Basic Limit Theorems

$\forall n \geq 1$ $\int f_n \, d\mu$, $\int f d\mu$ exist and $f_n \xrightarrow{p.w.} f$. Could we deduce that

$$\int f_n \, d\mu \longrightarrow \int f d\mu$$

So does $\int \lim f_n \, d\mu \overset{?}{=} \lim \int f_n d\mu$.

It is not the case in general! But under some different type of conditions this will be the case.

Counterexample

$\left((0,1), \mathcal{B}(0,1), \lambda\right)$ ↳ Lebesgue

Take

$$f_n = n \cdot \mathbb{1}_{(0,\frac{1}{n})} = \begin{cases} n, & 0 < w < \frac{1}{n} \\ 0, & \frac{1}{n} \leq w < 1 \end{cases}$$

$\int f_n \, d\lambda$ is defined $\forall n \geq 1$ $(f_n \geq 0)$.

In particular, we have $f_n \xrightarrow{p.w.} 0$ (Take $0 < w < 1$ and then $\exists n_0(w)$: $\frac{1}{n_0} \leq w$. So $\forall n \geq n_0(w)$: $\frac{1}{n} \leq \frac{1}{n_0} \leq w$ and $f_n(w) = 0$). But

$$1 = n \cdot \frac{1}{n} = n \cdot \lambda\left((0,\tfrac{1}{n})\right) = \int f_n \, d\lambda \not\longrightarrow \int \underset{f}{\underset{\parallel}{0}} \, d\lambda$$

Monotone Convergence Theorem (MCT) Θεώρημα Μονότονης Σύγκλισης

Let (f_n) an increasing sequence of measurable functions $f_n : X \to [0, +\infty]$

and set

$$f = \lim f_n \quad \text{(it always exists)}, \text{(eventually } +\infty)$$

Then

$$\int \underbrace{f}_{\lim f_n} d\mu = \lim_n \int f_n \, d\mu$$

(For proof see Koumoulis & Negrepontis Chapter 6).

Applications of Monotone Convergence Theorem

Beppo-Levi Theorem

Let (f_n) be a sequence of measurable functions with $f_n : X \to [0, +\infty]$

and set

$$f = \sum_{n \geq 1} f_n : X \to [0, +\infty].$$

Then,

$$\int f \, d\mu = \sum_n \int f_n \, d\mu \qquad \left(\int \left(\sum_n f_n \right) d\mu = \sum_n \int f_n \, d\mu \right)$$

Proof

Set

$$g_n = \sum_{k=1}^{n} f_k .$$

Obviously, $(g_n) \uparrow$ sequence of measurable functions. So, if

$$f = \lim_n g_n = \sum_{n=1}^{+\infty} f_n,$$

then by the MCT

$$\int f \, d\mu = \int \sum_{n=1}^{+\infty} f_n \, d\mu = \lim_n \int \underbrace{g_n}_{= \sum_{k=1}^{n} f_k} d\mu = \lim_n \sum_{k=1}^{n} \int f_k \, d\mu =$$

$$= \sum_n \int f_n \, d\mu \quad \text{(eventually } +\infty\text{, but always exists)}$$

Θεώρημα Μονότονης Σύγκλισης
Θεώρημα Κυριαρχημένης Σύγκλισης
Beppo-Levi
Λήμμα Fatou

Applications

Borel-Cantelli 1^{st} Lemma

Let $(\Omega, \mathcal{A}, P)$ be a probability space and a sequence (A_n) in $\mathcal{A}$. Then,

$$\sum_{n=1}^{+\infty} P(A_n) < +\infty \Longrightarrow P(\limsup A_n) = 0$$ [probability that infinite A_n will be realised]

Ερμηνεία

Proof

Set $X = \sum_n \mathbb{1}_{A_n}$.

$\omega \in \limsup A_n \Leftrightarrow \omega \in A_n$ for infinite n $\Leftrightarrow \mathbb{1}_{A_n}(\omega) = 1$ for infinite n

$\Leftrightarrow X(\omega) = +\infty$. characterize limsup as the event that X will be $+\infty$.

