Strong Law of Large Numbers (SLLN)

Theorem (Strong Law of Large Numbers): Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables on $(\Omega, \mathcal{A}, P)$ with $X_n \in \overline{\mathbb{R}}$, $n \geq 1$ and $E|X_1| < +\infty$. Set $\mu = E(X_1)$ and $S_n = \sum_{k=1}^{n} X_k$. Then:

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu$$

Proof

We will assume that $E(X_1^2) < +\infty$ (to simplify proof)

$$E(X_1^2) < +\infty \Rightarrow V(X_1) < +\infty$$

We set $\sigma^2 = V(X_1)$.

Step 1: We prove theorem for $X_n \geq 0$

Set $Y_n = \overline{X}_n - \mu$, $n \geq 1$

$$E(Y_n) = E(\overline{X}_n - \mu) = E(\overline{X}_n) - \mu = E(X_1) - \mu = \mu - \mu = 0$$

$$E(Y_n^2) = V(Y_n) = V(\overline{X}_n) = \frac{V(X_1)}{n} = \frac{\sigma^2}{n}, \quad \sigma^2 < +\infty$$

Take the subsequence $(Y_{n^2})_{n\geq 1}$ of $(Y_n)_{n\geq 1}$.

$$E\left(\sum_{n\geq 1} Y_{n^2}^2\right) \overset{B-L}{=\!=} \sum_{n\geq 1} E(Y_{n^2}^2) = \sum_{n\geq 1} \frac{\sigma^2}{n^2} = \sigma^2 \sum_{n\geq 1} \frac{1}{n^2} < +\infty \Rightarrow$$

$$\Rightarrow \sum_1^{\infty} Y_{n^2}^2 < +\infty \text{ with prob } 1$$

$$\Rightarrow Y_{n^2}^2 \xrightarrow{a.s.} 0 \xrightarrow{CMT} |Y_{n^2}| \xrightarrow{a.s.} 0 \Rightarrow Y_{n^2} \xrightarrow{a.s.} 0$$

$$\Rightarrow \frac{S_{n^2}}{n^2} - \mu \xrightarrow{a.s.} 0 \Rightarrow \boxed{\frac{S_{n^2}}{n^2} \xrightarrow{a.s.} \mu \quad (1)}$$

$\forall k \geq 1$, set $n(k) = [\sqrt{k}]$.

But $[\sqrt{k}] \leq \sqrt{k} \leq [\sqrt{k}] + 1 \Rightarrow n(k) \leq \sqrt{k} \leq n(k) + 1 \Rightarrow$

$$\Rightarrow \frac{1}{n(k)+1} \leq \frac{1}{\sqrt{k}} \leq \frac{1}{n(k)} \Rightarrow \boxed{\frac{1}{(n(k)+1)^2} \leq \frac{1}{k} \leq \frac{1}{n^2(k)} \quad (2)}$$

Since $n^2(k) \leq k \leq (n(k)+1)^2$, we have:

$$\boxed{S_{n^2(k)} \leq S_k \leq S_{(n(k)+1)^2} \quad (3)}$$

From (1), (2), we derive:

$$\frac{S_{n^2(k)}}{(n(k)+1)^2} \leq \frac{S_k}{k} \leq \frac{S_{(n(k)+1)^2}}{n^2(k)}$$

But:

$$\frac{S_{n^2(k)}}{(n(k)+1)^2} = \frac{S_{n^2(k)}}{n^2(k)} \left(\frac{n(k)}{n(k)+1}\right)^2 \xrightarrow[k\to\infty]{a.s.} \mu \cdot 1 = \mu$$

$$\left(n(k) \xrightarrow[k\to\infty]{} \infty\right)$$

$$\cdot \; \frac{S_{(n(k)+1)^2}}{n^2(k)} = \frac{S_{(n(k)+1)^2}}{(n(k)+1)^2} \left(\frac{n(k)+1}{n(k)}\right)^2 \xrightarrow{a.s.} \mu \cdot 1 = \mu$$

By sandwich theorem $\Rightarrow \frac{S_k}{k} \xrightarrow[k\to\infty]{a.s.} \mu$

Step 2 : $X_n \in \overline{\mathbb{R}}$: $X_i := X_i^+ - X_i^-$

σε κάθε περίπτωση ένα από τα X_i^+, X_i^- θα είναι 0.

$$\frac{S_n}{n} = \frac{X_1 + \ldots + X_n}{n} = \frac{X_1^+ + \ldots + X_n^+}{n} - \frac{X_1^- + \ldots + X_n^-}{n}$$

X_n = i.i.d. $\rightarrow (X_n^+)_{n \geq 1}$ and $(X_n^-)_{n \geq 1}$ are i.i.d. (ανεξ και ισόνομες)

Also,

$$\left.\begin{array}{l} X_1^+ = \max\{0, X_1\} \leq |X_1| \\ X_1^- = \max\{0, -X_1\} \leq |X_1| \end{array}\right\} \Rightarrow \left.\begin{array}{l} (X_1^+)^2 \leq X_1^2 \\ (X_1^-)^2 \leq X_1^2 \end{array}\right\} \Rightarrow \begin{array}{l} E(X_1^+)^2 < +\infty \\ E(X_1^-)^2 < +\infty \end{array}$$

So the conditions of SL hold (i.i.d + second moment finite)

$$\left.\begin{array}{l} \dfrac{X_1^+ + \ldots + X_n^+}{n} \xrightarrow{a.s.} E(X_1^+) \in \mathbb{R} \\ \\ \dfrac{X_1^- + \ldots + X_n^-}{n} \xrightarrow{a.s.} E(X_1^-) \in \mathbb{R} \end{array}\right\} \Rightarrow \frac{S_n}{n} \xrightarrow{a.s.} E(X_1^+) - E(X_1^-) = E(X_1) = \mu$$

Reminder: The theorem also holds for $E(X) = +\infty$ or $E(X) = -\infty$.

