Probabilities II / Tsiretas

Independence of Random Variables

If $(\Omega, \mathcal{A}, P)$ is a probability space, $(E_i, \mathcal{E}_i)_{i\in I}$ are measurable spaces and $X_i: \Omega \to E_i$, $i \in I$ are random variables, $(X_i)_{i\in I}$ are called independent if $(\sigma(X_i))_{i\in I}$ are independent.

__Remark__: $(X_i)_{i\in I}$ are independent $\Leftrightarrow$ $\forall n \geq 2$ $\forall i_1, i_2, \dots, i_n \in I$ and $B_{i_1} \in \mathcal{E}_{i_1}, \dots, B_{i_n} \in \mathcal{E}_{i_n}$ $P(X_{i_1} \in B_{i_1}, \dots, X_{i_n} \in B_{i_n}) = P(X_{i_1} \in B_{i_1}) \cdots P(X_{i_n} \in B_{i_n})$.

It follows easily from the fact that:

$$\sigma(X_i) = X_i^{-1}(\mathcal{E}_i) = \{X_i^{-1}(B) : B \in \mathcal{E}_i\}$$

For $n=2$, X independent of $Y \Leftrightarrow P(X\in A, Y\in B) = P(X\in A)\cdot P(Y\in B)$ $\forall A \in \mathcal{A}, \forall B \in \mathcal{B}$

__Theorem__: Let $(E, \mathcal{E}), (F, \mathcal{F})$ be two measure spaces and $X: \Omega \to E$, $Y: \Omega \to F$ random variables. Then X and Y are independent random variables $\Leftrightarrow P(X\in A, Y\in B) = P(X\in A)\cdot P(Y\in B)$ $\forall A \in \mathcal{C}$, $\forall B \in \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are families closed in finite intersections and $\sigma(\mathcal{C}) = \mathcal{E}$ and $\sigma(\mathcal{D}) = \mathcal{F}$.

__Corollary__: Let $X, Y: \Omega \to \mathbb{R}$ or $\overline{\mathbb{R}}$ random variables. X and Y are independent random variables $\leftrightarrow P(X\leq x, Y\leq y) = P(X\leq x)\cdot P(Y \leq y)$ $\forall x, y \in \mathbb{R}$

__Proof__ SOS: Homework

<u>Theorem</u>: Let $X, Y: \Omega \to \bar{\mathbb{R}}$ be random variables where $X, Y \geq 0$ or $E|X|, E|Y| < +\infty$. Then

$$E(X \cdot Y) = E(X) \cdot E(Y) \qquad (1)$$

In fact, if $f, g: \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ are measurable with $f, g \geq 0$ or $E|f(X)|$, $E|g(Y)| < +\infty$, then:

Αν ξεκινήσω με δείκτριες καταλήγει σε ιδιότητα και μετά με τυπική μηχανή

$$E[f(X) \cdot g(Y)] = E[f(X)] \cdot E[g(Y)]. \qquad (2).$$

(<u>Proof</u> with formal machine, see notes)

By induction, if $X_i: \Omega \to \bar{\mathbb{R}}$ independent random variables, $1 \leq i \leq n$ and f_i, $1 \leq i \leq n$, measurable, non-negatives or $E[|f_i(X_i)|] < +\infty$ $\forall 1 \leq i \leq n$, then:

$$E[f_1(X_1) \cdots f_n(X_n)] = E[f_1(X_1)] \cdots E[f_n(X_n)].$$

<u>Corollary</u>: If $X_1, \dots, X_n$ independent random variables with $E(X_i^2) < +\infty$, $\forall 1 \leq i \leq n$, then:

$$V(X_1 + \dots + X_n) = V(X_1) + \dots + V(X_n)$$

<u>Proof</u>

It suffices to show that $Cov(X_i, Y_i) = 0$ $\forall i \neq j$, since

$$Var(X_1 + \dots + X_n) = \sum_{i=1}^{n} V(X_i) + 2 \sum_{i<j}' Cov(X_i, X_j).$$

Indeed, $Cov(X_i, X_j) = E(X_i X_j) - E(X_i) E(X_j) \overset{\text{Theor.}}{=\!=\!=} E(X_i)E(X_j) - E(X_i)E(X_j)$
$= 0$, $i \neq j$

