Probabilities II | Trevezas

Covariance of 2 random variables

Definition: Let $X, Y \in \mathcal{L}^1(P)$ and $E(X, Y)$ well defined. Then the quantity

$$C(X,Y) \text{ or } Cov(X,Y) = E[(X - \underbrace{E(X)}_{\in \mathbb{R}})(Y - \underbrace{E(Y)}_{\in \mathbb{R}})] \in \overline{\mathbb{R}}$$

is said to be the covariance of X and Y. (συνδιακύμανση/συνδιασπορά)

Properties

(1) $C(X,Y) = E(XY) - E(X) \cdot E(Y)$

(2) $C(aX+\beta, \gamma Y + \delta) = a \cdot \gamma \cdot C(X,Y)$

(3) $C(X+Y, Z+W) = C(X,Y) + C(X,W) + C(Y,Z) + C(Y,W)$

Proposition

$$X, Y \in \mathcal{L}^2(P) \rightarrow |C(X,Y)| \leq \sqrt{V(X)}\sqrt{V(Y)}$$

Proof

$$|C(X,Y)| = |E[\underbrace{(X-\mu_X)}_{\tilde{X}}\underbrace{(Y-\mu_Y)}_{\tilde{Y}}]| \overset{C-S}{\leq} (\underbrace{E(\tilde{X})^2}_{V(X)})^{1/2}(\underbrace{E(\tilde{Y})^2}_{V(Y)})^{1/2}$$

Modes of Convergence of sequences of random variables (Τρόποι σύγκλισ

i) $X_n \xrightarrow{a.s.} X \overset{def}{\Leftrightarrow} P(\lim X_n = X) = 1$, where

$$\{\lim X_n = X\} = \{\omega \in \Omega : \exists \lim X_n(\omega) \text{ και } \lim X_n(\omega) = X(\omega)\}$$

$X_n \xrightarrow{\mathcal{L}^p} X \leftrightarrow \forall p \in (0,+\infty),\ E(|X_n - X|^p) \xrightarrow[n\to\infty]{} 0$ ίσα $(E(|X_n - X|^p))^{1/p} \to 0 \to E(|X_n - X|^p \to 0$

$(d_p(X_n, X) \to 0)$ or $\forall p \in (0, +\infty],\ \|X_n - X\|_p \to 0$

($p \in (0,1)$ δεν είναι νόρμα)

Για $p=0$, $E|X_n - X|^0 = P(X_n \neq X)$, $\|X_n - X\|_p = (E|X_n - X|^p)^{1/p}$.

(We had defined ~~esssup~~ $\|X_n - X\|_p$ $= \text{ess sup } |X_n - X|$

$(\|X\|_\infty = \text{ess sup}|X| = \inf\{M \geq 0 : |X| \overset{a.s.}{\leq} M\}$.

All r.v.'s are defined on the same probability space $(\Omega, \mathcal{A}, P)$.

<u>Particular cases of interest</u>

$p=1$, $E|X_n - X| \to 0$ (convergence in mean - σύγκλιση κατά μέσο)

$p=2$, $E(X_n - X)^2 \to 0$ (σύγκλιση κατά μέσο τετράγωνο- convergence in mean square)

$p=\infty$ $\|X_n - X\|_\infty \to 0$ (συγκλίνει σχεδόν ομοιόμορφα ή ομοιόμορφα με πιθανότητα 1 - converges almost uniformly)

$p=0$ $E|X_n - X|^0 = P(X_n \neq X) \to 0$ (converges in $\mathcal{L}^0$).

$\quad\quad\quad\quad\quad\quad\quad \| \quad d_0(X_n, X)$

iii) $X_n \xrightarrow{P} X \leftrightarrow \forall \varepsilon > 0,\ P(|X_n - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$ ποσοστό είναι πιθανότητα εκτός της λωρίδας πλάτους (βάζοντας όλα τα ω) από την ε-ζώνη

or equiv. $\forall \varepsilon > 0$ $P(|X_n - X| \leq \varepsilon) \xrightarrow[n\to\infty]{} 1$

(Since those properties are limit properties, we have:

$$P(|Y_n - X| > \varepsilon) = P(|X_n - X| \geq \varepsilon) \to 0$$)