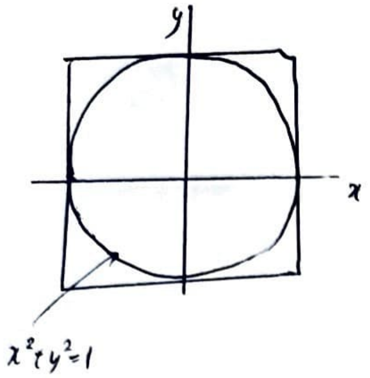
Only exception if $E(X_1^+) = E(X_1^-) = +\infty$

Application (Monte Carlo Integration Method)

It's an integral approximation method, meaning its a method of approximating the value of an integral, which can be expressed as a mean value of some random variable.

It requires the possibility of simulation from the distribution of the random variable.

Example: Approximation of $\pi \simeq 3,1415, \ldots$



$$\text{Area} = \pi \cdot \rho^2 \overset{\rho=1}{=} \pi$$

$$\pi = \iint_{x^2+y^2\le 1} 1\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \mathbb{1}_{\{x^2+y^2\le 1\}}\,dx\,dy$$

Idea: (X,Y) such that $X, Y \sim Unif(-1,1)$ and X independent of $Y \Rightarrow f_{(X,Y)} = \frac{1}{4}$ $(|X|,|Y|\le 1)$.

Rewrite $\pi = 4\cdot\frac{\pi}{4} = 4\cdot\int_{-1}^{1}\int_{-1}^{1} \mathbb{1}_{\{x^2+y^2\le 1\}}\cdot \underset{f_{(X,Y)}}{\underset{\shortparallel}{\frac{1}{4}}}\,dx\,dy =$

$= 4 E[\mathbb{1}_{Disk}(X,Y)]$, $Disk = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\} \Rightarrow$

$\rightarrow \frac{\pi}{4} = E[\underbrace{\mathbb{1}_{Disk}(X,Y)}_{g(X,Y)}]$

Approximation via Monte-Carlo

Draw $X_1,\dots,X_n \sim Unif(-1,1)$
$Y_1,\dots,Y_n \sim Unif(-1,1)$ i.i.d. (from your favorite simulator)

Then $Z_1 = \mathbb{1}_{Disk}(X_1,Y_1)$, $Z_2 = \mathbb{1}_{Disk}(X_2,Y_2)$, ..., $Z_n = \mathbb{1}_{Disk}(X_n,Y_n)$

are i.i.d. with $Z_n \sim Ber(p)$, $p = \underset{E[\mathbb{1}_{Disk}(X_n,Y_n)]}{\underset{\shortparallel}{P(Z_n=1)}} = P(X_n^2+Y_n^2\le 1) = \frac{\pi}{4}$.

Take $\overline{Z}_n = \frac{Z_1+\dots+Z_n}{n} \xrightarrow[SLLN]{a.s.} \frac{\pi}{4}$ (for large n: $\pi \simeq 4\cdot\overline{Z}_n$

percentage of points inside disk.

Corollary (Weak Law of Large Numbers): Let $(X_n)_{n\geq 1}$ i.i.d. random variables with $E|X_1|<+\infty$. Then:

$$\bar{X}_n = \frac{S_n}{n} \xrightarrow{P} \mu = E(X_1).$$

Proof

From SLLN we have: $\bar{X}_n \xrightarrow{a.s.} \mu$.

However $\xrightarrow{a.s.} \to \xrightarrow{P} \Rightarrow \bar{X}_n \xrightarrow{P} \mu$.

Alternatively:

With $E(X_1^2)<+\infty$, let $\varepsilon>0$. Then:

$$P(|\bar{X}_n-\mu|>\varepsilon) = P(|\bar{X}_n - E(\bar{X}_n)|>\varepsilon) \overset{\text{Chebyshev}}{\leq} \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{Var(X_1)}{n\varepsilon^2} \xrightarrow[n]{} 0$$

Therefore: $\bar{X}_n \to \mu$.

Exercise 12.6: Let $(U_i)_{i\geq 1}$ i.i.d random variables with $U_1 \sim U(0,1)$.

Show that:

a) $(U_1 \cdots U_n)^{1/n} \xrightarrow{a.s.} e^{-1}$

b) $U_1 \cdots U_n \xrightarrow{a.s.} 0$

c) $\frac{U_1^a + \ldots + U_n^a}{n} \xrightarrow{a.s.} \begin{cases} \frac{1}{1+a}, & a>-1 \\ +\infty, & a\leq -1 \end{cases}$

Proof

a) $(U_1 \cdots U_n)^{1/n} = e^{\log(U_1\cdots U_n)^{1/n}} = e^{\frac{1}{n}\sum_{i=1}^{n} \log U_i}$

It's easy to show that $-\log U \sim Exp(1)$ if $U \sim U(0,1)$ (hw)

So :

$$E[-\log U] = E[Exp(1)] \Rightarrow E[\log U] = -1 \xrightarrow{SLLN}$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} \log U_i \xrightarrow{a.s.} E[\log U] = -1 \xrightarrow{CMT}$$

$$\Rightarrow e^{\frac{1}{n}\sum_{i=1}^{n} U_i} \xrightarrow{a.s.} e^{-1}$$

b) $$U_1 \cdots U_n = e^{\sum_{i=1}^{n} \log U_i} = e^{n \underbrace{\frac{1}{n}\sum_{i=1}^{n} \log U_i}_{\xrightarrow{a.s.} -1}} \xrightarrow{a.s.} e^{-\infty} = 0$$

c) $E[U^a]$ $\forall a \in \mathbb{R}$.

