# Strong Law of Large Numbers (SLLN) 1.

**Theorem**/ Ισχυρός Νόμος Των Μεγάλων Αριθμών (ΙΝΜΑ)

Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. r.v.'s on $(\Omega, \mathcal{A}, P)$ with $X_n \in \overline{\mathbb{R}}$, $n\geq 1$ and $E|X_1| < +\infty$. Set $\mu = E(X_1)$ and $S_n = \sum_{k=1}^{n} X_k$, $\forall n\geq 1$. Then $\boxed{\overline{X_n} = \frac{S_n}{n} \xrightarrow{a.s.} \mu}$.

**Proof**: We will assume that $E(X_1^2) < +\infty$ (to simplify the proof)

$E(X_1^2) < +\infty \Rightarrow V(X_1) < +\infty$. We set $\sigma^2 = V(X_1)$.

S1) (step 1). We prove the theorem for $X_n \geq 0$.

Set $Y_n = \overline{X_n} - \mu$, $n \geq 1$

$$E(Y_n) = E(\overline{X_n} - \mu) = E(\overline{X_n}) - \mu = E(X_1) - \mu = \mu - \mu = 0$$

$$E(Y_n^2) = V(Y_n) = V(\overline{X_n}) = \frac{V(X_1)}{n} = \frac{\sigma^2}{n}, \quad \sigma^2 < +\infty$$

Take the subsequence $(Y_{n^2})_{n\geq 1}$ of $(Y_n)_{n\geq 1}$.

$$E\left(\sum_{n\geq 1} Y_{n^2}^2\right) \overset{B-L}{=} \sum_{n\geq 1} E(Y_{n^2}^2) = \sum_{n\geq 1} \frac{\sigma^2}{n^2} = \sigma^2 \sum_{n\geq 1} \frac{1}{n^2} < +\infty$$

$\Rightarrow \sum_{n\geq 1} Y_{n^2}^2 < +\infty$ with prob. 1.

$$\Rightarrow Y_{n^2}^2 \xrightarrow{a.s.} 0 \overset{CMT}{\Rightarrow} |Y_{n^2}| \xrightarrow{a.s.} 0 \Rightarrow Y_{n^2} \xrightarrow{a.s.} 0$$

$$\Rightarrow \frac{S_{n^2}}{n^2} - \mu \xrightarrow{a.s.} 0 \rightarrow \boxed{\frac{S_{n^2}}{n^2} \xrightarrow{a.s.} \mu} \quad (1)$$

$\forall k \geq 1$, set $n(k) = [\sqrt{k}]$

But, $[\sqrt{k}] \leq \sqrt{k} \leq [\sqrt{k}] + 1 \Rightarrow n(k) \leq \sqrt{k} \leq n(k)+1$

$$\Rightarrow \frac{1}{n(k)+1} \leq \frac{1}{\sqrt{k}} \leq \frac{1}{n(k)} \Rightarrow \boxed{\frac{1}{(n(k)+1)^2} \leq \frac{1}{k} \leq \frac{1}{n^2(k)}} \quad (2)$$

Since $n^2(k) \leq k \leq (n(k)+1)^2$, we have

$$\boxed{S_{n^2(k)} \leq S_k \leq S_{(n(k)+1)^2}} \quad (3)$$

$(X_i \geq 0)$

$$(4) \quad \boxed{\frac{S_{n^2(k)}}{(n(k)+1)^2} \leq \frac{S_k}{k} \leq \frac{S_{(n(k)+1)^2}}{n^2(k)}}$$

But

- $$\frac{S_{n^2(k)}}{(n(k)+1)^2} = \frac{S_{n^2(k)}}{n^2(k)} \cdot \left(\frac{n(k)}{n(k)+1}\right)^2 \xrightarrow[k\to\infty]{a.s.} \mu \cdot 1 = \mu$$
$$\left(n(k) \xrightarrow[k\to\infty]{} \infty\right)$$

- $$\frac{S_{(n(k)+1)^2}}{n^2(k)} = \frac{S_{(n(k)+1)^2}}{(n(k)+1)^2} \cdot \left(\frac{n(k)+1}{n(k)}\right)^2 \xrightarrow{a.s.} \mu \cdot 1 = \mu$$

By sandwich $\frac{S_k}{k} \xrightarrow[k\to\infty]{a.s.} \mu$

S2] $X_n \in \overline{\mathbb{R}}$ // $X_i = X_i^+ - X_i^-$

$$\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} = \frac{X_1^+ + \dots + X_n^+}{n} - \frac{X_1^- + \dots + X_n^-}{n}$$

$X_n$ i.i.d $\Rightarrow (X_n^+)_{n\geq 1}$ and $(X_n^-)_{n\geq 1}$ are i.i.d.