So,

$$\limsup A_n = \{X = +\infty\}.$$

By assumption,

$$\sum_n P(A_n) < +\infty \Rightarrow \sum_{n=1}^{+\infty} E(\mathbb{1}_{A_n}) < +\infty \overset{B\text{-}L}{\Leftrightarrow} E\Big(\underbrace{\sum_n \mathbb{1}_{A_n}}_{=X}\Big) < +\infty$$

(We know $\int \underset{\geq 0}{f}\, d\mu < +\infty \Rightarrow \mu(f = +\infty) = 0$)

Therefore,

$$E(X) < +\infty \Rightarrow P(X = +\infty) = 0$$

$$\Rightarrow P(\limsup A_n) = 0$$

reformulation: if $\sum_n P(A_n)$ converges fast:

Questions

(1) $P(\limsup A_n) = 0 \overset{?}{\Longrightarrow} \sum_{n=1}^{+\infty} P(A_n) < +\infty$

(2) $P(\limsup A_n) = 0 \overset{?}{\Longrightarrow} P(A_n) \overset{?}{\longrightarrow} 0$

(3) $P(A_n) \longrightarrow 0 \overset{?}{\Longrightarrow} P(\limsup A_n) = 0$

Q1: No, the converse of Borel-Cantelli 1st Lemma is not true

$((0,1), \mathcal{B}(0,1), \lambda)$

$A_n = (0, \frac{1}{n})$ and $A_n \downarrow \emptyset$.

Take

$$P(A_n) = \lambda((0, \tfrac{1}{n})) = \tfrac{1}{n} \longrightarrow 0$$

$$\sum_{n=1}^{+\infty} P(A_n) = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

So,

$$\underset{\lim A_n}{\underset{\shortparallel}{P(\limsup A_n)}} = P(\emptyset) = 0.$$

But $\sum_{n=1}^{+\infty} P(A_n) = +\infty$.

So, if $P(\limsup A_n) = 0$, could we at least deduce that $P(A_n) \to 0$

Q2 : Indeed it holds!

$$P(\limsup A_n) = 0 \Longrightarrow P(A_n) \longrightarrow 0.$$

Let us assume that $P(\limsup A_n) = 0$ and $P(A_n) \not\to 0$. So $\exists \varepsilon > 0$ and a sequence (n_k):

$$P(A_{n_k}) \geq \varepsilon.$$

Take this sequence $\{A_{n_1}, A_{n_2}, \ldots, A_{n_k}, \ldots\}$ and form

$$\limsup_k A_{n_k} = \bigcap_{k \geq 1} \bigcup_{m \geq k} A_{n_m}$$

Obviously

$$\limsup A_{n_k} \subset \limsup A_n \xRightarrow{P(\limsup A_n)=0} P(\limsup_k A_{n_k}) = 0.$$

We know that:

$$\limsup P(A_{n_k}) \leq P(\limsup A_{n_k})$$

$$\underbrace{\limsup P(A_{n_k}) \geq \varepsilon}_{\text{bound.}}$$ contradiction!

Q3) is not True. $((0,1), \mathcal{B}(0,1), \lambda)$ Take partitions of $(0,1)$

$A_1 \xrightarrow{\text{e.g.}} (0,1)$

$A_2\ A_3 \xrightarrow{\text{e.g.}} (0,1/2) \cup [1/2, 1)$

$A_4\ A_5\ A_6 \xrightarrow{\text{e.g.}} (0, 1/3) \cup [\frac{1}{3}, \frac{2}{3}) \cup [\frac{2}{3}, 1)$

$\vdots$

$A_{\frac{n(n-1)}{2}+1} \cdots A_{\frac{n(n+1)}{2}} \xrightarrow{\text{e.g.}} (0, \frac{1}{n}) \cup [\frac{1}{n}, \frac{2}{n}) \cup \dots \cup [\frac{n-1}{n}, 1)$

(A_n) is defined like this. Obviously,

$P(A_n) \to 0$ (e.g. $n_0 \geq 1$ there exists an n_1: $\forall n \geq n_1$

$P(A_n) \leq \frac{1}{n_0}$.