When the random variable is well defined, from SLLN (U_i^a, $i \geq 1$, iid random variables):

$$\frac{U_1^a + \ldots + U_n^a}{n} \xrightarrow{a.s.} E(U^a) \quad , \quad U \sim U(0,1)$$

$$E[U^a] = \int_0^1 u^a 1 du = \begin{cases} \left[\frac{u^{a+1}}{a+1}\right]_0^1 , & a \neq -1 \\ [\log u]_0^1 , & a = -1 \end{cases} =$$

$$= \begin{cases} \frac{1}{a+1} - \frac{1}{a+1} \lim_{u \to 0^+} u^{a+1} , & a \neq -1 \\ +\infty , & a = -1 \end{cases} = \begin{cases} \frac{1}{a+1} , & a > -1 \\ +\infty , & a = -1 \\ \infty , & a < -1 \end{cases}$$

Fourier Transformation of Probability Measure

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f: \Omega \to \mathbb{C}$ Borel-measurable. Then $f = Re(f) + Im(f)$, where $Re(f), Im(f): \Omega \to \mathbb{R}$ Borel-measurable

We define

$$\int f \, d\mu = \int Re(f) \, d\mu + i \int Im(f) \, d\mu \in \mathbb{C} \quad \text{if} \quad \int |Re(f)| \, d\mu, \int |Im(f)| \, d\mu < \infty$$

Properties

(1) $\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$

(2) $\int \bar{f} \, d\mu = \overline{\int f \, d\mu}$, where $|f|, \bar{f}$ measure and conjugate complex func;

Proof

(1) Firstly: $\int f \, d\mu \xlongequal[\text{tion of complex numbers}]{\text{polar representa}} e^{i\vartheta} \left| \int f \, d\mu \right| \Rightarrow \quad (z = |z| e^{i\vartheta}, \ \vartheta \in [0, 2\pi))$

$$\to \underbrace{\left| \int f \, d\mu \right|}_{\in \mathbb{R}} = e^{-i\vartheta} \int f \, d\mu =$$

$$= \int e^{-i\vartheta} f \, d\mu = \int Re(e^{-i\vartheta} f) \, d\mu + i \int Im(e^{-i\vartheta} f) \, d\mu \xlongequal{Re(z) \leq |z|}$$

$$\leq \int |e^{-i\vartheta} f| \, d\mu \leq \int |e^{-i\vartheta}| |f| \, d\mu =$$

$$= \int |f| \, d\mu \qquad \left(\begin{array}{l} e^{i\vartheta} = \cos\vartheta + i \sin\vartheta \\ |e^{i\vartheta}| = 1, \ \forall \vartheta \end{array} \right)$$

(2) Obvious

<u>Definition</u> (Fourier transformation of a probability measure): Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We call <u>Fourier transformation of μ</u>, the function $\hat{\mu}: \mathbb{R} \to \mathbb{C}$ such that ($e^{itx} = \cos tx + i \sin tx$)

$$\hat{\mu}(t) = \int e^{itx} d\mu(x) = \int \cos(tx) d\mu(x) + i \int \sin(tx) d\mu(x).$$

<u>Theorem</u>: Let μ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then

i) $|\hat{\mu}(t)| \leq 1 \quad \forall t \in \mathbb{R}$

ii) $\hat{\mu}(0) = 1$

iii) $\hat{\mu}(t)$ is uniformuly continuous function

<u>Proof</u>

i) $$|\hat{\mu}(t)| = \left|\int e^{itx} d\mu(x)\right| \overset{P1}{\leq} \int \underbrace{|e^{itx}|}_{=1} d\mu(x) = \int 1 \, d\mu(x) = \mu(\mathbb{R}) = 1$$

ii) $$|\hat{\mu}(0)| = \int e^{i0x} d\mu(x) = \int 1 \, d\mu(x) = \mu(\mathbb{R}) = 1$$

iii) It suffices to show that

$$\sup_{t \in \mathbb{R}} |\hat{\mu}(t+h) - \hat{\mu}(t)| \xrightarrow[h \to 0]{} 0.$$

Indeed,

$$|\hat{\mu}(t+h) - \hat{\mu}(t)| = \left|\int e^{i(t+h)x} d\mu(x) - \int e^{itx} d\mu(x)\right| =$$

$$= \left|\int e^{itx}(e^{ihx} - 1) d\mu(x)\right| \leq \int \underbrace{|e^{itx}|}_{=1} |e^{ihx} - 1| \, d\mu(x) =$$

$= \int |e^{ihx}-1|\, d\mu(x) \Rightarrow \sup_{t\in\mathbb{R}} |\hat{\mu}(t+h)-\hat{\mu}(t)| \le \int |e^{ihx}-1|\, d\mu(x)$

Remark: We observe that:

i) $|e^{ihx}-1| \xrightarrow[h\to 0]{} 0 \quad \forall x\in\mathbb{R}$

ii) $|e^{ihx}-1| \le |e^{ihx}|+1 = 2 = g(x) \quad \forall x\in\mathbb{R}$.

iii) $\int g(x)\, d\mu(x) = \int 2\, d\mu(x) = 2\mu(\mathbb{R}) = 2 < \infty$ Άρα:

$$\int |e^{ihx}-1|\, d\mu(x) \xrightarrow{(\gamma.\Theta K\Sigma)} \int 0\, d\mu(x) = 0$$

Χαρακτηριστικές συναρτήσεις (Eigenfunction)

Definition: Let X be a real random variable in probability space $(\Omega,\mathcal{A},P)$. We call eigenfunction of X the function:

$$\varphi_X : \mathbb{R} \to \mathbb{C} : \varphi_X(t) = E(e^{itX})$$

Remark: $\varphi_X(t) = E[e^{itX}] = \int e^{itX}\, dP \underset{(T.A.M.)}{=} \int e^{itx}\, dP_X(x) = \hat{P}_X(t)$.

Therefore the eigenfunction of a random variable X is the Fourier transformation of a probability measure P, or in other words ~~the~~ of the distribution of X.

