Probabilities II / Therezos

<u>Exercise</u>: Let $(X_n),(Y_n)$ real random variables on $(\Omega,\mathcal{A},P)$. Then

(X_n) tight $\wedge$ $Y_n \xrightarrow{d} 0 \Rightarrow X_n\cdot Y_n \xrightarrow{d} 0$ (αντίστοιχο μηδενικής επί φραγμένης)

<u>Solution</u>

We will show that if $(X_nY_n)^{(1)} = X_n^{(1)}\cdot Y_n^{(1)} \subset (X_n, Y_n)$, then:

$\exists (X_nY_n)^{(2)} = X_n^{(2)} Y_n^{(2)} \subset (X_nY_n)^{(1)}$: $X_n^{(2)}\cdot Y_n^{(2)} \xrightarrow{d} 0$ (and then $X_nY_n \xrightarrow{d} 0$)

Let $X_n^{(1)} Y_n^{(1)}$ be such a sequence.

(X_n) tight $\Rightarrow (X_n^{(1)})$ tight $\Rightarrow \exists X_n^{(2)} \subset X_n^{(1)}$ and X: $X_n^{(2)} \xrightarrow{d} X$,

$Y_n \xrightarrow{d} 0 \Rightarrow Y_n^{(1)} \xrightarrow{d} 0 \Rightarrow Y_n^{(2)} \xrightarrow{d} 0$ $\Bigg\} \Rightarrow$

$\xRightarrow{\text{Slutsky}} X_n^{(2)}\cdot Y_n^{(2)} \xrightarrow{d} X\cdot 0 = 0$

Theorem (Lévy Continuity Theorem - LCT) Θεώρημα Συνέχειας Lévy

Let $(X_n), X$ be random variables and φ_{X_n}, φ_X be their characteristic functions (eigenfunctions) in Prev. Lecture. Then

i) $X_n \xrightarrow{d} X \Rightarrow \varphi_{X_n}(t) \longrightarrow \varphi_X(t)$, $\forall t\in\mathbb{R}$

ii) If $\varphi_{X_n}(t) \longrightarrow \varphi(t)$ $\forall t \in \mathbb{R}$ and φ is continuous at 0, then:

$\exists X: \varphi_X(t) = \varphi(t)$ and $X_n \xrightarrow{d} X$

Before the proof we need a Lemma.

<u>Lemma</u>: Let X random variable, φ_X its characteristic function. Then:

$$P\left(|X| > \frac{2}{u}\right) \leq \frac{1}{u} \cdot \int_{-u}^{u} Re(1-\varphi_X(t))\,dt$$

<u>Proof</u>

$$\frac{\sin(ux)}{ux} \leq \left|\frac{\sin(ux)}{ux}\right| \leq \frac{1}{|ux|} \leq \frac{1}{2}, \text{ when } |ux| \geq 2.$$

$$\Rightarrow 1 - \frac{\sin(ux)}{ux} \geq 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow 2\left(1 - \frac{\sin(ux)}{ux}\right) \geq 1 \quad (|ux| \geq 2)$$

$$\Rightarrow \mathbb{1}_{\{|ux| \geq 2\}} \leq 2 \cdot \left(1 - \frac{\sin(ux)}{ux}\right) = \frac{1}{u} \cdot \left(2u - \frac{2\sin(ux)}{x}\right)$$

$\sin(ux) - \sin(-ux)$

$$= \frac{1}{u}\left(\int_{-u}^{u} 1\,dt - \int_{-u}^{u} \cos(tx)\,dt\right) = \frac{1}{u} \int_{-u}^{u} (1 - \cos(tx))\,dt.$$

So by adding $x=0$, we have

$$\mathbb{1}_{\{|ux| \geq 2\}} \leq \frac{1}{u} \int_{-u}^{u} (1 - \cos(tx))\,dt \Rightarrow$$

$$\Rightarrow \mathbb{1}_{\{|uX| \geq 2\}} \leq \frac{1}{u} \int_{-u}^{u} (1 - \cos(tX))\,dt \xRightarrow{E(\cdot)}$$

$$\Rightarrow P\left(|X| \geq \frac{2}{u}\right) \leq \frac{1}{u} E \int_{-u}^{u} (1 - \cos(tX))\,dt$$

