Probabilities II / Trevezas

Remark: We showed that $X_n \xrightarrow{a.s.} X \not\Rightarrow X_n \xrightarrow{\mathcal{L}^p} X$. However, the DCT, for $p=1$, if $X_n \xrightarrow{a.s.} X$ and $|X_n| \leq Y \in \mathcal{L}^1 \Rightarrow$

i) $X_n, X \in \mathcal{L}^1$

ii) $X_n \xrightarrow{\mathcal{L}} X$

iii) $E(X_n) \to E(X)$

Weaker statement of the DCT

If $X_n \xrightarrow{P} X$ and $\exists p > 0 : |X_n| \leq Y \in \mathcal{L}^p$, then:

i) $X_n, X \in \mathcal{L}^p$

ii) $X_n \xrightarrow{\mathcal{L}^p} X$

iii) $E|X_n|^p \to E|X|^p$ (Exercise)

Proof

i) $E|X_n|^p \leq EY^p$. But $Y \in \mathcal{L}^p$, therefore: $E|Y|^p \overset{Y \geq 0}{=} EY^p < +\infty \Rightarrow$

$\Rightarrow E|X_n|^p < +\infty \Rightarrow X_n \in \mathcal{L}^p$.

Now we prove that $X \in \mathcal{L}^p$.

$X_n \xrightarrow{P} X \xrightarrow{\text{Lect 18}} \exists (Y_n) \subset (X_n) : Y_n \xrightarrow{a.s.} X$.

But, by the CMT (Continuous Mapping Theorem), since $|Y|^p$ is contin.

$|Y_n|^p \xrightarrow{a.s.} |X|^p$.

Also,

ομοιόμορφα φραγμένα από $\mathcal{L}^1$

$$|X_n| \leq Y \xRightarrow{(Y_n) \subset (X_n)} |X_n| \leq |Y \longrightarrow |Y_n|^p \leq Y^p \in \mathcal{L}^1, \text{ since } EY^p < +\infty.$$

So the conditions of DCT are satisfied and:

$$|X|^p \in \mathcal{L}^1 \Rightarrow E|X|^p < +\infty \rightarrow X \in \mathcal{L}^p.$$

ii) Assume that $X_n \overset{\mathcal{L}^p}{\not\rightarrow} X$.

Then:

$$\exists \varepsilon > 0 \ \& \ \exists (Y_n) \subset (X_n) : E|Y_n - X|^p \geq \varepsilon, \ \forall n \geq 1.$$

Moreover,

$$\forall (Z_n) \subset (Y_n), \ E|Z_n - X|^p \geq \varepsilon \quad \forall n \geq 1.$$

But, by assumption $X_n \xrightarrow{P} X$, so for $(Y_n) \subset (X_n)$, $(Y_n \xrightarrow{P} X)$

$$\exists (Z_n) \subset (Y_n) : \ Z_n \xrightarrow{a.s.} X \overset{CMT}{\Longrightarrow} W_n := |Z_n - X|^p \xrightarrow{a.s.} 0.$$

But, $|Z_n - X|^p \leq (Y + |X|)^p$ and $E(Y + |X|)^p < +\infty \Longrightarrow$

$$\mathcal{L}^p + \mathcal{L}^p \in \mathcal{L}^p$$

κάτι ανεξάρτητο του n κυριαρχεί σε αυτό $|Z_n - X|^p$

$$\rightarrow E|Z_n - X|^p \underset{\substack{\text{by DCT} \\ \text{and } W_n \xrightarrow{a.s.} 0}}{\longrightarrow} E(0) = 0$$, <u>contradiction</u>! $(E|Z_n - X|^p \geq \varepsilon \ \forall n \geq 1)$

So, finally: $X_n \xrightarrow{\mathcal{L}^p} X$.

Exercise: Let $X, Y : \Omega \to \mathbb{R}$ be real random variables and we define:

$$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right] \in [0,1).$$

a) If $X, Y, Z : \Omega \to \mathbb{R}$ random variables, show that:

$$\left.\begin{array}{l} \text{i) } d(X,Y) = 0 \leftrightarrow P(X=Y) = 1 \\ \text{ii) } d(X,Y) = d(Y,X) \\ \text{iii) } d(X,Y) \leq d(X,Z) + d(Z,Y) \end{array}\right\} \Rightarrow d \text{ is a pseudometric.}$$

b) Let $X, (X_n)_{n\geq 1}$ random variables. Show that:

$$X_n \xrightarrow{P} X \leftrightarrow d(X_n, X) \to 0 \ (\text{So } \xrightarrow{P} \text{ is metrizable}).$$