If we take $n_1 = n_0 \cdot \frac{n_0+1}{2}$, so here

$P(A_n) \leq \frac{1}{n_0}\ \forall n \geq n_1$

But $\limsup A_n = (0,1)$, because for any $w \in (0,1)$, we have:

$w \in A_n$ for infinite n (infinite lines and it appears in each line).

$$\frac{1}{n^*} \leq \varepsilon \qquad n \geq n_0(\varepsilon)$$

n^* γραμμή $\quad P(A_n)$ έχει πιθ $\frac{1}{n^*}$

Άρα n^* γραμμή και κάτω ισχύει.

Fatou lemma

Let (f_n) be a set of measurable functions with $f_n : X \to [0, +\infty]$, $n \geq 1$.

Then:

$$\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu$$

Proof

$$\liminf f_n = \sup_{n \geq 1} \underbrace{\inf_{k \geq n} f_k}_{g_n} \ , \quad (g_n) = \left(\inf_{k \geq n} f_k\right) \uparrow \quad g_n : X \to [0, +\infty] \text{ measurable}$$

$$= \lim_n g_n$$

So,

$$\int \liminf_n f_n \, d\mu = \int \lim_n g_n \, d\mu \overset{MCT}{=\!=} \lim_n \int g_n \, d\mu = \quad \text{I don't know if } \exists \lim_n \int f_n$$

$$\leq f_n$$

$$\overset{f_n \geq g_n}{=} \liminf_n \underset{\exists \liminf}{\int g_n \, d\mu} \leq \liminf_n \int f_n \, d\mu$$

Dominated Convergence Theorem (Θεώρημα Κυριαρχημένης Σύγκλισης)

Let (f_n) be a sequence of measurable functions with $f_n : X \to \overline{\mathbb{R}}$, $n \geq 1$, $f : X \to \overline{\mathbb{R}}$ such that $f = \lim_n f_n$ (we assume that $f_n \xrightarrow{p.w.} f$). If $\exists g : X \to [0, +\infty]$ measurable with $|f_n| \leq g$, $\forall n \geq 1$ and $\int g \, d\mu < +\infty$. Then:

i) $\int |f_n| d\mu, \int |f| d\mu < +\infty$

ii) $\lim_n \int |f_n - f| d\mu = 0$

iii) $\int f d\mu = \lim_n \int f_n d\mu$

(For proof see Chapter 6 of Koumoulis & Negrepontis)

$|a_n| \to 0 \Leftrightarrow a_n \to 0.$

ii) $\Rightarrow$ iii) $0 \leq \left| \int (f_n - f) d\mu \right| \leq \int |f_n - f| d\mu \xrightarrow{ii)} 0$

$\rightarrow \int (f_n - f) d\mu \to 0 \xRightarrow[\text{functions}]{\text{integr.}} \int f_n d\mu - \int f d\mu = 0 \Rightarrow$

$\Rightarrow \int f_n d\mu = \int f d\mu.$

Remarks for random variables

MCT: $X_n \xrightarrow{p.w.} X$ + assumptions of MC $\Rightarrow E(X_n) \to E(X)$

DCT: $X_n \xrightarrow{p.w.} X$ + assumptions of DC $\Rightarrow$ i) $E|X_n|, E|X| < +\infty$ (they exist, in $\mathbb{R}$)

ii) $E|X_n - X| \to 0$

iii) $E(X_n) \to E(X)$

we don't like p.w. in random variables

The previous theorems hold if we replace p.w. with <u>a.e.</u> or <u>a.s.</u> and conditions to hold a.e or a.s.

$$f_n \xrightarrow{a.e.} f \text{ (or } \mu\text{-a.e.) if } \mu\left(\{\nexists \lim f_n\} \cup \underbrace{\{\lim f_n \neq f\}}_{\text{measurable set.}}\right) = 0$$

For $X_n \xrightarrow[\substack{\text{σύγκλιση} \\ \text{(με πιθ. 1)}}]{a.s.} X \iff P(\{\exists \lim X_n \text{ και } \lim X_n = X\}) = 1.$

<u>Example for DCT for Random Variables</u>

If $X_n \xrightarrow{a.s.} X$ and $\exists Y$ random variable: $|X_n| \overset{a.s}{\leq} Y$ $\forall n \geq 1$, such that $E(Y) < +\infty$, then

i) $E|X_n|, E|X| < +\infty$

ii) $E|X_n - X| \to 0$

iii) $E(X_n) \to E(X)$

In any theorem written "p.w.", we can substitute it with "a.s."