We have that

$$E\int_{-u}^{u}\underbrace{|1-\cos(tX)|}_{\leq 2}dt < +\infty$$

So, from Fubini:

$$P\left(|X|\geq \frac{2}{u}\right)\leq \frac{1}{u}E\int_{-u}^{u}(1-\cos(tX))dt = \frac{1}{u}\int_{-u}^{u}(1-E[\cos(tX)])dt$$

Recall that:

$$\varphi_X(t)=E[e^{itX}]=\underbrace{E[\cos(tX)]}_{Re(\varphi_X(t))}+iE[\sin(tX)].$$

So,

$$P\left(|X|\geq \frac{2}{u}\right)\leq \frac{1}{u}\int_{-u}^{u}Re(1-\varphi_X(t))dt$$

Proof of Lévy continuity Theorem (LCT)

"$\Longrightarrow$" $\left(X_n \xrightarrow{d} X \Rightarrow \varphi_{X_n}(t)\longrightarrow \varphi_X(t)\right)$

$$\varphi_{X_n}(t)=E[\cos(tX_n)]+iE[\sin(tX_n)]. \quad (1)$$

We set $f_t(x)=\cos(t\cdot x)$. Obviously $f_t(x)\in C^{B}(\mathbb{R})$. Since $X_n\xrightarrow{d}X$ (by assumption) $\overset{\text{characteristic}}{\overset{d}{\Rightarrow}} E(f_t(X_n))\longrightarrow E(f_t(X))$

$$\Rightarrow E[\cos(tX_n)]\longrightarrow E[\cos(t\cdot X)] \quad (2)$$

Similarly,

$$E[\sin(tX_n)]\longrightarrow E[\sin(tX)] \quad (3)$$

By (2)+(3), (1) gives

$$\varphi_{X_n}(t) \xrightarrow[n\to+\infty]{} E[\cos(tX)] + iE[\sin(tX)] = \varphi_X(t)$$

"$\Leftarrow$"

Step 1: We will show that (X_n) is tight, so

$\forall \varepsilon>0 : \exists M>0 : P(|X_n|>M)<\varepsilon \quad \forall n\geq 1.$

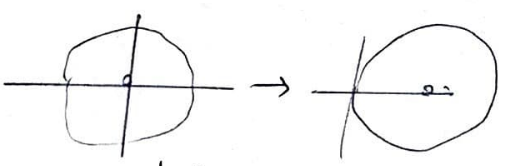
Let $u>0$, then $\forall n\geq 1$, from the previous Lemma

$$P\left(|X_n|>\frac{2}{u}\right) \leq \frac{1}{u}\int_{-u}^{u} Re\left(1-\varphi_{X_n}(t)\right) dt \Rightarrow$$

$$\Rightarrow \limsup_n P\left(|X_n|>\frac{2}{u}\right) \leq \frac{1}{u} \limsup_n \int_{-u}^{u} Re\left(1-\varphi_{X_n}(t)\right) dt.$$

In fact the second $\limsup_n = \lim_n$

$$\varphi_{X_n}(t) \longrightarrow \varphi(t) \Longrightarrow Re\left(1-\varphi_{X_n}(t)\right) \xrightarrow[n\to\infty]{} Re\left(1-\varphi(t)\right).$$

Additionally  $0 \leq Re\left(1-\varphi_{X_n}(t)\right) \leq 2$

$|\varphi_{X_n}(t)| \leq 1$

So,

$$\int_{-u}^{u} Re\left(1-\varphi_{X_n}(t)\right) dt \leq \int_{-u}^{u} 2 \cdot dt < +\infty \xRightarrow[\text{BCT}]{\text{Bounded Conv. Theorem}}$$

$$\Rightarrow \limsup_n \int_{-u}^{u} Re\left(1-\varphi_{X_n}(t)\right) dt = \int_{-u}^{u} Re\left(1-\varphi(t)\right) dt$$

Consequently,

$$\limsup_n P\left(|X_n| > \frac{2}{u}\right) \leq \frac{1}{u}\int_{-u}^{u} Re\,(1-\varphi(t))\,dt$$

