Probabilities II / Trevezas

Exercise 1: Show that

i) $X_n \xrightarrow{P} X \overset{(a)}{\Rightarrow} \exists (Y_n)$ subsequence of (X_n): $Y_n \xrightarrow{c} X$

$\overset{(b)}{\Rightarrow} \exists (Y_n)$ subsequence of (X_n): $Y_n \xrightarrow{a.s.} X$

ii) $X_n \xrightarrow{P} X \overset{(a)}{\Leftrightarrow} (\forall$ subsequence (Y_n) of (X_n), $\exists$ subsequence (Z_n) of (Y_n): $Z_n \xrightarrow{c} X)$

$\overset{(b)}{\Longleftrightarrow} (\forall$ subsequence (Y_n) of (X_n), $\exists$ subsequence (Z_n) of (Y_n): $Z_n \xrightarrow{a.s.} X)$

$\overset{(c)}{\Longleftrightarrow} \forall$ subsequence (Y_n) of (X_n), $\exists$ subsequence (Z_n) of (Y_n): $Z_n \xrightarrow{P} X)$

iii) Deduce that the $\xrightarrow{a.s.}$ convergence and $\xrightarrow{c}$ convergence are not metrizable (μετρικοποιήσιμη)

Solution

$$X_n \xrightarrow{P} X \Leftrightarrow \forall \varepsilon > 0, \ P(|X_n - X| > \varepsilon) \xrightarrow{n\to\infty} 0 \Leftrightarrow$$

$$\Leftrightarrow \forall \varepsilon > 0, \forall \delta > 0, \exists n_0(\varepsilon, \delta) : \forall n \geq n_0 \quad P(|X_n - X| > \varepsilon) < \delta.$$

Let $\varepsilon > 0$.

$$Y_n \xrightarrow{c} X \Leftrightarrow \sum_{n\geq 1} P(|Y_n - X| > \varepsilon) < +\infty.$$

Take $\varepsilon = \frac{1}{m}$, $\delta = \frac{1}{2^m}$, $m = 1, 2, 3, \ldots$

So $\forall m \geq 1$, $\exists n_0(m)$: $\forall n \geq n_0(m)$

$$P(|X_n - X| > \tfrac{1}{m}) < \frac{1}{2^m}.$$

We also have that: $n_0(1) < n_0(2) < \dots < n_0(m) < \dots$

Take $Y_1 = X_{n_0(1)}$, $Y_2 = X_{n_0(2)}$, $\dots$, $Y_m = X_{n_0(m)}$, $\dots$

So,

$$P(|Y_n - X| > \tfrac{1}{n}) < \frac{1}{2^n} \quad (*), \quad n = 1, 2, \dots$$

$Y_n \xrightarrow{c} X$.

Let $\varepsilon > 0$. We should have: $\sum_{n=1}^{+\infty} P(|Y_n - X| > \varepsilon) < +\infty$.

$\exists n_0 : \frac{1}{n_0} < \varepsilon$. So $\forall n \geq n_0$ $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$. Then

$$P(|Y_n - X| > \varepsilon) \leq P(|Y_n - X| > \tfrac{1}{n}) \quad \forall n \geq n_0.$$

$$\sum_{n \geq n_0} P(|Y_n - X| > \varepsilon) \leq \sum_{n \geq n_0} P(|Y_n - X| > \tfrac{1}{n}) \overset{*}{\leq} \sum_{n \geq n_0} \frac{1}{2^n} < +\infty.$$

So,

$$Y_n \xrightarrow{c} X \Rightarrow Y_n \xrightarrow{a.s.} X$$

ii) "$\longrightarrow$" Let $X_n \xrightarrow{P} X$ and take an arbitrary sequence $(Y_n) \subset (X_n)$. Obviously, $Y_n \xrightarrow{P} X$. By (1),

$$\exists (Z_n) \subset (Y_n) : Z_n \xrightarrow{c} X \rightarrow Z_n \xrightarrow{a.s.} X \Rightarrow Z_n \xrightarrow{P} X.$$

"$\longleftarrow$" Assume that $X_n \not\xrightarrow{P} X$.

Then $\exists \varepsilon > 0, \delta > 0$ and a subsequence $(Y_n) \subset X_n : P(|Y_n - X| > \varepsilon) \leq \delta$, $\forall n \geq 1. \rightarrow \forall (Z_n) \subset (Y_n)$ we have: $Z_n \not\xrightarrow{P} X \Rightarrow Z_n \not\xrightarrow{a.s.} X \rightarrow Z_n \not\xrightarrow{c} X$.

So we have a contradiction.