Proposition: If X, Y real random variables in $(\Omega,\mathcal{A},P)$ and $a,b\in\mathbb{R}$

then $\forall t \in \mathbb{R}$:

i) $\varphi_X(-t) = \overline{\varphi_X(t)}$

ii) $\varphi_{aX+b}(t) = e^{itb} \cdot \varphi_X(at)$

iii) If X, Y independent random variables, $\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t)$

Proof

$$\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = e^{-i\theta}$$

$$= \cos(-\theta) + i\sin(-\theta)$$

i) $\varphi_X(-t) = E[e^{i(-t)X}] = E[e^{-itX}] = E[\overline{e^{itX}}] = \overline{E[e^{itX}]} = \overline{\varphi_X(t)}$

ii) $\varphi_{aX+b}(t) = E[e^{it(aX+b)}] = E(e^{itb} \cdot e^{iatX}] = e^{itb} E[e^{i(at)X}] =$

$$= e^{itb} \varphi_X(at)$$

iii) $\varphi_{X+Y}(t) = E[e^{it(X+Y)}] = E[\underbrace{e^{itX}}_{g(X)} \cdot \underbrace{e^{itY}}_{g(Y)}] \overset{X,Y \text{ independent}}{=\!=}$

$$= E[e^{itX}] \cdot E[e^{itY}] = \varphi_X(t) \cdot \varphi_Y(t)$$

Example (Calculation of eigenfunction):

i) $X \sim Bin(n,p) \Rightarrow \varphi_X(t) = (pe^{it} + (1-p))^n$

Proof

$$\varphi_X(t) = E[e^{itX}] = \sum_{k=0}^{n} e^{itk} \cdot P(X=k) =$$

$$= \sum_{k=0}^{n} e^{itk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (\underbrace{pe^{it}}_{a})^k (\underbrace{1-p}_{b})^{n-k} =$$

$$= (pe^{it} + 1-p)^n$$

<u>Alternatively</u>:

$$X \sim Bin(n,p) \Rightarrow X = \sum_{i=1}^{n} X_i, \; X_i \sim Ber(p), \text{ independent.}$$

$$\varphi_X(t) = \varphi_{\sum_{i=1}^{n} X_i}(t) \overset{\text{independent}}{=\!=} \prod_{i=1}^{n} \varphi_{X_i}(t) \overset{X_i \sim Ber(p)}{=\!=} \varphi_{X_1}^{n}(t) =$$

$$= E^n[e^{tX_1}] = [pe^{t} + (1-p)\cdot 1]^n$$

ii) $X \sim U(-a,a) \Rightarrow \varphi_X(t) = \begin{cases} \dfrac{\sin(at)}{at}, & t \neq 0 \\ 1, & t = 0. \end{cases}$

<u>Proof</u>

If $X \sim U(-a,a)$, then $f_X(x) = \begin{cases} \frac{1}{2a}, & x \in (-a,a) \\ 0, & \text{otherwise} \end{cases}$

$$\varphi_X(t) = E[e^{itX}] = \int_{-a}^{a} e^{itx} \cdot \frac{1}{2a}\, dx = \frac{1}{2a} \int_{-a}^{a} \underbrace{e^{itx}}_{\cos(tx) + i\sin(tx)} dx =$$

$$= \frac{1}{2a} \underbrace{\int_{-a}^{a} \cos(tx)\, dx}_{\text{even function in symmetric interval around } 0} + \frac{i}{2a} \underbrace{\int_{-a}^{a} \sin(tx)\, dx}_{\text{odd function in symmetric interval around } 0}$$

$$= \frac{2}{2a}\int_0^a \cos(tx)\,dx = \frac{1}{at}\left[\sin tx\right]_0^a = \frac{1}{at}\sin(at),$$

<u>Alternatively</u>; $\varphi_X(t) = \frac{1}{2a}\int_{-a}^{a} e^{itx}\,dx \overset{t\neq 0}{=} \frac{1}{2ait}\left[e^{itx}\right]_{-a}^{a}$ $\underline{\underline{(e^{itx})' = ite^{itx}}}$

$$= \frac{1}{2ait}\left(e^{ita} - e^{-ita}\right) = \frac{1}{2ait}\, 2i\, \mathrm{Im}\left(e^{ita}\right) =$$

$$= \frac{\cancel{i}}{a\cancel{i}t}\sin(ta) = \frac{\sin(ta)}{at} \qquad \left(z - \bar{z} = 2i\,\mathrm{Im}(z)\right)$$

<u>Remark</u>: i) If $X \sim Bin(n,p) \Rightarrow M_X(t) = \left(pe^t + (1-p)\right)^n$

ii) If $X \sim U(-a,a) \Rightarrow M_X(t) = \frac{1}{2at}\left(e^{at} - e^{-at}\right),\ t \neq 0$

<u>Proof</u>

i) $t \to it \Rightarrow M_X(it) = \left(pe^{it} + (1-p)\right)^n = \varphi_X(t)$.

It's $M_X(t) = E[e^{tX}]$

$\varphi_X(t) = E[e^{itX}]$.

We will see when its possible to get $\varphi_X(t) = M_X(it)$ after we have calculated $M_X(t)$.

ii) hw: replace "t" with "it" and get the eigenfunction

iii) If $X \sim N(0,1)$, then $M_X(t) = e^{t^2/2}$, $\forall t \in \mathbb{R}$.

Assume that $\varphi_X(t) = M_X(it)$ holds. Then:

$$\varphi_X(t) = e^{(it)^2/2} = e^{-t^2/2}, \quad \forall t \in \mathbb{R}$$

<u>Fourier Transformation in $\mathbb{R}^n$:</u>

<u>Notation</u>: $\langle x, y \rangle := \sum_{k=1}^{n} x_k y_k, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n$
$\qquad\qquad\qquad\qquad\qquad\qquad y = (y_1, \dots, y_n) \in \mathbb{R}^n$

<u>Definition</u>: Let μ be a probability measure in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. We call <u>Fourier transformation of μ</u> the function $\hat{\mu} : \mathbb{R}^n \to \mathbb{C}$ with

$$\hat{\mu}(u) = \int e^{i\langle u, x \rangle} d\mu(x) = \int \cos(\langle u, x \rangle) d\mu(x) + i \int \sin(\langle u, x \rangle) d\mu(x) \quad \forall u \in \mathbb{R}^n.$$

<u>Properties</u>

1) $|\hat{\mu}(u)| \le 1$
2) $\hat{\mu}(0) = 1$
3) $\hat{\mu}(u)$ is uniformly distributed function.