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Αν ξεκινήσω με διακριτές καταλήγει σε πιθανότητα και μετά με τυπική μηχανή

$$E[f(X) \cdot g(Y)] = E[f(X)] \cdot E[g(Y)]. \qquad (2).$$

(Proof with formal machine, see notes)

By induction, if $X_i : \Omega \to \bar{\mathbb{R}}$ independent random variables, $1 \leq i \leq n$ and f_i, $1 \leq i \leq n$, measurable, non-negatives or $E[|f_i(X_i)|] < +\infty$ $\forall 1 \leq i \leq n$, then:

$$E[f_1(X_1) \cdots f_n(X_n)] = E[f_1(X_1)] \cdots E[f_n(X_n)].$$

Corollary: If $X_1, \ldots, X_n$ independent random variables with $E(X_i^2) < +\infty$, $\forall 1 \leq i \leq n$, then:

$$V(X_1 + \ldots + X_n) = V(X_1) + \ldots + V(X_n)$$

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It suffices to show that $Cov(X_i, Y_i) = 0$ $\forall i \neq j$, since

$$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} V(X_i) + 2 \sum_{i<j}' Cov(X_i, X_j).$$

Indeed, $Cov(X_i, X_j) = E(X_i X_j) - E(X_i) E(X_j) \overset{\text{Theor.}}{=\!=} E(X_i)E(X_j) - E(X_i)E(X_j)$
$= 0$, $i \neq j$

Independence and grouping of random variables

δε χρειάζεται να είναι σ-αλγ.

Theorem: Let $(\mathcal{A}_i)_{i\in I}$ independent families of events, with $\mathcal{A}_i \subset \mathcal{A}$ $\forall i \in I$ and $(I_j)_{j\in J}$ a partition of I. If $\mathcal{B}_j = \sigma\left(\bigcup_{i\in I_j} \mathcal{A}_i\right)$ $\forall j \in J$. Then

$(\mathcal{B}_j)_{j\in J}$ are independent σ-algebras.

Corollary: If $A_1, \dots, A_n$ are n independent events, then $A_1^c, \dots, A_n^c$ are independent.

Proof

If $A_1, \dots, A_n$ are independent events $\xrightarrow{\text{Theorem}}$

$$\left.\begin{array}{l} \sigma(\{A_1\}), \dots, \sigma(\{A_n\}) \text{ are independent } \sigma\text{-algebras} \\ (I = \{1, 2, \dots, n\} \text{ and } I_j = \{j\},\ j = 1, 2, \dots, n) \end{array}\right\} \Rightarrow$$

$\Rightarrow \{\emptyset, A_1, A_1^c, \Omega\}, \dots, \{\emptyset, A_n, A_n^c, \Omega\}$ are independent families of events $\Rightarrow$

$\Rightarrow A_1^c, \dots, A_n^c$ are independent

Example: If A_1, A_2, A_3, A_4 independent events $\Rightarrow$

$\rightarrow A_1 \cup A_2,\ A_3 \cup A_4$ are independent events

Theorem: Let $(X_i)_{i\in I}$ be independent random variables and $(I_j)_{j\in J}$ be a partition of I with $f_j : \mathbb{R}^{I_j} \to \mathbb{R}$ measurable functions $\forall j \in J$. If

$Y_j = f_j\left[(X_i)_{i\in I_j}\right]$, then $(Y_j)_{j\in J}$ are independent random variables

Examples:

i) If X, Y, Z are independent random variables, then:

$$\underbrace{X^2+Y^2}_{f_1(X,Y)} \text{ independent of } \underbrace{e^Z}_{f_2(Z)}$$

ii) If $(X_n)_{n\geq 1}$ is a sequence of random variables, then

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \underbrace{\min\{|X_{2n+1}|, 1\}}_{\leq 1} \quad \text{and} \quad \underbrace{(\sin X_{2n})_{n\geq 1}}_{(g_n(X_{2n}))_{n\geq 1}}$$

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$f_2(X_1, X_3, \dots)$

are independent

Independence - Product measure

Proposition: Let $X = (X_1, \dots, X_n)$ be a random vector. The random variables $X_1, \dots, X_n$ are independent $\Leftrightarrow P_X = P_{X_1} \otimes \dots \otimes P_{X_n}$.

