geq n} B_k\Big)$$

<u>b' solution</u>: $A_n \to A \iff \mathbb{1}_{A_n} \to \mathbb{1}_A$.

$$A_n \to A \iff \liminf A_n = \limsup A_n = A \iff$$

$$\iff \mathbb{1}_{\liminf A_n} = \mathbb{1}_{\limsup A_n} = \mathbb{1}_A . \quad (*)$$

In general,

$$\mathbb{1}_{\liminf A_n} = \liminf \mathbb{1}_{A_n} \quad \& \quad \mathbb{1}_{\limsup A_n} = \limsup \mathbb{1}_{A_n}$$

So, $(*)$

$$A_n \to A \Leftrightarrow \liminf \mathbb{1}_{A_n} = \limsup \mathbb{1}_{A_n} = \mathbb{1}_A \Longleftrightarrow \mathbb{1}_{A_n} \to \mathbb{1}_A.$$

$\mathbb{1}_{\liminf}(\omega) = 1 \Leftrightarrow \omega \in \liminf A_n \Leftrightarrow \omega \in A_n$ eventually for all n.

$$\Leftrightarrow \mathbb{1}_{A_n}(\omega) = 1, \text{ eventually for all } n$$

$$\Longleftrightarrow \liminf \mathbb{1}_{A_n}(\omega) = 1.$$

So,

$$\mathbb{1}_{\liminf A_n} = \liminf \mathbb{1}_{A_n}.$$

By changing, "eventually for all n" to "infinite n", we have:

$$\mathbb{1}_{\limsup A_n} = \limsup \mathbb{1}_{A_n}$$

We proved that

$$A_n \to A \Leftrightarrow \mathbb{1}_{A_n} \to \mathbb{1}_A$$

So

$$A_n \cup B_n \to A \cup B \Longleftrightarrow \mathbb{1}_{A_n \cup B_n} \to \mathbb{1}_{A \cup B}$$

and indeed

$$\mathbb{1}_{A_n \cup B_n} = \mathbb{1}_{A_n} + \mathbb{1}_{B_n} - \mathbb{1}_{A_n B_n} = \mathbb{1}_{A_n} + \mathbb{1}_{B_n} - \mathbb{1}_{A_n} \cdot \mathbb{1}_{B_n} \xrightarrow[\text{asumption}]{\text{by}} \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \mathbb{1}_B$$

$$= \mathbb{1}_{A \cup B}.$$

Question 3: On the probability space $((0,1), \mathcal{B}(0,1), \lambda)$ we have that $X = \mathbb{1}_{(0,1)\setminus\mathbb{Q}}$

i) Show that X is a random variable

ii) Examine if X is a continuous function with probability 1 (a.s.) or/and equal with a continuous function with probability 1.

iii) Examine if X is R-integrable or/and L-integrable and compute $E(X)$ and $V(X)$.

<u>Solution</u>

i) X is an indicator function, so

X is a random variable (measurable on the corresponding probability space) $\Leftrightarrow (0,1)\setminus\mathbb{Q} \in \mathcal{B}(0,1)$.

But $(0,1)\setminus\mathbb{Q} = (0,1)\cap(\underbrace{\mathbb{R}\setminus\mathbb{Q}}_{\in \mathcal{B}(\mathbb{R})}) \in \mathcal{B}(0,1)$

it is a complement of a countable set (which is Borel)

(inverse image $((-\infty, x])$ either empty or irrational....)

ii) $X: (0,1) \to \mathbb{R}$

$D_X = \{\omega \in (0,1) : X \text{ is discontinuous at } \omega\} = (0,1) \to X$ is ~~every~~

everywhere discontinuous

$$\lambda(D_X) = 1 \neq 0,$$

so it is not.