<u>Definition</u>: Let X be a random variable in a probability space $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}^n$. We call <u>eigenfunction of X</u>, the function:

$$\varphi_X : \mathbb{R}^n \to \mathbb{C} \text{ such that} : \varphi_X(u) = E[e^{i\langle u, X \rangle}]$$

<u>Proposition</u>: i) $\varphi_X(-u) = \overline{\varphi_X(u)} \quad \forall u \in \mathbb{R}^n$

ii) $\varphi_{AX+b}(v) = e^{i\langle v,b\rangle} \cdot \varphi_X(A^t v)$, $A \in \mathbb{R}^{m\times n}$

$b \in \mathbb{R}^m$

$u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$

iii) If X, Y independent n-dimensional random variables, then:

$$\varphi_{X+Y}(u) = \varphi_X(u)\varphi_Y(u)$$

Uniqueness Theorem: Let μ, ν probability measures in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.
Then $\hat{\mu}(u) = \hat{\nu}(u)$ $\forall u \in \mathbb{R}^n \to \mu = \nu$.

Corollary: If $X, Y \in \mathbb{R}^n$ random variables, then:

$$\varphi_X(u) = \varphi_Y(u), \ \forall u \in \mathbb{R}^n \Rightarrow P_X = P_Y \quad (X \overset{d}{=} Y)$$

Corollary: Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ random variable. Then:

$(X_j)_{1\leq j\leq n}$ independent random variables $\Leftrightarrow$ $\varphi_X(u_1, \dots, u_n) =$

$= \varphi_{X_1}(u_1) \cdots \varphi_{X_n}(u_n)$ $\forall u \in \mathbb{R}^n$

Applications

i) If $X \sim Bin(n,p)$, $Y \sim Bin(m,p)$ and X, Y independent $\Rightarrow$

$\Rightarrow X+Y \sim Bin(n+m, p)$

ii) If $X \sim Gamma(a_1, \theta)$, $Y \sim Gamma(a_2, \theta)$ $\xrightarrow[\text{independent}]{X,Y}$

$\Rightarrow X+Y \sim Gamma(a_1+a_2, \theta)$, $a_1, a_2, \theta > 0$

iii) If $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$ $\xrightarrow[\text{independence}]{X,Y}$ $X+Y \sim N(\mu_X+\mu_Y, \sigma_Y^2+\sigma_X^2)$

Proof

i), ii) Homework

iii) $Z \sim N(0,1) \Rightarrow \varphi_Z(t) = e^{-t^2/2}$ $t \in \mathbb{R}$.

If $X \sim N(\mu, \sigma^2) \Rightarrow X = \mu + \sigma Z$, where $Z \sim N(0,1)$.

So: $\varphi_X(t) = \varphi_{\mu+\sigma Z}(t) \overset{\text{Properties}}{=\!=\!=} e^{it\mu} \varphi_Z(\sigma t) = e^{it\mu} \cdot e^{-\frac{\sigma^2}{2}t^2} = e^{it\mu - \frac{1}{2}\sigma^2 t^2}$.

Therefore:

$$\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t) = e^{it\mu_X - \frac{\sigma_X^2}{2}t^2} \cdot e^{it\mu_Y - \frac{\sigma_Y^2}{2}t^2} =$$

$$= e^{it(\mu_X+\mu_Y) - \frac{\sigma_X^2+\sigma_Y^2}{2}t^2} = e^{it\mu - \frac{\sigma^2}{2}t^2} = \quad \text{, where} \quad \begin{array}{l} \mu = \mu_X + \mu_Y \\ \sigma^2 = \sigma_X^2 + \sigma_Y^2 \end{array}$$

$=$ eigenfunction of $N(\mu, \sigma^2)$

$\Rightarrow X+Y \sim N(\mu, \sigma^2)$ (From Uniqueness Theorem)

Homework (eigenfunctions)

- $X_i \sim P(\lambda_i)$, $1 \le i \le n$, independent $\Rightarrow \sum_{i=1}^n X_i \sim P(\sum_{i=1}^n \lambda_i)$
- $X_i \sim Geo(p)$, $1 \le i \le n$, independent $\Rightarrow \sum_{i=1}^n X_i \sim NegBin(n, p)$
- $X_i \sim Exp(\theta)$, $1 \le i \le n$, independent $\Rightarrow \sum_{i=1}^n X_i \sim Erlang(n, \theta)$

∘ $X = \sum_1^n z_i^2$, where $z_i \sim N(0,1)$, $1 \le i \le n \Rightarrow X \sim X_n^2 \equiv G(\frac{n}{2}, \frac{1}{2})$

Moment generation functions (Ροπογεννήτριες)

Definition: If X real random variable, we call moment generating function the function:

$$M_X : \mathbb{R} \to [0, +\infty] \text{ with } M_X(t) = E(e^{tX})$$

Lemma: i) If $\varepsilon > 0$ and $M_X(\varepsilon) < +\infty \Rightarrow M_X(t) < +\infty \quad \forall t \in [0, \varepsilon]$

Additionally: $E[X^+]^k < +\infty \quad \forall k \ge 1$.

ii) If $\varepsilon > 0$ and $M_X(-\varepsilon) < +\infty \Rightarrow M_X(t) < +\infty \quad \forall t \in [-\varepsilon, 0]$

and $E[X^-]^k < +\infty$, $\forall k \ge 1$

Proof

i) We will show that: $M_X(t) = E[e^{tX}] < +\infty$, $\forall t \in [0, \varepsilon]$

If $\omega : X(\omega) \ge 0$ then for $t \in [0, \varepsilon]$ we have:

$$tX(\omega) \le \varepsilon \cdot X(\omega) \Rightarrow e^{tX(\omega)} \le e^{\varepsilon X(\omega)} < e^{\varepsilon X(\omega)} + 1$$

If $\omega : X(\omega) < 0$ then for $t \in [0, \varepsilon]$ we have:

$$tX(\omega) \le 0 \Rightarrow e^{tX(\omega)} \le e^0 = 1 < 1 + e^{\varepsilon X(\omega)}$$

Therefore: $e^{tX} \le 1 + e^{\varepsilon X}$

$$M_X(t) = E[e^{tX}] \le 1 + E[e^{\varepsilon X}] = 1 + \underbrace{M_X(\varepsilon)}_{< +\infty \text{ (assumption)}} < +\infty$$

- If $w: X(\omega) \geq 0 \to X^+(\omega) = X(\omega)$. Therefore:

$$\frac{\varepsilon^k (X^+(\omega))^k}{k!} = \frac{\varepsilon^k}{k!} X^k(\omega) \leq e^{\varepsilon X(\omega)}$$

If $w: X(\omega) < 0 \Rightarrow X^+(\omega) = 0 \to$ the above inequality also holds

So:

$$\frac{\varepsilon^k (X^+)^k}{k!} \leq e^{\varepsilon X} \Rightarrow (X^+)^k \leq \frac{k!}{\varepsilon^k} e^{\varepsilon X} \Rightarrow$$

$$\Rightarrow E[X^+]^k \leq \frac{k!}{\varepsilon^k} E[e^{\varepsilon X}] = \frac{k!}{\varepsilon^k} \underbrace{M_X(\varepsilon)}_{<+\infty \text{ (assumption)}} < +\infty \quad \forall k \geq 1$$

ii) Homework.

