Probabilities II / Trevezas

$\mathcal{L}^p(P) = \{X \in \mathcal{L}^0(P) : E|X|^p < +\infty\}$, $0 < p < +\infty$

Lemma: $\forall\ 0<p<q<+\infty$, $\mathcal{L}^q(P) \subset \mathcal{L}^p(P)$.

Proof

Let $0<p<q<+\infty$.

In general, if x: $|x| \leq 1 \Rightarrow |x|^p \leq 1 \Rightarrow |x|^p \leq 1 + |x|^q$

if x: $|x| > 1 \Rightarrow |x|^p < |x|^q \leq 1 + |x|^q$

$\Rightarrow \forall x \in \mathbb{R}$ $|x|^p \leq 1 + |x|^q$

Therefore,

$|X|^p \leq 1 + |X|^q \rightarrow E|X|^p \leq 1 + E|X|^q$ $(*)$.

$X \in \mathcal{L}^q(P) \Rightarrow E|X|^q < +\infty \Rightarrow E|X|^p < +\infty \Rightarrow X \in \mathcal{L}^p(P)$

Lyapunov Inequality: For $0<p<q<\infty$, $(E|X|^p)^{1/p} \leq (E|X|^q)^{1/q}$.

If also $p \geq 1$, then: $\|X\|_p \leq \|X\|_q$.

Proof

If $E|X|^q = +\infty$, then obviously it holds.

So, assume that $E|X|^q < +\infty$

$E|X|^q = E\left((|X|^p)^{\frac{q}{p}}\right) = E[Y^{q/p}]$, where $Y = |X|^p$

Set also, $g(y) = y^{q/p}$, $q/p > 1$, $y \geq 0$. (g is convex on $I = [0, +\infty)$)

So,

$$E|X|^q = E[g(Y)] \overset{\text{Jensen}}{\underset{(*)}{\geq}} g[E(Y)] = (E(Y))^{q/p} \Rightarrow$$

$$\Rightarrow (E|X|^q)^{1/q} \geq (E(Y))^{1/p} = (E|X|^p)^{1/p}.$$

(*) <u>Conditions for Jensen</u>

i) $P(Y \in \underset{\text{interval}}{\underset{\downarrow}{I}}) = 1$ ✓, since $Y \in \mathcal{L}^q(P)$.

ii) g is convex on I

iii) $E|Y| = E|X|^p < +\infty$, since $Y \in \mathcal{L}^q(P) \subset \mathcal{L}^p(P)$ and

$E|g(Y)| = EY^{q/p} = E|X|^q < +\infty$

$\blacksquare$

$$\mathcal{L}^0(P) \underset{\neq ?}{\supseteq} \mathcal{L}^p(P) \underset{\neq ?}{\supseteq} \mathcal{L}^q(P), \quad 0<p<q<\infty$$

<u>Remark:</u>

i) The inequalities would be strict. Take $U \sim Unif(0,1)$,

$(X \in \mathcal{L}^p(P)$ και $X \notin \mathcal{L}^q(P))$, $X = U^{-1/q} = \dfrac{1}{U^{1/q}}$

$$E|X|^q = E(X^q) = E\left(\frac{1}{U}\right) = \int_0^1 \frac{1}{u}\,du = [\log u]_0^1 = +\infty \Rightarrow X \notin \mathcal{L}^q(P)$$

Now, take

$$E|X|^p = E(X^p) = E(U^{-p/q}) = \int_0^1 u^{\overset{-1<-\frac{p}{q}}{-\frac{p}{q}}} \cdot 1\,du = \frac{1}{1-\frac{p}{q}} \underbrace{\left[u^{1-p/q}\right]_0^1}_{1} = \underline{\underline{\frac{1}{1-p/q} < +\infty}}$$

We showed here that $\mathcal{L}^p(P) \underset{\neq}{\supset} \mathcal{L}^q(P)$

$$\mathcal{L}^0(P) \overset{?}{=} \bigcup_{p>0} \mathcal{L}^p(P) \quad \text{or} \quad \overset{?}{\exists} X \in \mathcal{L}^0(P) : X \notin \mathcal{L}^p(P) \ \forall p>0 :$$

The answer is that

$$\exists X \in \mathcal{L}^0(P) : X \notin \mathcal{L}^p(P) : \forall p > 0.$$

e.g. $X : P(X = x) = \frac{c}{n^2}, \ n = 1, 2, \dots$ $\qquad c = \frac{6}{\pi^2}$

$$E(X) = c \cdot \sum_{n=1}^{+\infty} \frac{n}{n^2} = c \cdot \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$$

Take $Y = e^X$. Obviously, $Y \in \mathcal{L}^0(P)$. But $E|Y|^p \overset{Y>0}{=} E[Y^p] = +\infty \ \forall p > 0$.