Proof

"$\Rightarrow$" It suffices to show that $\underbrace{P_X(A)}_{A_1 \times \dots \times A_n \text{ measurable}} = P_{X_1}(A_1) \cdots P_{X_n}(A_n)$

$(= P_{X_1} \otimes \dots \otimes P_n(A))$.

From the uniqueness of the extension $\rightarrow P_X = P_{X_1} \otimes \dots \otimes P_{X_n}$.

Let $A = A_1 \times \dots \times A_n$ be a measurable rectangle.

$$P_X(A) = P(X \in A) = P((X_1, \dots, X_n) \in A_1 \times \dots \times A_n)$$

$$= P(X_1 \in A_1, \dots, X_n \in A_n) \xlongequal[\text{assumption}]{\text{independence}}$$

$$= P(X_1 \in A_1) \cdots P(X_n \in A_n) =$$

$$= P_{X_1}(A_1) \cdots P_{X_n}(A_n)$$

"$\leftarrow$" Let $A_1, \dots, A_n$ measurable sets

$$P(X_1 \in A_1, \dots, X_n \in A_n) \xlongequal{\text{from previous}} P_X(A_1 \times \dots \times A_n) = P_{X_1} \otimes \dots \otimes P_{X_n}(A_1 \times \dots \times A_n)$$

$$= P_{X_1}(A_1) \cdots P_{X_n}(A_n) = P(X_1 \in A_1) \cdots P(X_n \in A_n) \ \checkmark \ \text{independe}$$

<u>Exercise</u>: Let X, Y independent random variables with $E|X+Y| < +\infty$. Sho that $E|X|, E|Y| < +\infty$. Is this true for dependent random variables. (obviously not, e.g. take $X \sim \text{Cauchy}(0,1)$, with $E|X| = +\infty$, then if $Y = -$ then $E|X+Y| = 0$).

<u>Solution</u>

$$E|X+Y| = \int |X+Y| \, dP \xlongequal[\text{CVF}]{\text{Τύπος αλλαγής μεταβλ}} \int |x+y| \, dP_{(X,Y)}(x,y) \xlongequal[\text{assumption}]{\text{independence}}$$

$$= \int |x+y| \, d(P_X \otimes P_Y)(x,y) \xlongequal[\text{assumption}]{\text{Tonelli}}$$

$$= \int \Big(\underbrace{\int |x+y| \, dP_X(x)}_{g(y)} \Big) dP_Y(y) < +\infty$$

$\xrightarrow[\text{χρειάζεται}]{\text{μόνο αυτό}} \exists y_0 \in \mathbb{R} : g(y_0) < +\infty \Rightarrow$

$$\Rightarrow \int |x+y_0| \, d_x P(x) < +\infty \Rightarrow E|X+y_0| < +\infty.$$

So, $E|X| = E|X+y_0-y_0| \leq E|X+y_0| + |y_0| < +\infty$. Since X, Y are arbitrary obviously, symmetrically $E|Y| < +\infty$.

Construction of random variables with specified distributions

General Problem

Let I be an index set and $(E_i, \mathcal{E}_i, P_i)_{i\in I}$ a family of probability spaces. We want to construct a probability space $(\Omega, \mathcal{A}, P)$ and random variables $X_i : \Omega \to E_i$ such that:

a) $P_{X_i} = P_i \quad \forall i \in I$

b) (X_i) be independent random variables

For this construction

$$\Omega = \prod_{i\in I} E_i, \quad \mathcal{A} = \bigotimes \mathcal{E}_i, \quad P = \bigotimes_{i\in I} P_i,$$

so $(\Omega, \mathcal{A}, P)$ is the product space. For the random variables, set $X_i(\omega) = X_i((\omega_r)_{r\in I}) = \omega_i$ προβολή στην i συνιστώσα
the projection on the i-th coordinate.

This construction has the desired properties: a) and b).

Proposition: i) X_i is a random variable $\forall i \in I$ and $P_{X_i} = P_i$

ii) $(X_i)_{i\in I}$ are independent random variables.

Let $A_i \in \mathcal{E}_i$. $\underbrace{X_i^{-1}(A_i) = A_i^*}_{\text{the cylinder}} =$

$$= \prod_{r \in I} B_r, \text{ where } B_r = \begin{cases} A_r, & r = i \\ E_r, & r \neq i \end{cases}$$

So $X_i^{-1}(A_i) \in \mathcal{A} \Rightarrow X_i$ is a random variable.