<u>Corollary</u>: Let $D_X = \{t \in \mathbb{R} : M_X(t) < +\infty\}$. Therefore:

i) D_X is an interval with $0 \in D_X$

ii) $E(X^+) = E(X^-) = +\infty$ (when the mean value cannot be defined) $\to$

$\to D_X = \{0\}$.

<u>Proposition</u>: If $\exists \varepsilon > 0$: $M_X(\varepsilon) < +\infty$, $M_X(-\varepsilon) < +\infty$, then:

i) $M_X(t) < +\infty$, $\forall t \in [-\varepsilon, \varepsilon]$

ii) $E|X|^k < +\infty$, $\forall k \geq 1$ (moment function of k-rank exist $\forall k \geq 1$)

iii) $M_X(t) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k$ (with radius of convergence at least ε)

iv) $E[X^k] = M_X^{(k)}(0)$

<u>Proof</u>

i) $\exists \varepsilon>0: \ M_X(\varepsilon)<+\infty \xRightarrow{\text{Prop}} M_X(t)<+\infty \quad \forall t\in[0,\varepsilon]$

$\exists \varepsilon>0: \ M_X(-\varepsilon)<+\infty \xRightarrow{\text{Prop}} M_X(t)<+\infty \quad \forall t\in[-\varepsilon,0]$ $\Bigg\} \Rightarrow$

$\Rightarrow M_X(t)<+\infty \quad \forall t\in[-\varepsilon,\varepsilon]$

ii) $|X| = \begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases} \Rightarrow |x|^k = \begin{cases} x^k, & x\geq 0 \\ (-x)^k, & x<0 \end{cases} \Rightarrow$

$\Rightarrow |X|^k = X^k \mathbb{1}_{(X\geq 0)} + (-X)^k \mathbb{1}_{(X<0)} =$

$= X^k \mathbb{1}^k_{(X\geq 0)} + (-X)^k \cdot \mathbb{1}_{\{X<0\}} =$

$= (X\mathbb{1}_{\{X\geq 0\}})^k + (-X\mathbb{1}_{\{X<0\}})^k = (X^+)^k + (X^-)^k$

(especially for $k=1$, we get: $|X| = X^+ + X^-$)

$\Rightarrow |X|^k = (X^+)^k + (X^-)^k$

Because: $M_X(\varepsilon)<+\infty \xRightarrow{\text{Prop.}} E(X^+)^k<+\infty \quad \forall k$

$M_X(-\varepsilon)<+\infty \xRightarrow{\text{Prop}} E(X^-)^k<+\infty \quad \forall k.$

So: $E|X|^k = \underset{+\infty}{\underset{\wedge}{E|X^+|^k}} + \underset{+\infty}{\underset{\wedge}{E|X^-|^k}} < +\infty \quad \forall k.$

iii) $M_X(t) = E[e^{tX}] = E\left[\sum_{k=0}^{+\infty} \frac{(tX)^k}{k!}\right] \overset{?}{=} \sum_{k=0}^{+\infty} \frac{E(X^k)}{k!} t^k$

→ if this switch can be justified

We will apply Beppo-Levi (for extended functions) if $E(|\cdot|) < \infty$

$$E\left|\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right| \le E\left(\sum_{k=0}^{\infty} \frac{|tX|^k}{k!}\right) = E\left(e^{|tX|}\right) \le E\left(e^{tX} + e^{-tX}\right)$$

<u>Calculation of eigenfunction from moment generating function</u> (where applicable)

<u>Example</u>: $M_X(t) = e^{t^2/2}$ $\forall t \in \mathbb{R}$ when $X \sim N(0,1)$

We would like $\varphi_X(t) = M_X(it) = e^{-t^2/2}$ $\forall t \in \mathbb{R}$

$\varphi_X(t) = E(e^{itX})$ → the domain is $\mathbb{R}$, $it \notin \mathbb{R}$ (generally)

<u>Problem</u>: The domain of $M_X(t)$ needs to be extended and contain it. Observe that: $e^{z^2/2}$ and $e^{|z|^2/2}$ ~~can~~ are defined in $\mathbb{C}$ and for $z = t \in \mathbb{R}$: $e^{z^2/2} \to e^{t^2/2}$, $e^{|z|^2/2} \to e^{t^2/2}$.

So two different complex function coincide in $\mathbb{R}$. We need to find an extension to satisfy our needs.

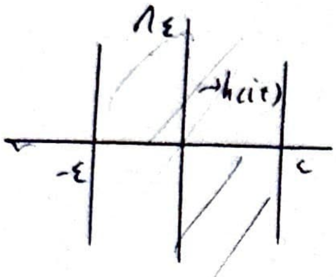
<u>Proposition</u>: Let X be a random variable with a moment generating function M_X. If $\exists \varepsilon > 0$:

i) $M_X(t) < +\infty$ $\forall t \in (-\varepsilon, \varepsilon)$

ii) $\exists$ analytical function $h: \Lambda_\varepsilon \to \mathbb{C}$ where:

$h(t) = M_X(t) \quad \forall t \in (-\varepsilon, \varepsilon)$ where $\Lambda_\varepsilon = \{ z \in \mathbb{C} : |Re(z)| < \varepsilon \}$ then

$\varphi_X(t) = h(it)$.



$$M_X(t) = E(e^{tX})$$

Proof

Let $g(z) = E[e^{zX}] \quad \forall z \in \Lambda_\varepsilon$ και $g: \Lambda_\varepsilon \to \mathbb{C}$.