$\Rightarrow Y \notin \mathcal{L}^p(P), \forall p > 0.$

Indeed, $E(X) = +\infty \Rightarrow E(1 + pX) = +\infty$.

$$e^{pX} = 1 + pX + \frac{p^2 X^2}{2} + \dots \geq 1 + pX \Rightarrow$$

$$\Rightarrow E(e^{pX}) \geq E(1 + pX) = +\infty$$

The space $\mathcal{L}^\infty(P)$

Definition: If $X \in \mathcal{L}^p(P)$, then we say that it is upper bounded with probability 1, if $\exists M \in \mathbb{R} : X \overset{a.s.}{\leq} M$. We define essential supremum

$$\text{ess sup} := \inf\{M \in \mathbb{R} : X \overset{a.s.}{\leq} M\} \ (\text{ουσιώδες supremum}).$$

Definition: If $X \in \mathcal{L}^p(P)$, then we say that it is lower bounded with probability 1, if $\exists M \in \mathbb{R} : X \overset{a.s.}{\geq} M$. We define essential infimum (ουσιώ ess infimum) of X:

$$\text{ess inf} := \sup\{M \in \mathbb{R} : X \overset{a.s.}{\geq} M\}.$$

<u>Definition</u>: If $X \in \mathcal{L}^0(P)$, we say that it is <u>bounded with probability 1</u>, if it is ~~bounded~~ both lower and upper bounded with probability 1 $\Leftrightarrow$

$|X|$ is upper bounded with probability 1.

We define

$$\|X\|_\infty = \text{ess sup}\,|X| = \inf\{M \geq 0 : |X| \overset{a.s.}{\leq} M\}$$

and we call it infinite norm (άπειρο-νόρμα) of X.

Then,

$$\mathcal{L}^\infty(P) = \{X \in \mathcal{L}^0(P) : \|X\|_\infty < +\infty\}$$ (it suffices to show its a norm)

we mean psaudonorm in $\mathcal{L}^\infty(P)$ and norm in $L^\infty(P)$

it is a pseudonorm on $\mathcal{L}^\infty(P)$ and a norm on $L^\infty(P)$. (Homework)

<u>Proposition</u>:

i) If $p > 0$, then $\|X\|_p \leq \|X\|_\infty$ $(\mathcal{L}^0(P) \supset \mathcal{L}^p(P) \supset \mathcal{L}^q(P) \supset \mathcal{L}^\infty(P)\ \forall 0<p<q<\infty)$

ii) $\|X\|_\infty = \lim\limits_{p \to +\infty} \|X\|_p$.

<u>Proof</u>

i) $\|X\|_p = (E|X|^p)^{1/p}$, $\|X\|_\infty = \inf\{M \geq 0 : |X| \overset{a.s.}{\leq} M\}$

First we show that:

$$|X| \overset{a.s.}{\leq} \|X\|_\infty$$

αν $\|X\|_\infty$ πεπερ., είναι μέσα στο σύνολο και είναι minimum. Όχι όμως αν είναι ∞ και γι αυτό παίρνω inf.

If $\|X\|_\infty = +\infty$, then it holds. If $\|X\|_\infty < +\infty$, then:

$$|X| \overset{a.s.}{\leq} \|X\|_\infty + \frac{1}{n}, \quad \forall n \geq 1.$$

Set

$$A_n = \left\{ |X| \leq \|X\|_\infty + \frac{1}{n} \right\} \text{ and } P(A_n) = 1 \quad \forall n \geq 1.$$

Also, $(A_n)\downarrow$. So,

$$P\left(\bigcap_n A_n\right) = P\left(\lim_n A_n\right) = \lim_n \underbrace{P(A_n)}_{1} = 1.$$

But,

$$\bigcap_n A_n = \left\{ |X| \leq \|X\|_\infty \right\} \quad \text{and} \quad P\left(\bigcap_n A_n\right) = 1,$$

therefore $|X| \overset{a.s.}{\leq} \|X\|_\infty \Rightarrow |X|^p \overset{a.s.}{\leq} \|X\|_\infty^p \Rightarrow E|X|^p \leq \|X\|_\infty^p \Rightarrow$

$$\Rightarrow \underbrace{\left(E|X|^p\right)^{1/p}}_{\|X\|_p} \leq \|X\|_\infty.$$

ii) Set $f(p) = \|X\|_p$. By Lyapunov inequality, for $0 < p < q < +\infty$

$$\left(E|X|^p\right)^{1/p} \leq \left(E|X|^q\right)^{1/q} \Rightarrow f(p) \leq f(q) \Rightarrow f(p)\uparrow \text{ with respect to } p.$$

Set $f(\infty) = \lim_{p \to +\infty} f(p)$.

We want to show that $f(\infty) = \|X\|_\infty$. $\quad (\|X\|_p \le \|X\|_\infty)$

First, assume that

$$\exists p > 0 : f(p) = +\infty \Rightarrow f(\infty) = +\infty = \|X\|_\infty .$$

Let us assume that $f(p) < +\infty$ $\forall p > 0$. We will show that

$$f(\infty) = \|X\|_\infty .$$

Let us assume that : $f(\infty) < \|X\|_\infty$. So, it exists $M > 0$:

$$f(\infty) < M < \|X\|_\infty .$$

πολλά με δείκτρια

$$E|X|^p \ge \underset{\substack{\| \\ E(|X|^p \cdot \mathbb{1}_{|X| \ge M})}}{E(|X|^p ; X \ge M)} \ge M^p \underbrace{P(|X| \ge M)}_{\substack{\| \\ E(\mathbb{1}_{|X| \ge M})}} \Rightarrow$$

$$\Rightarrow \underset{\substack{\| \\ \|X\|_p}}{(E|X|^p)^{1/p}} \ge M \cdot P^{1/p}(|X| \ge M). \qquad (1)$$

<u>Assertion</u> (Ισχυρισμός) : $P(|X| \ge M) > 0$.