Solution

Let $g(x) = \frac{x}{1+x}, \quad x \in [0, +\infty)$

Properties of g:

- $g(x) = \frac{x+1-1}{1+x} = 1 - \frac{1}{1+x}$, so $g \nearrow$ and obviously 1-1 from $[0,+\infty)$ to $[0,1)$, with $y = 1 - \frac{1}{1+x} \Leftrightarrow \frac{1}{1+x} = 1-y \Leftrightarrow x = \frac{1}{1-y} - 1 \Leftrightarrow \boxed{x = \frac{y}{1-y}} \Rightarrow$

$$\Rightarrow \boxed{g^{-1}(y) = \frac{y}{1-y}}$$

- for $u, v \geq 0$, we have $g(u+v) \leq g(u) + g(v) \Leftrightarrow$

$$\Leftrightarrow 1 - \frac{1}{1+u+v} \leq -\left(\frac{1}{1+u} + \frac{1}{1+v}\right) + 2 \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{1+u} + \frac{1}{1+v} \leq 1 + \frac{1}{1+u+v}$$

Indeed

$$\frac{1}{1+u}+\frac{1}{1+v}=\frac{1+1+u+v}{(1+u)(1+v)}\leq\frac{1+1+u+v+uv}{(1+u)(1+v)}=\frac{1+(1+u)(1+v)}{(1+u)(1+v)}=$$

$$=1+\frac{1}{(1+u)(1+v)}=1+\frac{1}{1+u+v+uv}\leq 1+\frac{1}{u+v+1}$$

We will now show that i), ii), iii) hold

i) $d(X,Y)=0 \Leftrightarrow E[\underbrace{g(|X-Y|)}_{\geq 0}]=0 \Leftrightarrow g(|X-Y|)\overset{a.s.}{=}0=g(0)$

$\overset{g\ 1\text{-}1}{\longleftrightarrow} |X-Y|\overset{a.s.}{=}0 \Leftrightarrow X\overset{a.s.}{=}Y \Leftrightarrow P(X=Y)=1.$

ii) $d(X,Y)=d(Y,X)$ obviously since $|X-Y|=|Y-X|$.

iii) $d(X,Y)=E[g(|X-Y|)]$.

Equivalently, we need to show that:

$$E[g(|X-Y|)]\leq E[g(|X-Z|)]+E[g(|Z-Y|)].$$

It suffices to show that:

$g(|X-Y|)\leq g(|X-Z|)+g(|Z-Y|)$, since then we take expectations

We have that:

$$g(|X-Y|)\overset{g\nearrow}{\leq} g(|X-Z|+|Z-Y|)\underset{\text{of } g}{\overset{\text{prop.}}{\leq}} g(|X-Z|)+g(|Z-Y|).$$

So the proof is complete.

b) $X_n \xrightarrow{P} X \iff d(X_n, X) \longrightarrow 0$. ($\iff E[g(|X_n - X|)] \longrightarrow 0$)

"$\Longrightarrow$" $X_n \xrightarrow{P} X \to |X_n - X| \xrightarrow{P} 0 \underset{CMT}{\overset{g \text{ cont}}{\longrightarrow}} g(|X_n - X|) \xrightarrow{P} g(0) = 0$.

But, since g is bounded by 1, by BCT: σχέση κυριαρχίας ισχύει τετριμμένα με ~~τα~~ το κοινό τους φράγμα

$$E[g(|X_n - X|)] \to E(0) = 0 \to$$

$$\to d(X_n, X) \to 0$$

"$\Longleftarrow$" Let $\varepsilon > 0$,

$$P(|X_n - X| > \varepsilon) \underset{g \nearrow}{=} P(g(|X_n - X|) > \underbrace{g(\varepsilon)}_{>0}) \overset{\text{Markov}}{\leq} \frac{E[g(|X_n - X|)]}{g(\varepsilon)} \xrightarrow{\text{assumption}} 0$$

$\varepsilon > 0 \overset{g \nearrow}{\Rightarrow}$

<u>Distribution of a random variable</u>

<u>Definition</u>: Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(E, \mathcal{E})$ be a measurable space and $X : \Omega \to E$ be a random variable. The probability measure $P_X : \mathcal{E} \to [0, 1]$ on E, with

$$P_X(B) = P(X^{-1}(B)) = P(X \in B) \quad \forall B \in \mathcal{E}$$

is called <u>distribution of X</u> (κατανομή της X) or <u>induced probability measure</u> (επαγόμενο μέτρο πιθανότητας). or <u>image measure</u> of P via X (μέτρο εικόνα του P ως προς X).

