Probabilities I | Trevezas

$\mathcal{L}^0(\Omega, \mathcal{A}, P) := \{ X : \Omega \to \overline{\mathbb{R}} \mid X \text{ is a random variable and } P(X \in \mathbb{R}) = 1 \}$

↳ vector space :

- $(X+Y)(\omega) = X(\omega) + Y(\omega)$
- $(\lambda X)(\omega) = \lambda \cdot X(\omega)$, $\lambda \in \mathbb{R}$

$A = \{X \in \mathbb{R}\}$, $B = \{Y \in \mathbb{R}\}$

If we take $A \cap B$, then $P(A \cap B) = 1$, if $P(A) = 1$ and $P(B) = 1$

$$\underbrace{P(A)}_{=1} = P(AB) + \underbrace{P(AB^c)}_{\subset B^c,\ =0}$$

$P(|X| < +\infty) = 1$

$$X^0 = \begin{cases} 1, & x \in \mathbb{R}^* \\ 0, & x = 0 \\ +\infty, & x = +\infty \\ -\infty, & x = -\infty \end{cases}$$

Then,

$$|X|^0 = \begin{cases} 1, & x \in \mathbb{R}^* \\ 0, & x = 0 \\ +\infty, & x \in \{-\infty, +\infty\} \end{cases}$$ and

$$E[|X|^0] = 1 \cdot P(X \in \mathbb{R}^*) + \cancel{0(+\infty)}^{\,0}.$$

$X, Y \in \mathbb{1}^0(P)$,

Let

$$d_0(X,Y) = E\left[\,|X-Y|^0\right] = P(|X-Y| \in \mathbb{R}^*) = P(X \neq Y)$$

i) $d(x,y) = 0 \Leftrightarrow x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,z) \leq d(x,y) + d(y,z)$

$\underbrace{(ii) + (iii)}$

pseudometric or pseudodistance

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ii) $P(X \neq Y) = P(Y \neq X)$

iii) $P(X \neq Z) \leq P(X \neq Y) + P(Y \neq Z)$

i) $P(X \neq Y) = 0 \Downarrow P(X = Y) = 1 \Rightarrow X \overset{a.s.}{=} Y$

$A \subset B \Leftrightarrow B^c \subset A^c$

$\{X \neq Z\} \subset \{X \neq Y\} \cup \{Y \neq Z\} \Leftrightarrow$

$\{X = Y\} \cap \{Y = Z\} \subset \{X = Z\} \Leftrightarrow \{X = Y = Z\} \subset \{X = Z\}$

If "a.s." could define an equivalence relation in our space, ~~we could~~ "d" would be a distance in the quotient space

Identify X and Y, if $X \overset{a.s}{=} Y$.

$X \sim Y \Leftrightarrow X \overset{a.s}{=} Y$ Is it an equivalence relation?

i) $X \sim X$

ii) if $X \sim Y \rightarrow Y \sim X$

iii) if $X \sim Y$ and $Y \sim Z \rightarrow X \sim Z$,

Proof

i) $X \overset{a.s.}{=} X$

ii) if $X \overset{a.s}{=} Y \Rightarrow Y \overset{a.s}{=} X$

iii) If $X \overset{a.s}{=} Y$ and $Y \overset{a.s.}{=} Z \Rightarrow X \overset{a.s}{=} Z$. We need to prove that:

$$P(X=Z)=1$$

$$P(X=Z) = P(X=Y=Z) + \underbrace{P(X=Z, Y\neq Z)}_{0} = P(X=Y, Y=Z) = 1$$

Therefore

$$\mathcal{L}^0(P)/_\sim \equiv L^0(P).$$ → κάνει την ταύτιση με τ.μ. ίσων με πιθ. 1.

$$d_0(X,Y) = P(X\neq Y) = E[\,|X-Y|^0]$$

e.g. $X = \mathbb{1}_A$, $Y = \mathbb{1}_B$

$$d_0(\mathbb{1}_A, \mathbb{1}_B) = P(\mathbb{1}_A \neq \mathbb{1}_B)$$

$\mathbb{1}_A \neq \mathbb{1}_B \leftrightarrow (w\in A$ and $w\notin B)$ or $(w\in B$ and $w\notin A)$

$\Leftrightarrow w\in A\setminus B$ or $w\in B\setminus A$

$\Leftrightarrow w \in A \triangle B$

$$\mathcal{L}^p(\Omega, \mathcal{A}, P) \equiv \mathcal{L}^p(P) = \{X\in \mathcal{L}^0(P) : E|X|^p < +\infty\} \quad \forall p\in(0,+\infty)$$

↓

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Suppose $0 < p < 1$.