Indeed, if not, in other words, if $P(|X| \ge M) = 0 \Rightarrow X \overset{a.s.}{\le} M$ (infeasible

because $M < \|X\|_\infty$.

So $(1) \Rightarrow$

$$\Rightarrow \lim_{p \to +\infty} \|X\|_p \ge \lim_{p \to +\infty} M \cdot P^{1/p}(|X| \ge M) \Rightarrow$$

$$\Rightarrow f(\infty) \ge M \cdot \underbrace{\lim_{p \to +\infty} P^{1/p}(|X| \ge M)}_{1} \Rightarrow$$

$\Rightarrow f(\infty) \geq M \cdot 1$ <u>contradiction</u>! (since by our hypothesis $f(\infty) < M$)

Therefore $f(\infty) < \|X\|_\infty$ does not hold. So finally,

$$f(\infty) = \|X\|_\infty$$

$$\mathcal{L}^p(P) \supset \mathcal{L}^q(P) \supset \mathcal{L}^\infty(P).$$

$$\bigcap \mathcal{L}^p(P) \overset{?}{=} \mathcal{L}^\infty(P)$$

Σος

(think about it) Εκθετική είναι φραγμένη με πθ. 1?

or

$$\bigcap \mathcal{L}^p(P) \overset{?}{\neq} \mathcal{L}^\infty(P)$$

<u>Exercise</u>(DCT) : Let $(X, \mathcal{A}, \mu)$ be a measure space, $T \subset \mathbb{R}$ and t_0 an accumulation point (σ.σ.) of T (eventually $\pm\infty$).

Assume that $\forall t \in T \setminus \{t_0\}$, $\exists$ measurable function $f_t : X \to \bar{\mathbb{R}}$ and $\exists \lim_{t \to t_0} f_t(x) =: f(x)$, $\forall x \in X$ and $g : X \to [0, +\infty]$ measurable with

$$|f_t(x)| \leq g(x) \quad \forall x \in X, \ \forall t \in T \setminus \{t_0\}.$$

Then, show that i) f is measurable, integrable

f_t is integrable $\forall t \in T \setminus \{t_0\}$

ii) $\lim_{t \to t_0} \int |f_t(x) - f(x)| \, d\mu = 0$

iii) $\lim_{t \to t_0} \int f_t(x) \, d\mu(x) = \int f(x) \, d\mu(x)$ (Homework)

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μdx ίδιος συμβολισμός

Exercise: If $X \geq 0$; show that:

a) $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E[X; X<\varepsilon] = 0$

b) $\lim_{M \to \infty} \frac{1}{M} E[X; X<M] = 0$.

Solution

Set $X_\varepsilon = \frac{X \cdot \mathbb{1}_{\{X<\varepsilon\}}}{\varepsilon}$, $\forall \varepsilon > 0$. Obviously, X_ε are random variables

and

$$|X_\varepsilon| = \left| \frac{X \mathbb{1}_{\{X<\varepsilon\}}}{\varepsilon} \right| \leq \frac{\varepsilon \cdot 1}{\varepsilon} = 1 \quad \text{uniformly bounded.}$$

Take

$$X_\varepsilon(\omega) = X(\omega) \cdot \mathbb{1}_{\{X(\omega)<\varepsilon\}} / \varepsilon \leq \frac{\varepsilon}{\varepsilon} \mathbb{1}_{\{0<X(\omega)<\varepsilon\}} = \mathbb{1}_{\{0<X(\omega)<\varepsilon\}} \xrightarrow[\varepsilon \to 0^+]{} \mathbb{1}_{\emptyset} = 0$$

Set $A_\varepsilon = \{0 < X < \varepsilon\} \downarrow \varepsilon \downarrow 0$. Let $\varepsilon = \frac{1}{n} \Rightarrow A_\varepsilon \downarrow (\emptyset)$. So $X_\varepsilon \xrightarrow[\varepsilon \to 0^+]{} 0$

Therefore, by general DCT,
↪ also called Bounded CT

$$E(X_\varepsilon) \to E(0) = 0.$$

$\mathbb{1}_{\{0 \leq X(\omega) < \varepsilon\}}$ $\qquad$ $\lim A_\varepsilon = \{0\}$ and not $\emptyset$.

b) Set $X_M = \frac{X \mathbb{1}_{\{X \leq M\}}}{M} \Rightarrow \dots \quad |X_M| \leq \frac{M}{M} = 1.$

Show
$X_M(\omega) \to 0$ and BCT $\Rightarrow E(X_M) \to 0 \dots$