<u>Remark</u>: P_X is indeed a probability measure on $(E, \mathcal{E})$

Obviously, $P_X(B) \geq 0, \ \forall B \in \mathcal{E}$

i) $P_X(\emptyset) = P(X^{-1}(\emptyset)) = P(\emptyset) = 0$

ii) If $(B_n)\dagger$ in $\mathcal{E}$, then

$$P_X\left(\bigcup_n B_n\right) = P\left(X^{-1}\left(\bigcup_n B_n\right)\right) = P\left(\bigcup_n X^{-1}(B_n)\right) \underset{\sigma\text{-addit.}}{\overset{X^{-1}(B_i)\cap X^{-1}(B_j) = X^{-1}(B_iB_j) = \emptyset}{=\!=}}$$

$$= \sum_n P(X^{-1}(B_n)) = \sum_n P_X(B_n)$$

Therefore P_X is a measure.

$P_X(E) = P(X^{-1}(E)) = P(\Omega) = 1$ it is a probability measure

$$\left(\begin{array}{l} P_X(B) \geq 0 \\ \text{i) } P_X(E) = 1 \\ \text{ii) } (B_n)\dagger \rightarrow \sigma\text{-addit} \end{array} \right)$$

Special case of interest: when $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

<u>Change of Variable Formula</u> (Τύπος αλλαγής μεταβλητής)

<u>Theorem</u>: Let $(\Omega, \mathcal{A}, P)$ be a probability space, $(E, \mathcal{E})$ be a measurable space and $X : \Omega \to E$ be a random variable with distribution P_X. Then:

όταν είσαι σε χώρο πιθανότητας, κάθε μετρήσιμη τη λες τ.μ.

$\forall h : E \to [0, +\infty]$ measurable, we have $E_P[h(X)] = E_{P_X}[h]$.

or equivalently:

$$\int_\Omega h(\underbrace{X(\omega)}_{x})\, dP(\omega) = \int_E h(x)\, dP_X(x) \qquad (P_X = P \circ X^{-1})$$

παίρνει κάτι έχει και το αντιστρέφει για να δει πόσο πιθανό είναι

If $h: E \to \overline{\mathbb{R}}$ is measurable, then either both members of the equation are well defined and are equal, or both members are not defined.

e.g. $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $E(X) = \int_{\Omega} X(\omega)\,dP(\omega) = \int_{\mathbb{R}} x\,dP_X(x)$

Proof

S1| Firstly, for h simple, measurable, ≥ 0. So

$$h = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}, \quad A_i \in \mathcal{E}, \quad a_i \in [0, +\infty] \quad 1 \leq i \leq n.$$

πέρασα σε μέτρο που πάω στον E

$$E_P[h(X)] = E_P\Big[\sum_{i=1}^{n} a_i \mathbb{1}_{A_i}(X)\Big] \overset{\text{simple}}{=} \sum_{i=1}^{n} a_i P(X \in A_i) = \sum_{i=1}^{n} a_i P_X(A_i) =$$

$$= E_{P_X}\Big[\sum_{i=1}^{n} a_i \mathbb{1}_{A_i}\Big] = E_{P_X}(h)$$

S2| For h measurable, ≥ 0.

For such an h there exists $(h_n) \uparrow$ of simple, measurable, ≥ 0 functions: $h = \lim_n h_n$. So,

$$E_P[h(X)] = E_P[\lim_n h_n(X)] \underset{\text{on } \Omega}{\overset{MCT}{=\!=}} \lim_n E_P[h_n(X)] \overset{(1)}{=}$$

$$= \lim_n E_{P_X}(h_n) \underset{\text{on } E}{\overset{MCT}{=\!=}} E_{P_X}(\lim_n h_n) = E_{P_X}(h)$$

S3| For h measurable.

$h = h^+ - h^-$, where h^+, h^- satisfy S2.

$$E_P[h^+(X)] = E_P[h^-(X)] = +\infty \overset{(52)}{\Longleftrightarrow} E_{P_X}(h^+) = E_{P_X}(h^-) = +\infty$$

$\Leftrightarrow E_P[h(X)]^{x}$ is not well defined $\Leftrightarrow$

$\Leftrightarrow E_{P_X}(h)$ is not well defined.

Otherwise,

$$E_P[h(X)] = E_P[h^+(X)] - E_P[h^-(X)] \overset{52}{=}$$

$$= E_{P_X}(h^+) - E_{P_X}(h^-) =$$

$$= E_{P_X}(h^+ - h^-) =$$

$$= E_{P_X}(h).$$