$\forall x, y \geq 0$, $(x+y)^p \leq x^p + y^p$

- if $y = 0$, $x^p \leq x^p$ ✓

- if $y > 0$, $\left(1 + \frac{x}{y}\right)^p \leq \left(\frac{x}{y}\right)^p + 1$

Set $z \geq 0$,

$(1+z)^p \leq z^p + 1$. ($f(z) = 1 + z^p - (1+z)^p \geq 0$ $\forall z \geq 0$

$f(0) = 0$.

If $X, Y \in \mathcal{L}^p(P)$, $0 < p < 1$

$E|X|^p, E|Y|^p < +\infty$.

We will prove that $X + Y \in \mathcal{L}^p(P)$

$$E|X+Y|^p \leq E(|X| + |Y|)^p \leq E(|X|^p + |Y|^p) = E|X|^p + E|Y|^p < +\infty.$$

If $X \in \mathcal{L}^p(P)$ and $\lambda \in \mathbb{R} \Rightarrow \lambda X \in \mathcal{L}^p(P)$

- $\underline{p \geq 1}$ $E|X+Y| \leq \underbrace{E|X|}_{<+\infty} + \underbrace{E|Y|}_{<+\infty} < +\infty$

$\underline{p > 1}$

$(x+y)^p \not\leq x^p + y^p$.

$$\underset{(x_1, \dots, x_d)}{\|x\|_p} = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$$

$\|X\|_p = (E|X|^p)^{1/p}$, $\forall p \geq 1$ we do not suppose it's a norm. It's just a symbol for us, yet.

We would like to prove that

$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$ $\quad (\mathcal{L}^p(P) = \{X \in \mathcal{L}^0(P) : E|X|^p < +\infty\})$

Cauchy-Schwarz inequality for random variables

$|E(XY)| \leq (E(X^2))^{1/2} (E(Y^2))^{1/2}$

$\hookrightarrow |E(XY)| \leq E|XY| \leq (E X^2)^{1/2} (E Y^2)^{1/2}$

$E(\lambda|X| - |Y|)^2 \geq 0 \quad \forall \lambda \in \mathbb{R}$

If EX^2 or $EY^2 = +\infty$, then the inequality holds.

If $EX^2, EY^2 < +\infty$

$|xy| \leq \frac{x^2+y^2}{2}$ bcs $(|x|-|y|)^2 \geq 0$

$\Downarrow$ r.v.

$|XY| \leq \frac{X^2+Y^2}{2}$.

Therefore, if $X, Y \in \mathcal{L}^2(P) \rightarrow X \cdot Y \in \mathcal{L}^1(P)$

Hölder inequality (Generalised C-S)

Def: p, q are said to be conjugate exponents (συζυγείς εκθέτες) if

$p, q > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$

$$E|XY| \leq (E|X|^p)^{1/p} \cdot (E|Y|^q)^{1/q}$$

$\|X \cdot Y\|_1 \leq \|X\|_p \cdot \|Y\|_q$, if you put $p = q = 2$ you get Cauchy-Schwarz

$\forall x, y \geq 0 \quad xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q.$

For random variables,

$$|XY| \leq \frac{1}{p}|X|^p + \frac{1}{q}|Y|^q.$$

So if $X \in \mathcal{L}^p, Y \in \mathcal{L}^q \Rightarrow X \cdot Y \in \mathcal{L}^1$

<u>Homework</u>: Proof of Hölder inequality (Real Analysis) and how to apply it here.