Also

$$\left.\begin{array}{l} X_1^+ = \max\{0, X_1\} \leq |X_1| \\ X_1^- = \max\{0, -X_1\} \leq |X_1| \end{array}\right\} \Rightarrow \left.\begin{array}{l} (X_1^+)^2 \leq X_1^2 \\ (X_1^-)^2 \leq X_1^2 \end{array}\right\} \Rightarrow \begin{array}{l} E(X_1^{+2}) < +\infty \\ E(X_1^{-2}) < +\infty \end{array}$$

So, the conditions of S1 hold (i.i.d + second moment finite)

$$\left.\begin{array}{l} \dfrac{X_1^+ + \dots + X_n^+}{n} \xrightarrow{a.s.} E(X_1^+) \in \mathbb{R} \\[2ex] \dfrac{X_1^- + \dots + X_n^-}{n} \xrightarrow{a.s.} E(X_1^-) \in \mathbb{R} \end{array}\right\} \Rightarrow \frac{S_n}{n} \xrightarrow{a.s.} E(X_1^+) - E(X_1^-) = E(X_1)$$

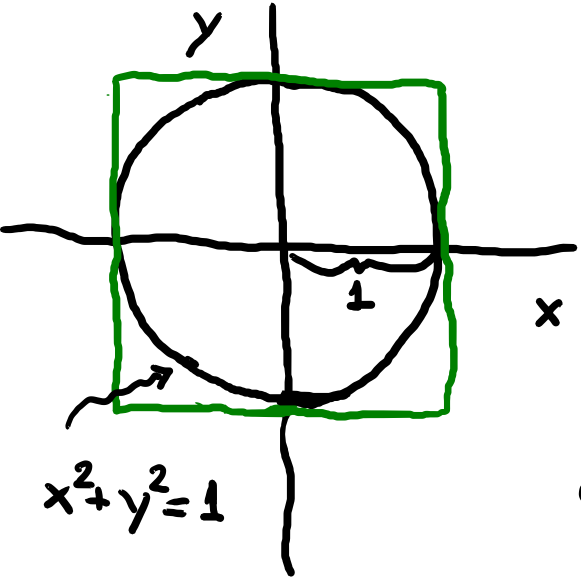
<u>Rem</u> The theorem also holds for $E(X_1) = +\infty$ or $E(X_1) = -\infty$. Only exception if $E(X_1^+) = E(X_1^-) = +\infty$

Εφαρμογή (Μέθοδος Ολοκλήρωσης Monte-Carlo)

Είναι μία μέθοδος προσέγγισης ολοκληρωμάτων, δηλ. της τιμής ενός ολοκληρώματος, που μπορεί να εκφραστεί ως μέση τιμή κάποιας τ.μ.
Προϋποθέτει τη δυνατότητα προσομοίωσης από την κατανομή αυτής της τ.μ.

Παράδειγμα : προσεγγιστικός υπολογισμός του π

$$\pi \simeq 3{,}1415\ldots$$






$$\text{Εμβ.} = \pi \cdot \rho^2 \overset{\rho=1}{=} \pi$$

$$\pi = \iint_{x^2+y^2\le 1} 1 \cdot dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \mathbb{1}_{\{x^2+y^2\le 1\}}\,dx\,dy$$

- Idea: $(X,Y)$ such that $X, Y \sim \mathcal{U}nif(-1,1)$, $X$ indep. of $Y$. $\Big\}$ $f(x,y) = \frac{1}{4}$ $(|x|,|y|\le 1)$

Rewrite $\pi = 4 \cdot \frac{\pi}{4} = 4 \cdot \int_{-1}^{1}\int_{-1}^{1} \mathbb{1}_{\{x^2+y^2\le 1\}} \cdot \underbrace{\frac{1}{4}}_{= f(x,y)}\,dx\,dy$

$= 4\, E\left[\mathbb{1}_{Disk}(X,Y)\right]$, $Disk = \{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$

$$\Rightarrow \quad \frac{\pi}{4} = E\Big[\underbrace{1_{Disk}(X,Y)}_{g(X,Y)}\Big]$$

Approximation via Monte-Carlo

Draw $X_1, X_2, \ldots, X_n \sim Unif(-1,1)$
$Y_1, Y_2, \ldots, Y_n \sim Unif(-1,1)$, all i.i.d.
(from your favorite simulator)

Then $Z_1 = 1_{Disk}(X_1,Y_1), Z_2 = 1_{Disk}(X_2,Y_2), \ldots, Z_n = 1_{Disk}(X_n,Y_n), \ldots$
are i.i.d, $Z_n \sim Be(p)$, $p = \underbrace{P(Z_n=1)}_{= E[1_{Disk}(Z_n)]} = P(X_n^2+Y_n^2 \le 1) = \frac{\pi}{4}$

Take $\bar{Z}_n = \underbrace{\frac{Z_1+\ldots+Z_n}{n}}_{=\text{percentage of points inside disk}} \xrightarrow[SLLN]{a.s.} \frac{\pi}{4}$

For large $n$: $\pi \simeq 4 \cdot \bar{Z}_n$