We will show that g is well defined and analytical in Λ_ε.

g well defined (έχει νόημα η έκφραση $\forall\, \in \mathbb{C}$)

It must hold that:

$$E[e^{zX}] < +\infty.$$

Then obviously $E[e^{zX}] \in \mathbb{C}$.

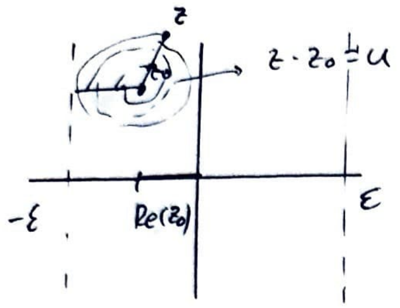
$$E[e^{zX}] = E|e^{(Re(z) + i Im(z))X}| = E[e^{Re(z)X} \cdot e^{i Im(z) X}] \underline{\underline{|e^{i Im(z) X}| = 1}}$$

$$= E[e^{Re(z)X}] = \cancel{E}E[e^{Re(z)X}] = M_X(Re(z)) < +\infty$$

From assumption i), because $|Re(z)| < \varepsilon \Rightarrow Re(z) \in (-\varepsilon, \varepsilon)$

$g(z)$ analytical

Let $z_0 \in \Lambda_\varepsilon$



I must be able to write

$$g(z) = g(\underbrace{z_0 + u}_{z - z_0}) = \sum_{k \geq 0}' a_k u^k \quad \left(a_k = \frac{g^{(k)}(z_0)}{k!} \right)$$

and this will hold for $\forall u \in \Lambda_\varepsilon$, with $\underbrace{u < \varepsilon - |Re(z_0)|}_{\text{circle radius}}$

$$g(z) = g(z_0 + u) = E[e^{(z_0+u)X}] = E[e^{z_0 X} \cdot e^{uX}] = E\left[e^{z_0 X} \cdot \sum_{k \geq 0}' \frac{(uX)^k}{k!}\right] =$$

$$= E\left[\sum_{k \geq 0} \frac{e^{z_0 X} \cdot X^k}{k!} u^k\right] \overset{?}{=} \sum_{k \geq 0} \frac{E[e^{z_0 X} X^k]}{k!} u^k = \sum_{k \geq 0} a_k u^k, \text{ με:}$$

$$a_k = \frac{E[e^{z_0 X} X^k]}{k!}$$

Αρκεί:

$$E\left[\sum_{k=0}^{+\infty} \left| \frac{e^{z_0 X} \cdot (uX)^k}{k!} \right| \right] < +\infty$$

$$E\left[\sum_{k \geq 0} |e^{z_0 X}| \cdot \frac{|uX|^k}{k!}\right] = E\left[|e^{z_0 X}| \cdot \sum_{k \geq 0} \frac{|uX|^k}{k!}\right] = E\left[|e^{z_0 X}| e^{|uX|}\right] =$$

$$= E\left[|e^{Re(z_0) X}| \underbrace{|e^{i Im(z_0) X}|}_{1} \cdot e^{|uX|}\right] = E\left[e^{Re(z_0) X} \cdot e^{|u||X|}\right] \leq$$

$$\leq E\left[e^{|Re(z_0)||X| + |u| \cdot |X|}\right] = E\left[e^{(|Re(z_0)| + |u|)|X|}\right] = E[e^{u^* X}] \leq$$

$$\leq E[e^{u^* X} + e^{-u^* X}] = M_X(u^*) + M_X(-u^*)$$

where $u^* = |Re(z_0)| + |u|$, $|u| < \varepsilon - |Re(z_0)|$

$u^* = |u| + |Re(z_0)| < \varepsilon - |Re(z_0)| + |Re(z_0)| < \varepsilon \Rightarrow$

$\Rightarrow -\varepsilon < -u^* < u^* < \varepsilon \Rightarrow M_X(u^*),\ M_X(-u^*) \underset{i)}{\overset{\text{assumpt}}{<}} +\infty \Rightarrow$

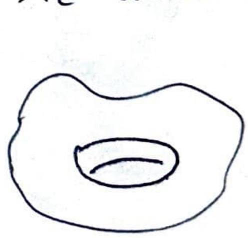
$\Rightarrow E[\Sigma' |\cdot|] < +\infty$.

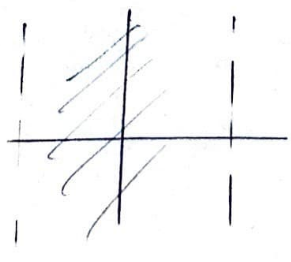
We conclude that $g(z)$ is analytical in Λ_ε. ■

From our assumption ii) we have that h is also analytic in Λ_ε and that $h(t) = M_X(t)$. Additionally, g is analytic in Λ_ε and $g(t) = E[e^{tX}] = M_X(t)$ $\forall t \in (-\varepsilon, \varepsilon)$, so from analytic continuity principle, if g and h are defined in a region (open and connected subset of $\mathbb{C}$) and they ~~are equal~~ coincide in an arc within that region, then $g = h$, when they are analytic in that region.

The g and h are analytic in the region Λ_ε and we have:

$g(t) = h(t)$ $\forall t \in (-\varepsilon, \varepsilon) \subset \Lambda_\varepsilon$
(arc)





Therefore: $g = h$

Therefore: $\varphi_X(t) \overset{def}{=} E[e^{itX}] = g(it) = h(it)$

Applications

1) If $X \sim N(0,1) \Rightarrow \varphi_X(t) = e^{-t^2/2}$ $\forall t \in \mathbb{R}$.

Solution

We have : $X \sim N(0,1) \Rightarrow M_X(t) = e^{t^2/2}$ $\forall t \in \mathbb{R}$. Then:

i) $M_X(t) < +\infty$ $\forall t \in \mathbb{R}$

ii) $h(z) = e^{z^2/2}$ is analytic in $\mathbb{C}$ and $h(t) = M_X(t)$ $\forall t \in \mathbb{R}$

$\xrightarrow{\text{Proposition}}$ $\varphi_X(t) = h(it) = e^{(it)^2/2} = e^{-t^2/2}$ $\forall t \in \mathbb{R}$

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2) If $X \sim Exp(\theta) \Rightarrow \varphi_X(t) = \dfrac{\theta}{\theta - it}$, $\forall t \in \mathbb{R}$.

