Integration on a product space

Theorem: Let $(X,\mathcal{A}),(Y,\mathcal{B})$ be two measurable spaces and $f: X\times Y \to \bar{\mathbb{R}}$, $\mathcal{A}\otimes\mathcal{B}/\mathcal{B}(\bar{\mathbb{R}})$ measurable. Then:

i) $\forall x\in X$ the partial (μερική) function $f_x(y): Y\to\bar{\mathbb{R}}$ with $f_x(y)=f(x,y)$ is $\mathcal{B}/\mathcal{B}(\bar{\mathbb{R}})$ measurable

ii) $\forall y\in Y$ the partial function $f^Y(x): X\to\bar{\mathbb{R}}$ with $f^Y(x)=f(x,y)$ is $\mathcal{A}/\mathcal{B}(\bar{\mathbb{R}})$ measurable

Tonelli Theorem (for positive functions)

Let $(X,\mathcal{A},\mu),(Y,\mathcal{B},\nu)$ be σ-finite measure spaces, $(X\times Y,\mathcal{A}\otimes\mathcal{B},\mu\otimes\nu)$ is the product space and $f: X\times Y \to [0,+\infty]$ measurable function, then the functions:

$$x\longmapsto \int f(x,y)\,d\nu(y),\quad x\in X \quad \text{and}$$

$$y\longmapsto \int f(x,y)\,d\mu(x),\quad y\in Y$$

are $\mathcal{A}/\mathcal{B}([0,+\infty])$ and $\mathcal{B}/\mathcal{B}([0,+\infty])$ measurable respectively

and

$$\int f(x,y)\,d(\mu\otimes\nu)(x,y)=\int\left(\int f(x,y)\,d\nu(y)\right)d\mu(x)=\int\left(\int f(x,y)\,d\mu(x)\right)d\nu(x)$$

The integrals are well defined, since we have non negative and measurable functions.

Fubini Theorem

Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces, $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ the product space and $f: X \times Y \to \bar{\mathbb{R}}$ a measurable function. Then, if $\int |f(x,y)| \, d(\mu \otimes \nu)(x,y) < +\infty$, then the results of Tonelli hold

Corollary: When the conditions (Fubini) hold, or in any case with Tonelli,

i) $\sum_{n \geq 1} \sum_{k \geq 1} a_{n,k} = \sum_{k \geq 1} \sum_{n \geq 1} a_{n,k}$ (αριθμητικό μέτρο)

ii) $\int_{\mathbb{R}} \sum_{n \geq 1} g_n(x) dx = \sum_{n \geq 1} \int g_n(x) dx$

iii) $E\left(\sum_{n \geq 1} X_n\right) = \sum_{n \geq 1} E(X_n)$

iv) $E\left[\int_{\mathbb{R}} f(x, \cdot) dx\right] = \int_{\mathbb{R}} E[f(x, \cdot)] dx$, where $\forall x \in \mathbb{R}$ $f(x, \cdot)$ is a random variable

ex. Let $X \geq 0$ be a random variable. Then

$$\int_0^{+\infty} P(X \geq t) dt = E(X).$$

Indeed,

$$\int_0^{+\infty} P(X>t)\,dt = \int_0^{+\infty} E(\underbrace{\mathbb{1}_{\{X>t\}}}_{f(t,\cdot)})\,dt \overset{\text{Tonelli}}{=\!=} E\int_0^{+\infty} \mathbb{1}_{\{X>t\}}\,dt = E\left[\int_0^{X} 1\,dt\right] = E(X)$$

Av $X \sim Geo(p)$ where $f(x) = p\underset{1-p}{\underset{\sim}{q}}^{x-1}$, $x = 1, 2, \dots$ We know $E(X) = \frac{1}{p}$.