Also, φ is continuous at 0 and obviously:

$$Re\,(1-\varphi_{X_n}(0)) = Re\,(1-1) = 0 = Re\,(1-\varphi(0))$$

Let $\varepsilon > 0$. Because of continuity at 0,

$$\exists \delta > 0 : \forall t \in [-\delta, \delta) : 0 \leq Re\,(1-\varphi(t)) \leq |1-\varphi(t)| < \frac{\varepsilon}{2}$$

So for $u = \delta$,

$$\limsup_n P\left(|X_n| > \frac{2}{\delta}\right) \leq \frac{1}{\delta}\int_{-\delta}^{\delta} Re\,(1-\varphi(t))\,dt \leq \frac{1}{\delta}\int_{-\delta}^{\delta} \frac{\varepsilon}{2}\,dt = \frac{1}{\delta}\cdot 2\delta \cdot \frac{\varepsilon}{2} = \varepsilon$$

We conclude that:

$$\limsup_n P\left(|X_n| > \frac{2}{\delta}\right) < \varepsilon.$$

Set $M_0 = \frac{2}{\delta}$. Then $\exists n_0 : \forall n \geq n_0$

$$P(|X_n| > M_0) < \varepsilon.$$

Also, X_n is tight $\forall n = 1, 2, \ldots, n_0 - 1 \Longrightarrow$

$$\exists M_n : P(|X_n| > M_n) < \varepsilon, \quad n = 1, 2, \ldots n_0 - 1.$$

Take $M = \max\{M_0, M_1, \ldots, M_{n_0-1}\}$. Then

$$P(|X_n| > M) < \varepsilon \quad \forall n \geq 1.$$

So, we conclude that (X_n) is tight.

Step 2: We will find X: $\varphi_X(t) = \varphi(t)$.

(X_n) is tight $\to \exists (X_n^{(1)}) \subset (X_n)$ and X random variable:

$$X_n^{(1)} \xrightarrow{d} X.$$

But $\varphi_{X_n^{(1)}}(t) \longrightarrow \varphi(t)$ $\forall t \in \mathbb{R}$ (since $\varphi_{X_n}(t) \longrightarrow \varphi(t) \wedge (X_n^{(1)}) \subset (X_n)$).

Also, by (i)

$$X_n^{(1)} \xrightarrow{d} X \to \varphi_{X_n^{(1)}}(t) \longrightarrow \varphi_X(t) \ \forall t \in \mathbb{R}.$$

So $\varphi_X(t) = \varphi(t)$ $\forall t \in \mathbb{R}$.

Step 3: Now we show that: $X_n \xrightarrow{d} X$. Take

$(X_n^{(2)}) \subset (X_n)$ arbitrary subsequence.

We proved that (X_n) tight $\Rightarrow (X_n^{(2)})$ tight $\to \exists (X_n^{(3)}) \subset (X_n^{(2)})$ and Y random variable such that: $X_n^{(3)} \xrightarrow{d} Y \overset{i)}{\Longrightarrow} \varphi_{X_n^{(3)}}(t) \underset{n\to\infty}{\longrightarrow} \varphi_Y(t)$ $\forall t \in \mathbb{R}$.

So, by Step 2, we have that $\varphi_Y(t) = \varphi(t) = \varphi_X(t) \Rightarrow Y \overset{d}{=} X$. So

$$X_n \xrightarrow{d} X$$

Application

If $X_n \sim Bin(n, p_n)$: $\forall n \geq 1$ such that $np_n \xrightarrow{n\to\infty} \lambda > 0 \Rightarrow X_n \xrightarrow{d} X \sim Poisson(\lambda)$

We will show this using Lévy Continuity Theorem.

$$\varphi_{X_n}(t) = (1 - p_n + p_n e^{it})^n$$

$$\varphi_{X_n}(t) = \Big(1 + \frac{\underbrace{np_n}_{\lambda}(e^{it}-1)}{n}\Big)^n \xrightarrow[n\to\infty]{} e^{c}, \text{ where } c_n \to c$$

$$c_n = np_n(e^{it}-1) \to \lambda(e^{it}-1)$$

$$\Rightarrow \varphi_{X_n}(t) \to e^{\lambda(e^{it}-1)} = \varphi_X(t)$$

where $X \sim \text{Poisson}(\lambda)$ (<u>Homework</u>: show this)

We could show this

(i) directly

(ii) $M_X(t)$, it is valid $t \to it$

Analytical Continuity
Θεώρ. κάτι τέτοιο
Μέση τιμή e^{zX} είναι αυτή που είναι η επέκταση, αναλυτική και κάποια αντίστοιχη επέκταση στο έχει μόνο.
στην θέση του t να βάλω $z \in \mathbb{C}$ και να πω ότι αυτή είναι αναλυτική και λόγω αναλυτικής συνέχειας θα συμπίπτει με την χαρακτηρ.