We want to prove that:

~~$E|X+Y|^p$~~

$\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$, so we want to prove:

$$(E|X+Y|^p)^{1/p} \leq (E|X|^p)^{1/p} + (E|Y|^p)^{1/p}.$$

So,

$$E|X+Y|^p \overset{p>1}{=} E[|X+Y|^{p-1} \underbrace{|X+Y|}_{\leq |X|+|Y|}] \leq E[|X+Y|^{p-1}|X|] + E[|X+Y|^{p-1}|Y|]$$

$$\leq (E(|X+Y|^{p-1})^q)^{1/q} \cdot (E|X|^p)^{1/p} + (E(|X+Y|^{p-1})^q)^{1/q} (E|Y|^p)^{1/p} \xRightarrow{(p-1)q = p}$$

$(p-1)q = \underset{\substack{\parallel \\ p+q}}{pq} - q = p$

$$E|X+Y|^p \le \left(E|X+Y|^p\right)^{1/q}\left(\|X\|_p + \|Y\|_p\right) \longrightarrow$$

$$\Rightarrow \underbrace{\left(E|X+Y|^p\right)^{1/p}}_{\|X+Y\|_p} \le \|X\|_p + \|Y\|_p$$

$\mathcal{L}^p(P)$, $0<p<+\infty$, $E|X|^p<+\infty$

$\underline{0<p<1}$ we proved it

$\underline{1\le p<+\infty}$. For $p\ge 1$ we defined $\|X\|_p = \left(E|X|^p\right)^{1/p}$. These $\mathcal{L}^p$ spaces are also vector spaces,

<u>Norm properties</u>

i) $\|x\|=0 \Rightarrow x=0$

ii) $\|\lambda x\| = |\lambda|\cdot\|x\|$

iii) $\|x+y\| \le \|x\|+\|y\|$

We ha We have proved (ii) and (iii)

$$\|X\|_p = 0 \overset{p\ge 1}{\Longrightarrow} \left(E|X|^p\right)^{1/p}=0 \rightarrow \underset{\ge 0}{E|X|^p=0} \Rightarrow |X|^p \overset{a.s}{=} 0 \Rightarrow |X| \overset{a.s}{=} 0 \Rightarrow$$

$$\Rightarrow X \overset{a.s.}{=} 0$$

$\mathcal{L}^p(P)/_\sim \equiv L^p(P)$. (here they will be norms)

So we defined $\mathcal{L}^p$ spaces

$\mathcal{L}^p(P)$, $p\ge 1$ $\longrightarrow L^p(P)$

$$d_p(X,Y) = E|X-Y|^p$$

$$\|X-Y\|_p$$

$0<p<1$ $\quad \mathcal{L}^p(P)$, $L^p(P)$

$d_p(X,Y) = E|X-Y|^p$ is a distance, of course symmetry holds, but we need to prove triangular inequality

But we have,

$E|X+Y|^p \le E|X|^p + E|Y|^p$.

$d_p(X,Z) = E|X-Z|^p = E|(X-Y)+(Y-Z)|^p \le E(|X-Y|+|Y-Z|)^p \le$

$\le E|X-Y|^p + E|Y-Z|^p = d_p(X,Y) + d_p(Y,Z)$

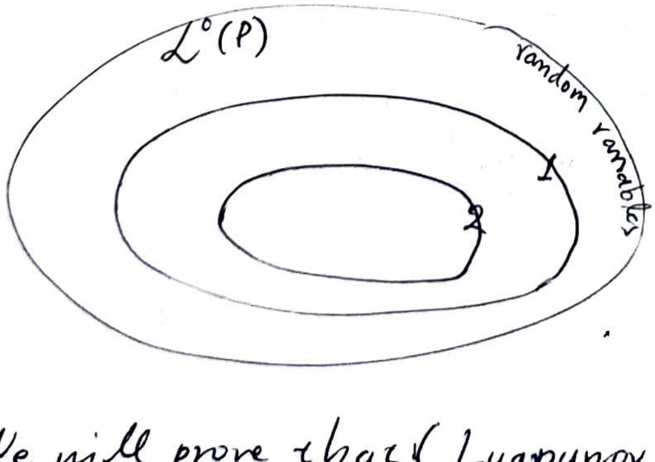
so its a metric

Cauchy Schwarz $\rightarrow$ Hölder $\rightarrow$ Minkowski (inequality for norms)

$\mathcal{L}^p(P) \supsetneq \mathcal{L}^q(P)$, $0<p<q<+\infty$

We have proved that: $EX^2<+\infty \Rightarrow E|X|<+\infty$

$(X\in\mathcal{L}^2 \Rightarrow X\in\mathcal{L}^1)$



We will prove that (Lyapunov Inequality)

$\mathcal{L}^p(P) \supset \mathcal{L}^q(P)$ $\quad \forall\, 0<p<q<+\infty$ $\quad$ (or $\mathcal{L}^q(P) \subset \mathcal{L}^p(P)$)