We conclude that: $X_n \xrightarrow{P} X$.

iii) If d is a distance function,

$$d(x_n, x) \longleftrightarrow x_n \xrightarrow{d} x.$$

$\forall \varepsilon > 0, \exists n_0 : \forall n \geq n_0, d(x_n, x) < \varepsilon$

- $x_n \overset{d}{\not\to} x \Leftrightarrow \exists \varepsilon > 0$ and a $(y_n) \subset (x_n)$ $d(y_n, x) \geq \varepsilon$.

$x_n \xrightarrow{d} x \leftrightarrow \forall (y_n) \subset (x_n), \exists (z_n) \subset (y_n) : z_n \xrightarrow{d} x$

"$\Rightarrow$" ✓

"$\Leftarrow$" Assume that $x_n \overset{d}{\not\to} x$. So,

$\exists \varepsilon > 0$ and a $(y_n) \subset (x_n) : d(y_n, x) \geq \varepsilon$.

Therefore,

$\forall (z_n) \subset (y_n)$, we have that: $z_n \overset{d}{\not\to} x$, contradiction.

Since ii-a) holds, $\xrightarrow{c}$ is not metrizable, since then:

$\xrightarrow{c} \Leftrightarrow \xrightarrow{P}$.

Similarly: ii)-b) holds, therefore: $\xrightarrow{a.s.}$ is not metrizable, since then:

$\xrightarrow{a.s.} \Leftrightarrow \xrightarrow{P}$

Exercise 2 : i) Show that if $p > 0$: $E|X|^p < +\infty$, then $E|X|^p \xrightarrow[p \to 0^+]{} P(X \neq 0)$

ii) Deduce that if $p > 0$, $X, Y \in \mathcal{L}^p(P)$, then:

$$d_p(X,Y) \xrightarrow[p\to 0^+]{} d_0(X,Y)$$

Solution

Set $X_p = |X|^p \xrightarrow[p\to 0^+]{a.s.} |X|^0$, $|X|^0 = \begin{cases} 0, & X=0 \\ 1, & 0<|X|<+\infty \\ +\infty, & |X|=+\infty \end{cases}$

By assumption, $\exists p_0 > 0 : E|X|^{p_0} < +\infty$.

Then $\forall 0 < p \le p_0$,

$$|X|^p \le \underbrace{1+|X|^{p_0}}_{\text{domination}} \Rightarrow E|X|^p \le 1 + E|X|^p < +\infty$$

From the generalised DCT

$$E(X_p) = E|X|^p \longrightarrow E|X|^0 = P(X \neq 0) \text{ since } X \in \mathcal{L}^0$$

ii) Similarly, $\exists p_0 > 0$: $E|X|^{p_0}, E|Y|^{p_0} < +\infty$.

For $0 < p < \min\{1, p_0\}$ we have:

$$d_p(X,Y) = E\underbrace{|X-Y|^p}_{\in \mathcal{L}^p \ \forall p \ge p_0} \xrightarrow[p\to 0^+]{i)} E|X-Y|^0$$

$$E|X-Y|^0 = P(X-Y\neq 0) = P(X\neq Y) = d_0(X,Y)$$

The Continuous Mapping Theorem

$\xrightarrow{a.s.}$ and $\xrightarrow{P}$ for continuous g.

If g is continuous, then

i) $X_n \xrightarrow{a.s.} X \rightarrow g(X_n) \xrightarrow{a.s.} g(X)$

ii) $X_n \xrightarrow{P} X \rightarrow g(X_n) \xrightarrow{P} g(X)$

This theorem also holds if g is continuous and a set $B \in \mathcal{B}(\mathbb{R})$:

: $P(X \in B) = 1$.

<u>Proof</u>

i) $X_n \xrightarrow{a.s.} X \leftrightarrow P(X_n \rightarrow X) = 1$.

If $w \in \Omega$: $X_n(w) \rightarrow X(w)$. Then $g(X_n(w)) \rightarrow g(X(w))$ by the

transfer principle (αρχή μεταφοράς). So

$\{X_n \rightarrow X\} \subset \{g(X_n) \rightarrow g(X)\}$, so:

$1 = P(X_n \rightarrow X) \leq P(g(X_n) \rightarrow g(X)) \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$.

ii) $X_n \xrightarrow{P} X$. Suppose that: $g(X_n) \not\xrightarrow{P} g(X)$. Therefore:

$\exists (g(Y_n)) \subset (g(X_n)) : \forall (g(Z_n)) \subset (g(Y_n)) : g(Z_n) \not\xrightarrow{a.s.} g(X) \xrightarrow{i)} \underset{\forall (Z_n) \subset (Y_n)}{Z_n \not\xrightarrow{a.s.} X}$

contradiction since $X_n \xrightarrow{P} X$.

<u>Exercise</u>: Show that $X_n \xrightarrow{\mathcal{L}^p} X \nRightarrow g(X_n) \xrightarrow{\mathcal{L}^p} g(X)$, even if $g(X_n), g(X) \in \mathcal{L}^p$

What happens if g is assumed to be continuous and bounded?