<u>Remark</u>: Lévy continuity Theorem also holds for sequences of random vectors

<u>Cramer-Wold device</u> (τέχνασμα Cramer-Wold)

<u>Corollary</u>: If (X_n), $X \in \mathbb{R}^k$ random vectors. Then:

$$X_n \xrightarrow{d} X \iff \langle u, X_n\rangle \xrightarrow{d} \langle u, X\rangle, \ \forall u \in \mathbb{R}^k$$

<u>Proof</u>

<u>"$\Rightarrow$"</u> $X_n \xrightarrow{d} X$ holds. Let $u \in \mathbb{R}^k$

$$\langle u, X\rangle = \sum_{i=1}^{k} u_i X_{n,i} = f_u(X_n) \quad f_u \text{ is continuous.}$$

From Continuous Mapping Theorem

$$f_u(X_n) \xrightarrow{d} f_u(X) = \langle u, X\rangle$$

"$\Leftarrow$" By assumption, $\forall u \in \mathbb{R}^k$, $\langle u, X_n\rangle \xrightarrow{d} \langle u, X\rangle$.

We want to show that $X_n \xrightarrow{d} X$. It suffices to show that

$\varphi_{X_n}(u) \longrightarrow \varphi_X(u)$ $\forall u \in \mathbb{R}^k$. Let $u \in \mathbb{R}^k$.

$$\varphi_{X_n}(u) = E\big(e^{i\cdot\underbrace{\langle u, X_n\rangle}_{r.v.}}\big) = \varphi_{\langle u, X_n\rangle}(1) \xrightarrow[n\to\infty]{} \varphi_{\langle u, X\rangle}(1), \text{ since}$$

$\hookrightarrow$ assumption that it's real random var.

$$\varphi_{\langle u, X_n\rangle}(t) \longrightarrow \varphi_{\langle u, X\rangle}(t) \quad \forall t \in \mathbb{R} \quad \varphi_{\langle u, X_n\rangle} = E[e^{it\langle u, X_n\rangle}]$$

$$\left(\langle u, X_n\rangle \xrightarrow{d} \langle u, X\rangle \text{ Lévy Continuity Theorem}\right)$$

and $\quad \varphi_{\langle u, X\rangle} = E[e^{i\langle u, X\rangle}] = \varphi_X(u)$

<u>Theorem</u> (Central Limit Theorem - CLT) Κεντρικό οριακό Θεώρημα.

Let $(X_n)_{n\geq 1}$ i.i.d. with $Var(X_1) \equiv \sigma^2 \in (0, +\infty)$ and $E(X_1) \equiv \mu$. If

$\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$ and $S_n = X_1 + X_2 + \dots + X_n$, then:

$$\sqrt{n}\cdot \frac{\overline{X}_n - \mu}{\sigma} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} Z \sim N(0,1)$$

<u>Remark</u>: It suffices to show this result for $\mu = 0$ and $\sigma^2 = 1$.

Indeed, $\quad \rightarrow$ i.i.d $= z_i = \frac{X_i - \mu}{\sigma}$, $E[z_1] = 0$, $V[z_1] = 1$

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma}\right)}{\sqrt{n}}.$$

Lemma: If X random variable and $E(X^2) < +\infty$, then for $\mu = E(X)$, $\beta = E(X^2)$, $\varphi_X(t) = 1 + it\mu - \frac{t^2}{2}\beta + o(t^2)$, $t \to 0$

Proof

$$\varphi_X(t) = E[e^{itX}] = E\big(1 + itX - \frac{t^2}{2}X^2 + \underbrace{u(t,X)}_{\text{Taylor remainder of order 2}}\big), \text{ so}$$

$$u(t,X) = e^{itX} - 1 - itX + \frac{t^2}{2}X^2.$$

So,

$$\varphi_X(t) = 1 + it \cdot E(X) - \frac{t^2}{2} \cdot E(X^2) + E[u(t,X)]$$

$$= 1 + it\mu - \frac{t^2}{2} \cdot \beta + E[u(t,X)].$$

we should show $= o(t^2)$

$$\cdot \; E[u(t,X)] = t^2 \underbrace{E\left[\frac{u(t,X)}{t^2}\right]}_{\text{need to show} \to 0}$$

family of r.v ~~fts~~ with parameter t.