Now,

$$P_{X_i}(A_i) = P(X_i^{-1}(A_i)) = P(A_i^*) = P_i(A_i) \Rightarrow P_{X_i} = P_i$$

ii) Exercise. ($P(X_i \in A_i \ ) = P(X_1 \in A_1) \cdots P$ για υποσυλλογή) $i \in J$ και αντιστοιχία με πεπερασμένους κυλίνδρους.

Exercise: Let X, Y be independent random variables and $X = Y$ with probability 1. Then X is a degenerate r.v. ($X = c$ with prob 1).

Solution

Let $x \in \mathbb{R}$.

$$P(X \le x) = P(X \le x, \underbrace{X = Y}_{\substack{\text{event of} \\ \text{prob. 1}}}) = P(X \le x, X = Y, Y \le x)$$

$= P(X \leq x, Y \leq x) \xrightarrow[\text{assumption}]{\text{independence}} P(X \leq x) P(Y \leq x) = \cancel{P(X \leq x)}$

$$= P(X \leq x) \cdot P(Y \leq x, X = Y) \rightarrow$$

$\Rightarrow P(X \leq x) = P(X \leq x) \cdot P(X \leq x) = P^2(X \leq x) \Rightarrow$

$\Rightarrow P(X \leq x) \in \{0, 1\} \xRightarrow[\text{distribution function}]{\text{properties of}}$

$\exists c \in \mathbb{R}:$

$\rightarrow \underset{P(X \leq x)}{\overset{\parallel}{F(x)}} = \begin{cases} 0, & x < c \\ 1, & x \geq c \end{cases} \Rightarrow X = c$ with probability 1.

<u>Reminder</u> (1st Lemma Borel-Cantelli)

If $\sum_{n \geq 1} P(A_n) < +\infty \Rightarrow P(\limsup A_n) = 0$

<u>Proposition</u> (2nd Lemma Borel-Cantelli): For an independent sequence of events $(A_n)_{n \geq 1}$, $\sum_{n \geq 1} P(A_n) = +\infty \Rightarrow P(\limsup A_n) = 1$.

<u>Proof</u>

$\limsup_n A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k \Rightarrow P(\limsup A_n) = 1 \Leftrightarrow P[(\limsup A_n)^c] = 0$.

We will show that $P[(\limsup A_n)^c] = 0$.

But $(\limsup A_n)^c = \bigcup_{n \geq 1} \underbrace{\bigcap_{k \geq n} A_k^c}_{\text{increasing sequence}}$

So, $P\left(\bigcup_{n\geq 1} \bigcap_{k\geq n} A_k^c\right) = \lim_{n\to\infty} P\left(\bigcap_{k\geq n} A_k^c\right)$.

αν δεν ήταν θα $\exists n_0 : P(A_{n_0}^c) \neq 0$ και αφού αύξουσα όλα τα υπόλοιπα

$$\lim_{n\to\infty} P\left(\bigcap_{k\geq n} A_k^c\right) = 0 \iff \forall n\geq 1,\ P\left(\bigcap_{k\geq n} A_k^c\right) = 0.$$

($\Rightarrow$) from contradiction

Let $n\in\mathbb{N}$ be arbitrary.

$$\bigcap_{k\geq n} A_k^c = \lim_{m\to\infty} \bigcap_{k=n}^{m} A_k^c.$$

So, $P\left(\bigcap_{k\geq n} A_k^c\right) = \lim_{m\to\infty} P\left(\bigcap_{k=n}^{m} A_k^c\right)$. But,

$$P\left(\bigcap_{k=n}^{m} A_k^c\right) \underset{\Rightarrow (A_k^c)\text{ independence}}{\overset{(A_k)\text{ independence}}{=\!=\!=}} \prod_{k=n}^{m} P(A_k^c) = \prod_{k=n}^{m} (\underbrace{1-P(A_k)}_{1-x_k}) \Longrightarrow$$

(We know that: $1+x\leq e^x\ \forall x\in\mathbb{R} \Rightarrow 1-x\leq e^{-x}\ \forall x\in\mathbb{R}$)

$$\Longrightarrow P\left(\bigcap_{k=n}^{m} A_k^c\right) \leq \prod_{k=n}^{m} e^{-P(A_k)} = e^{-\sum_{k=n}^{m} P(A_k)}. \quad (*)$$

But, by assumption, $\sum_{n\geq 1} P(A_n) = +\infty \Rightarrow \sum_{k=n}^{m} P(A_k) = +\infty\ \ \forall n\geq 1$.