For the second question,

$$X \overset{a.s.}{=} 1 \text{ continuous}$$

since;

$$P(X \neq 1) = \lambda(\{X \neq 1\}) = \lambda(\{X = 0\}) = \lambda(\mathbb{Q} \cap (0,1)) \leq \lambda(\mathbb{Q}) = 0 \Rightarrow$$

$$\Rightarrow P(X \neq 1) = 0.$$

ii) $\lambda(D_X) = 1 \neq 0 \Rightarrow X$ is not R-integrable (R-integrable $\Leftrightarrow \lambda(D_f) = 0$)

$X \geq 0$ and $E(X) \leq 1$ since $X \leq 1$. Therefore $E(X) < +\infty \Rightarrow X$ is L-integrable.

$$E(X) = E[\mathbb{1}_{(0,1)\setminus\mathbb{Q}}] = \lambda((0,1)\setminus\underbrace{\mathbb{Q}}_{\text{measure } 0}) = \lambda((0,1)) = 1.$$

For the variance,

$$\underbrace{E(X^2) = E(X)}_{X^2 = X \equiv \text{indicator}} = 1 \Rightarrow V(X) = E(X^2) - E^2(X) = 1 - 1 = 0$$

($X \overset{a.s.}{=} 1$, so a degenerate random variable)

Question 4: Justify the following statements

i) if X is a random variable, then $E(X^2) \geq E^2(X)$

ii) the function $F(x) = \frac{1}{2}, x \in \mathbb{R}$, is a distribution function of a probability measure on $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$. To which measure is it associated with?

iii) on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, if $f(x) = e^x$, then $\sigma(f) = \mathcal{B}(\mathbb{R})$

Solution

i) Firstly, let us assume that $E|X| < +\infty$ $(E(X) = \mu \in \mathbb{R})$. In this case, if $E(X^2) = +\infty$, the inequality holds.

If $E(X^2) < +\infty$, then by Jensen's inequality we have that

$$E(X^2) = E(\underbrace{g}_{\text{convex}}(X)) \geq g(E(X)) = E^2(X).$$

But of course $V(X) = \underbrace{E(X-\mu)^2}_{\geq 0} = E(X^2) - E^2(X) \geq 0 \Rightarrow E(X^2) \geq E^2(X)$.

More generally, we have an inequality that if $\boxed{E(X^2) < +\infty \Rightarrow E|X| < +\infty}$

$$E|X| = E(|X| \mathbb{1}_{|X|<1}) + E(|X| \mathbb{1}_{|X| \geq 1})$$

$$\leq 1 + E(X^2 \cdot \mathbb{1}_{|X| \geq 1}) \leq 1 + E(X^2)$$

Indeed, if $E(X^2) < +\infty$, this implies $E|X| < +\infty$

ii) $F(x)=\frac{1}{2},\ \forall x\in\mathbb{R}$ $\qquad (F:\bar{\mathbb{R}} \ : F(x)=\mu([-\infty,x])$

i) $F\uparrow$ ✓

ii) F right continuous

iii) $0\le F(-\infty)\le F(+\infty)\le 1$

$\qquad\quad \frac{1}{2} \qquad\quad \frac{1}{2}$

What is the measure?

$$\mu(\{-\infty\}) = \lim_{x\to-\infty} F(x) = \frac{1}{2}$$

$$\mu(\{+\infty\}) = 1-\mu([-\infty,+\infty)) = 1-\lim_{x\to+\infty} F(x) = 1-\frac{1}{2}=\frac{1}{2}$$

So

$$\mu = \frac{1}{2}\delta_{-\infty} + \frac{1}{2}\delta_{+\infty}$$

iii) $f(x)=e^x$, $f:\mathbb{R}\to(0,+\infty)$, homeomorphism.

$y=e^x \Leftrightarrow x=\log y$ continuous

$$\sigma(f) = f^{-1}(\mathcal{B}(0,+\infty)) = f^{-1}(\sigma(\mathcal{T}_{|\cdot|})) = \sigma(f^{-1}(\mathcal{T}_{|\cdot|})) = \sigma(\mathcal{T}_{|\cdot|}) = \mathcal{B}(\mathbb{R})$$

<u>Quiz</u>

3) $X=\{1,2,3,4\}$. Examine if $\sigma(\mathcal{C})=\delta(\mathcal{C})$ Πρέπει να είναι π-σύστημα

○ $\{\{1,4\}\}$ $\quad$ ⊘ $\{\{1\},\{1,2\}\}$ $\quad$ ○ $\{\{1,2\},\{2,3\}\}$ $\quad$ ⊘ $\{\{1,2\},\{3,4\}\}$

4) which of the families generate Borel

☑ open ☑ bounded intervals ○ Lebesgue measurable ○ uncountable sets

⊗ ⊗

$\mathcal{B}(\mathbb{R}) = \sigma(\text{open sets})$

Borel $\subsetneq$ Lebesgue