We will apply κυριαρχημένης σύγκλισης

Facts

i) $\lim_{t \to 0} \frac{u(t,X)}{t^2} = 0$ (from property of Taylor remainder)

ii) $\left| \frac{u(t,X)}{t^2} \right| \overset{\text{we accept}}{\leq} 2X^2$ (see book)

iii) $E(X^2) = 2\beta^2 < +\infty$ by assumption

From DCT; due (i), (ii), (iii)
dominating

$$\lim_{t\to 0} E\left[\frac{u(t,X)}{t^2}\right] = E\left[\lim_{t\to 0}\frac{u(t,X)}{t^2}\right] = 0$$

So $E[u(t,X)] = o(t^2)$ and $\varphi_X(t) = 1 + i\cdot t\cdot \mu - \frac{t^2}{2}\cdot \beta + o(t^2)$

For $\mu = 0,\ \sigma^2 = 1 = \beta \Rightarrow \varphi_X(t) = 1 - \frac{t^2}{2} + o(t^2)$. ∎

Proof of the CLT

We need to show that:

$$\frac{S_n}{\sqrt{n}} = \frac{\sum_{i=1}^{n} Z_i}{\sqrt{n}} \xrightarrow{d} Z \sim N(0,1).$$

So it suffices to show that

$$\varphi_{\frac{S_n}{\sqrt{n}}}(t) \longrightarrow \underbrace{\varphi_Z(t)}_{= e^{-t^2/2}} \quad \forall t \in \mathbb{R}$$

$$\varphi_{\frac{S_n}{\sqrt{n}}}(t) = E\left[e^{it\frac{S_n}{\sqrt{n}}}\right] = E\left[e^{i\frac{t}{\sqrt{n}}\cdot S_n}\right] = \varphi_{S_n}\left(\frac{t}{\sqrt{n}}\right), \text{ with } S_n = \sum_{i=1}^{n} Z_i, \text{ where}$$

Z_i are i.d.d with $\varphi_{Z_i}(t) = 1 - \frac{t^2}{2} + o(t^2)$.

So:

πετάω t^2, το θεωρώ σταθερά γιατί το βλέπω ως προς n

$$\varphi_{\frac{S_n}{\sqrt{n}}}(t) \overset{\text{i.i.d.}}{=} \varphi_{Z_i}^{n}\left(\frac{t}{\sqrt{n}}\right) = \left(1 - \frac{t^2}{2n} + o\left(\frac{1}{n}\right)\right)^n \Longrightarrow$$

$$\Longrightarrow \varphi_{\frac{S_n}{\sqrt{n}}}(t) = \left(1 + \frac{-t^2/2 + n \cdot o\left(\frac{1}{n}\right)}{n}\right)^n = \left(1 + \frac{-t^2/2 + \overbrace{o(1/n)/(1/n)}^{\to 0}}{n}\right)^n \xrightarrow[n\to\infty]{}$$

$$\to e^{-t^2/2} \quad \forall t \in \mathbb{R}.$$

όχι εξωτ. μέτρα
όχι Fourier transform

κυριαρχημένης σύγκλ.
Υπολογισμοί ολοκληρωμάτων } SOS

Inverse Fourier Transform (Αντίστροφος Μετασχ. Fourier)

If $a<\beta$, μ is a probability measure, $\varphi(t) = \int e^{itx} d\mu(x)$, then:

$$\mu((a,\beta)) + \frac{\mu(\{a\}) + \mu(\{\beta\})}{2} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-it\beta}}{it} \cdot \varphi(t)\, dt$$

Proof (See Book)

Remark: If $\mathcal{C} = \{(a,\beta) : a,\beta$, where $\mu(\{a\}) = \mu(\{\beta\}) = 0\}$ (it includes $\emptyset$), then μ is uniquely defined on $\mathcal{C}$.