By $(*)$ and since $\lim_{m\to\infty} e^{-\sum_{k=n}^{m} P(A_k)} = e^{-\sum_{k=n}^{\infty} P(A_k)} = e^{-\infty} = 0$, we conclude that:

$$\lim_{m\to\infty} P\left(\bigcap_{k=n}^{m} A_k^c\right) = 0$$

Remarks:

1) The independence assumption cannot be dropped. Indeed, take $(\Omega, \mathcal{A}, P) = ((0,1), \mathcal{B}(0,1), \lambda)$ and $A_n = (0, \frac{1}{n})$. Then:

$$\sum_{n\geq 1} P(A_n) = \sum_{n\geq 1} \lambda(A_n) = \sum_{n\geq 1} \frac{1}{n} = +\infty.$$

But, $\limsup A_n = \lim A_n = \emptyset$, since $(0, \frac{1}{n}) \downarrow \emptyset$. So $P(\limsup A_n) = 0$.

2) The converse ~~is~~ of the 2nd Lemma holds.

$$P(\limsup A_n) = 1 \xrightarrow{+\text{ indep.}} \sum_{n\geq 1} P(A_n) = +\infty.$$

Indeed, if $\sum_{n\geq 1} P(A_n) < +\infty \underset{\text{Lemma}}{\overset{1st}{\Longrightarrow}} P(\limsup A_n) = 0$ (Contradiction, $=1$)

So, if (A_n) independent then $\sum_{n\geq 1} P(A_n) = +\infty \Rightarrow P(\limsup A_n) = 1$.

(3) By (2) we have that for an independent sequence $(A_n)_{n\geq 1}$

$$P(\limsup A_n) \in \{0,1\}.$$

It is a consequence also of the 0-1 law of Kolmogorov for an independent set of events, that actually limits $P(A) \in \{0,1\}$ if A does not depend on the behavior of any finite collection of them but only depends on the <u>final</u> behavior of the sequence of events (final σ-algebra: τελική σ-αλγ)

Αν πάρω σ-αλγ από όλα αυτά ενδεχ και βγάλω ενδεχ, έχω φθίνουσα → όριο και την λέω τελική σ-αλγ.

Exercise: If $X_n \sim Ber(\frac{1}{n}), n \geq 1$ independent random variables.

Show that:

a) $X_n \xrightarrow{P} 0$

b) $X_n \overset{a.s.}{\not\to} 0$ and $P(\lim X_n = 0) = 0$

Solution

a) Take $\varepsilon > 0$ with no loss of generality $0 < \varepsilon < 1$.

$$P(|X_n - 0| \geq \varepsilon) = P(X_n \geq \varepsilon) = P(X_n = 1) = \frac{1}{n} \to 0 \Rightarrow X_n \xrightarrow{P} 0$$

b) $A_n = \{X_n = 1\}$. Since X_n are independent random variables, then $A_n = X_n^{-1}(\{1\})$ are independent events.

$$\sum_{n \geq 1} P(A_n) = \sum_{n \geq 1} P(X_n = 1) = \sum_{n \geq 1} \frac{1}{n} = +\infty \xrightarrow[\text{Borel Cantelli}]{\text{2nd lemma}}$$

$\Rightarrow P(\limsup A_n) = 1.$ $(*)$

$$\{\lim X_n = 0\} \subset (\limsup A_n)^c$$

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$(\lim X_n(\omega) = 0 \Rightarrow X_n(\omega) = 0$ finally for all $n \Rightarrow X_n(\omega) \in (\limsup A_n)^c)$

$\Rightarrow P(\lim X_n = 0) \leq P((\limsup A_n)^c) \overset{(*)}{=} 0$

$\Rightarrow P(\lim X_n = 0) = 0 \Rightarrow X_n \overset{a.s.}{\not\to} 0.$