Solution

If $X \sim Exp(\theta)$, $\theta > 0 \rightarrow M_X(t) = \dfrac{\theta}{\theta - t}$, $\forall t < \theta$ $(\theta > 0)$.

i) $M_X(t) < +\infty$ $\forall t \in (-\theta, \theta)$ (θ-περιοχή)

ii) $h(z) = \dfrac{\theta}{\theta - z}$, is analytic in U_θ.

and $h(t) = M_X(t)$ $\forall t \in (-\theta, \theta)$

$\xrightarrow{\text{Proposition}}$ $\varphi_X(t) = h(it) = \dfrac{\theta}{\theta - it}$, $\forall t \in \mathbb{R}$

Theorem: Let X, Y be real random variables. If $\exists \varepsilon > 0$:

$M_X(t) = M_Y(t) < +\infty$, $\forall t \in (-\varepsilon, \varepsilon)$ then $X \overset{d}{=} Y$.

Proof

It suffices to show that: $\varphi_X(t) = \varphi_Y(t) \quad \forall t \in \mathbb{R} \xrightarrow[\text{χαρακτηρ. ταυτισης}]{\text{Θεωρ.}} X \overset{d}{=} Y.$

From ~~the~~ previous proposition proof we have that:

$g_X(z) = E[e^{zX}]$ is well defined and analytic in Λ_ε (bcs of assumption i)).

Respectively: $g_Y(z) = E[e^{zY}]$ is well defined and analytic in Λ_ε.

However, $g_X(t) = E[e^{tX}] = M_X(t) \overset{\text{assump. ii}}{=\!=} M_Y(t) = E[e^{tY}] = g_Y(t) \quad \forall t \in (-\varepsilon, \varepsilon)$

Because of $(-\varepsilon, \varepsilon) \subset \Lambda_\varepsilon$ we conclude from the analytic continuity principle that $g_X = g_Y$ in Λ_ε.

$$\varphi_X(t) = E[e^{itX}] = g_X(it) \underset{\text{in the imaginary axis}}{\overset{\text{they coincide}}{=\!=\!=}} g_Y(it) = E[e^{itY}] = \varphi_Y(t) \quad \forall t \in \mathbb{R}$$

<u>Exercise 13.3</u>: If X a real random variable such that $E|X| < \infty$ ($\exists$ mean value of X), then show that φ_X is differentiable in ($\mathbb{R}$) and $\varphi_X'(0) = iE[X]$.

<u>Proof</u>

Let $t \in \mathbb{R}$. We want to find $\lim_{h \to 0} \frac{\varphi_X(t+h) - \varphi_X(t)}{h}$.

$$\lim_{h\to 0} \frac{\varphi_X(t+h) - \varphi(t)}{h} = \lim_{h \to 0} E\left(\frac{e^{i(t+h)X} - e^{itX}}{h}\right) = \lim_{h\to 0} E\left[e^{itX} \frac{e^{ihX} - 1}{h}\right] \overset{?}{=}$$

$$= E\left[\lim_{h\to 0} e^{itX} \left(\frac{e^{ihX} - 1}{h}\right)\right]$$

We will apply the Generalized Theorem of Dominance Convergence

i) $\lim\limits_{h\to 0}\left(e^{itX}\cdot\frac{e^{ihX}-1}{h}\right)=e^{itX}\cdot\lim\limits_{h\to 0}\frac{e^{ihX}-1}{h}$.

We set $g(t)=e^{itX}$ $\forall t\in\mathbb{R}$.

$$g'(t)=iXe^{itX}=\lim_{h\to 0}\frac{g(t+h)-g(t)}{h}=\lim_{h\to 0}\frac{e^{i(h+t)X}-e^{itX}}{h}=$$

$$=e^{itX}\cdot\underbrace{\boxed{\lim_{h\to 0}\frac{e^{ihX}-1}{h}}}_{=iX}\Rightarrow$$

$$\Rightarrow \lim_{h\to 0}\left(e^{itX}\cdot\frac{e^{ihX}-1}{h}\right)=iX\cdot e^{itX}.$$

ii) $\left|e^{itX}\cdot\frac{e^{ihX}-1}{h}\right|=\underbrace{|e^{itX}|}_{=1}\cdot\left|\frac{e^{ihX}-1}{h}\right|\overset{?}{\leq}|X|$.

It suffices to show that $|e^{i\theta}-1|\leq|\theta|$ because if that holds

$$|e^{ihX}-1|\leq|h|\cdot|X|\Rightarrow\left|\frac{e^{ihX}-1}{h}\right|\leq|X|\Rightarrow$$

$$\Rightarrow\int_0^\theta e^{it}dt=\frac{1}{i}\left[e^{it}\right]_0^\theta=\frac{1}{i}\left(e^{i\theta}-e^{0}\right)=\frac{1}{i}\left(e^{i\theta}-1\right)\Rightarrow e^{i\theta}-1=i\int_0^\theta e^{it}dt\Rightarrow$$

$$\Rightarrow|e^{i\theta}-1|=\underbrace{|i|}_{=1}\cdot\left|\int_0^\theta e^{it}dt\right|\leq\int_0^\theta\underbrace{|e^{it}|}_{=1}dt=\theta\Rightarrow\boxed{|e^{i\theta}-1|\leq\theta}$$

iii) $E|X|<+\infty$ from assumption, therefore: from the generalized Dominant Convergence Theorem:

$$\lim_{h\to 0}E\left(e^{itX}\frac{e^{ihX}-1}{h}\right)\overset{i)}{=}\underbrace{E\left[iX\cdot e^{itX}\right]}_{=\varphi_X'(t)}\Rightarrow\text{differentiable }\forall t\in\mathbb{R}$$

Especially for $t=0$:

$$\varphi_X'(0) = E[iX \cdot e^{i \cdot 0 X}] = iE(X).$$