The 2 properties that we need are:

i) $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$

ii) π-system

Proof

ii) $(a,\beta), (\gamma,\delta) \in \mathcal{C}$. Then $(a,\beta) \cap (\gamma,\delta) = (\max\{a,\gamma\}, \min\{\beta,\delta\})$

$\in \mathcal{C}$ since $\mu(\max\{a,\gamma\}) = \mu(\min\{\beta,\delta\}) = 0$

by $\mu(\{\varepsilon\})=0 \quad \forall \varepsilon \in \{a,\beta,\gamma,\delta\}$

If $\Delta=\{(a,\beta): a<\beta\}$, then $\sigma(\Delta)=\mathcal{B}(\mathbb{R})$.

If $\underbrace{\Delta \subset \sigma(\mathcal{C})}_{\downarrow} \Rightarrow \underbrace{\sigma(\Delta)}_{\mathcal{B}(\mathbb{R})} \subset \sigma(\mathcal{C}) \overset{\subset \mathcal{B}(\mathbb{R})}{\rightarrow} \sigma(\mathcal{C})=\mathcal{B}(\mathbb{R})$

Indeed, if $(a,\beta)\in\Delta$, then $(a,\beta)=\bigcup_n \underbrace{(a_n,\beta_n)}_{\in\mathcal{C}} \in \sigma(\mathcal{C})$

πλήθους

άρα αριθμήσιμο, μέτρο αυτών όταν είναι $\mu(\{i\})>0$, από πυκνότητα $\exists$ τόσοι να κατασκευάζω $a_n \to a$ φθίνουσα ...

Since we can choose $a_n \downarrow a$ and $b_n \uparrow b$ with $\mu(\{a_n\})=\mu(\{b_n\})=0$

Indeed, the set $\{\{a\}: \mu(\{a\})>0\}$ is countable, so it is possible.

So i) is true.

For random variables, $P(a<X<\beta)+\frac{1}{2}\left(P(X=a)+P(X=\beta)\right)=$

$$=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-it\beta}}{it}\varphi_X(t)\,dt$$

<u>Remark</u>: If X is a continuous random variable, then $P(X=x)=0$ $\forall x\in\mathbb{R}$. Then:

$$\forall a,\beta\in\mathbb{R}\ a<\beta,\quad P(a<X<\beta)=\lim\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-ita}-e^{-it\beta}}{it}\varphi_X(t)\,dt$$

<u>Remark</u>: We don't know in general if $\int_{\mathbb{R}}|\varphi_X(t)|dt<+\infty$ or not.

Exercise: Εναλλαγή ολοκλ. με χαρακτηριστική κλπ

If $\int |\varphi_X(t)|\,dt < +\infty$, then X has a probability density function f_X and f_X is bounded with

$$f_X(x) = \frac{1}{2\pi} \int_{R} e^{-itx} \varphi_X(t)\,dt$$

smooth τρόπος να αλλάξω τιμή συνάρτησης σε σημείο

inverse fourier ισχύει
έχει μέγα ισχύς

όσα δεν κάναμε ασκήσεις μέσω παραδ. ή ασκήσεις θα μπουν ταχαν

• θα ∃:

- σ-αλγ Dynkin πώς παίζουν ρόλο σε μέτρο

πώς φτιάχνω γεννητ

κλειστή σε πεπ. τομές έχεια τα prob meas.

ακόμα και σε πεπερ χώρο να φτιάχνει από σ-αλγ

Ποιες διαφορές ορισμοί

και πώς εmp. prob meas.

Lebesgue int

στις μέσες τιμές τ.μ. ολοκλ κατά Lebesgue. (ολοκλ Leb ως προς κάποιο μέτρο

κατανομή τ.μ.: Μέση τιμή και σχεδίαση

παίζει ρόλο σε σχεδίαση ακολουθ. τ.μ.

πλήρως συγκλ είναι βολικό για να δείχνω με πιθ. 1.

Σ συνεχή $\nearrow$ ιδιαίτ συνεχή

$\searrow$ διακρ