Compute:

$$\int_0^{+\infty} P(\underset{Geo(p)}{\underset{\wr}{X}}>t)\,dt = \sum_{0}^{\infty} \int_n^{n+1} P(X>t)\,dt = \dots$$

Example: Let $X \geq 0$ be a random variable

a) If $g: [0,+\infty) \to \mathbb{R}$ is differentiable with a continuous derivative on $(0,+\infty)$, show that:

$$E[g(X)] = g(0) + \int_0^{+\infty} g'(t) P(X>t)\,dt$$

if $g' \geq 0$ or if the integral converges absolutely

b) Show that

$$E(X^p) = p\int_0^{+\infty} t^{p-1} P(X>t)\,dt, \quad p>0$$

Solution

a) $g(x) = g(0) + g(x) - g(0) = g(0) + \int_0^x g'(t)\,dt \quad \forall x \geq 0.$

So, $\forall \omega \in \Omega$, $g(X(\omega)) = g(0) + \int_0^{X(\omega)} g'(t)\,dt \Rightarrow$

$$g(X) = g(0) + \int_0^X g'(t)\,dt \xrightarrow[\text{conditions}]{\text{under the}}$$

$$\Rightarrow E[g(X)] = g(0) + E\left[\int_0^{+\infty} g'(t) \mathbb{1}_{\{X>t\}}\,dt\right]$$

$$= g(0) + \int_0^{+\infty} g'(t) E(\mathbb{1}_{\{X>t\}})\,dt$$

$$= g(0) + \int_0^{+\infty} g'(t) P(X>t)\,dt$$

b) Set $g(x) = x^p$, $x \in [0, +\infty)$, $p > 0$. Then g is continuously differentiable on $(0, +\infty)$ and $g'(x) = p\,x^{p-1}$. So by (a)

$$E(X^p) = 0 + p\int_0^{+\infty} t^{p-1} P(X>t)\,dt.$$

Also note that; for $p=1$ and $p=2$

$$E(X) = \int_0^{+\infty} P(X>t)\,dt$$

$$E(X^2) = 2\int_0^{+\infty} t\,P(X>t)\,dt$$

Product of Probability spaces (arbitrary number)

Definition: Let $\{(\Omega_i, \mathcal{A}_i, P_i)\}_{i\in I}$, $I \neq \emptyset$ be a family of probability spaces. Set

$$\Omega = \prod_{i\in I} \Omega_i = \{(\omega_i)_{i\in I} : \omega_i \in \Omega_i, \forall i \in I\}$$

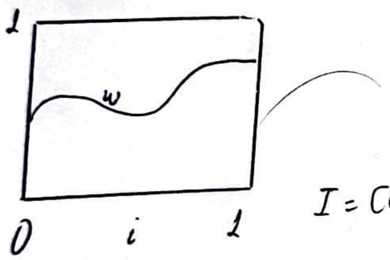
$= \{ w : I \longrightarrow \bigcup_{i \in I} \Omega_i \mid w_i \in \Omega_i, \ \forall i \in I \}$

Any set $A \subset \Omega$, $A = \prod_{i=1} A_i : A_i \in \mathcal{A}_i \ \forall i \in I$ and $J = \{ i \in I : A_i \neq \Omega_i \}$ is finite, is called a measurable cylinder on Ω.

Set $\bigotimes_{i \in I} \mathcal{A}_i := \sigma(\{ A \subset \Omega : A \text{ is a measurable cylinder} \})$ the σ-algeb[ra] generated by the measurable cylinders (the product σ-algebra)

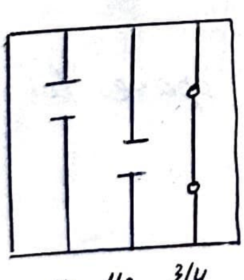
<u>Example</u>: $(\Omega_i, \mathcal{A}_i, P_i) = ([0,1], \mathcal{B}([0,1]), \lambda)$ $\forall i \in I = [0,1]$ (a continu[u]m of copies)

$[0,1]^{[0,1]} : [0,1] \to [0,1]$



$I = [0,1]$

An element w of Ω is a function from $[0,1]$ to $[0,1]$. How to create a measurable cylinder?



$A = \prod_{i \in I} A_i$, where all A_i are $[0,1]$ except for

$J = \{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \}$ and $A_{1/4} = (\frac{2}{5}, \frac{3}{5})$

$A_{1/2} = (\frac{1}{8}, \frac{1}{4})$ ενδεικτικά

$A_{3/4} = \{ \frac{1}{4}, \frac{1}{2} \}$

2) Take $\Omega = [0,1]^{\mathbb{N}} = [0,1] \times [0,1] \times \ldots \times [0,1] \times \ldots$

Each space is $([0,1], \mathcal{B}[0,1], \lambda)$

<u>Theorem</u>: If for any measurable cylinder, we define

$$P(A) = \prod_{i \in J} P_i(A_i) = \prod_{i \in I} P_i(A_i)$$

because for $i \notin J$, $A_i = \Omega \Rightarrow P_i(\Omega_i) = 1$.

then P extends uniquely to a probability measure on the measure space $(\Omega = \prod_{i \in I} \Omega_i, \bigotimes_{i \in I} \mathcal{A}_i)$ and we denote P by $\bigotimes_{i \in I} P_i$

<u>Example</u>: Let us consider the random experiment of tossing a coin an infinite number of times where

$$\underbrace{P_p(\text{"tails"})}_{\text{Γράμματα}} = p \quad \wedge \quad \underbrace{P_p(\text{"heads"})}_{\text{Κορώνα}} = 1 - p$$

We seek to construct an appropriate probability space. Here:

$$(\Omega_i, \mathcal{A}_i, P) = (\{H,T\}, \mathcal{P}(\{H,T\}), P_p), \quad 0 < p < 1.$$

By using the above theorem, there exists a probability space for this experiment which is:

$$(\Omega, \mathcal{A}, P) = (\{H,T\}^{\mathbb{N}^*}, \underbrace{\bigotimes_{i \in \mathbb{N}^*} \mathcal{P}(\{H,T\})}_{\neq \mathcal{P}(\{H,T\}^{\mathbb{N}^*})}, \bigotimes_{i \in \mathbb{N}^*} P_p)$$

HW

example: P("5th toss H", "10th toss T")

είναι μετρήσιμος κύλινδρος γιατί τα άλλα μπορεί να είναι όχι δεδο

$$= P_p(\text{"H"})\, P_p(\text{"T"}) = (1-p)\,p$$

sos $(H, T, H, T, H, T, \dots)$

έχω δεσμεύσει μέχρι n θέση και άλλα ελεύθερα

$A_n =$ "$H, T, H, T, \dots, H, T$" φθίνουσα ακολ. ενδεχομένων

A_{n+1} έχω δεσμεύσει μέχρι $n+1$.

$$P(A_n) = p^{\frac{n}{2}}(1-p)^{\frac{n}{2}} \xrightarrow{n \to \infty} 0$$

Independence of Events

Independence of a family of events (ανεξαρτησία μιας οικογένειας ενδεχομένων)

<u>Definition</u>: Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(A_i)_{i \in I}$ in $\mathcal{A}$ a family of events. The events $(A_i)_{i \in I}$ are called <u>independent</u> if for every finite $J \subset I$,

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

<u>Independence of a family of collections of events</u> (οικογένεια συλλογών ενδεχομένων)

Let $(\mathcal{A}_i)_{i \in I}$ be a family of collections $\mathcal{A}_i$ of events. The family $(\mathcal{A}_i)_{i \in I}$ is called independent, if for every finite $J \subset I$ and $A_i \in \mathcal{A}_i$, $i \in J$:

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

__Example__: Take A, B be two independent events ($A \perp\!\!\!\perp B$). A independent of $B \Leftrightarrow P(A \cap B) = P(A) \cdot P(B)$

(Exercise in Prob I : $A \perp\!\!\!\perp B \Leftrightarrow A \perp\!\!\!\perp B^c \Leftrightarrow A^c \perp\!\!\!\perp B \Leftrightarrow A^c \perp\!\!\!\perp B^c$)

So if $\mathcal{A}_1 = \{A, A^c\}$, $\mathcal{A}_2 = \{B, B^c\}$, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent collections of events.

Μπορώ να επεκτείνω με $\emptyset$ και Ω και να έχω αυτές σ-alg.

$\mathcal{A}_1' = \{\emptyset, A, A^c, \Omega\}$, $\mathcal{A}_2' = \{\emptyset, B, B^c, \Omega\}$ are independent collections

bcs κάθε ενδεχόμενο ανεξ από $\emptyset$ και Ω και η προσθήκη τους δεν αλλάζει κάτι

$$P(A \cap \emptyset) = P(\emptyset) = 0 = P(\emptyset) P(A)$$

$$P(A \cap \Omega) = P(A) = 1 \cdot P(A) = P(\Omega) \cdot P(A)$$

So every event A is independent of $\emptyset$ and Ω.