We say that (X_n) converges in probability (κατά πιθανότητα) to X or stochastic (στοχαστικά)

if $X_n \xrightarrow{c} X \Leftrightarrow \forall \varepsilon > 0 \quad \sum_{n \geq 1} P(|X_n - X| > \varepsilon) < +\infty$

c: completely

$$\Leftrightarrow \lim_{n \to \infty} \sum_{k \geq n} P(|X_k - X| > \varepsilon) = 0$$

$\left(\sum_{n} a_n \in \mathbb{R} \Leftrightarrow \lim_{k \geq n} \sum_{k \geq n} a_k = 0 \right.$ όρος ουράς μηδενικός$\left. \right)$, $a_n \geq 0$

Then we say that (X_n) converges completely to X. (it is used as a tool to deduce a.s. convergence)

<u>Proposition</u> (characterisation of $\xrightarrow{a.s.}$)

$$X_n \xrightarrow{a.s.} X \overset{(1)}{\Leftrightarrow} \forall \varepsilon > 0, P(\limsup_n |X_n - X| > \varepsilon) = 0$$

$$\overset{(2)}{\Leftrightarrow} \forall \varepsilon > 0, P(\underbrace{\sup_{k \geq n} |X_k - X|}_{Y_n} > \varepsilon) \to 0$$

$$\overset{(3)}{\Leftrightarrow} \sup_{k \geq n} |X_k - X| \xrightarrow{P} 0 \qquad \left(|X_n - X| \leq \sup_{k \geq n} |X_k - X| \right)$$

$$\overset{(4)}{\Leftrightarrow} \forall \varepsilon > 0 \; P\left(\bigcup_{k \geq n} \{ |X_k - X| > \varepsilon \} \right) \to 0$$

Proof

(1) $X_n \xrightarrow{a.s.} X \Leftrightarrow |X_n - X| \xrightarrow{a.s} 0 \underset{(*)}{\Leftrightarrow} \limsup_n |X_n - X| \overset{a.s}{=} 0$

$(*)$ $0 \le \liminf |X_n - X| \le \limsup |X_n - X|$. & $\underset{\Uparrow\ \limsup|X_n - X|}{\limsup} \overset{a.s.}{=} 0 \Rightarrow \underset{\Uparrow\ \text{ομοίως}}{\liminf} \overset{a.s}{=} 0 \Rightarrow$

$\Rightarrow |X_n - X| \to 0$.

$\Leftrightarrow \forall \varepsilon > 0, \limsup |X_n - X| \overset{a.s}{\le} \varepsilon$ (also for $\varepsilon = 1/n$ $1 + \forall \varepsilon > 0$ and $X \le 0$

$X \overset{a.s}{\le} 0 \Rightarrow X \overset{a.s}{=} 0$. Then for $\varepsilon = 1/n$, $X \overset{a.s.}{\le} 1/n$,

$P(X > 0) = \lim_n \underbrace{P(X > \tfrac{1}{n})}_{0} = 0$)

$\Leftrightarrow \forall \varepsilon > 0, P(\limsup |X_n - X| > \varepsilon) = 0$

(2) Let $\varepsilon > 0$.

$\{\limsup |X_n - X| > \varepsilon\} = \{\omega \in \Omega : \limsup_n |X_n^{(\omega)} - X^{(\omega)}| > \varepsilon\}$ $\overset{\text{we work}}{\underset{\text{with } \ge \varepsilon}{\Longleftrightarrow}}$

$\limsup |X_n(\omega) - X(\omega)| \ge \varepsilon \Leftrightarrow \underset{\text{inf}}{\lim_n} \sup_{k \ge n} |X_k(\omega) - X(\omega)| \ge \varepsilon$

(lim φθίνουσα με όριο inf αυτή)

$\Leftrightarrow \forall n \ge 1 \quad \sup_{k \ge n} |X_k(\omega) - X(\omega)| \ge \varepsilon$

$\Leftrightarrow \omega \in \bigcap_{n \ge 1} \{\sup_{k \ge n} |X_k - X| \ge \varepsilon\}$.

So,

$$\{\limsup |X_n - X| \geq \varepsilon\} = \bigcap_{n\geq 1} \underbrace{\{\sup_{k\geq n} |X_k - X| \geq \varepsilon\}}_{\text{decreasing seq. of events}}$$

We conclude that

$$P(\limsup |X_n - X| \geq \varepsilon) = P\left(\bigcap_{n\geq 1} \{\sup |X_k - X| \geq \varepsilon\}\right)$$

$$= P\left(\lim_n \{\sup |X_k - X| \geq \varepsilon\}\right)$$

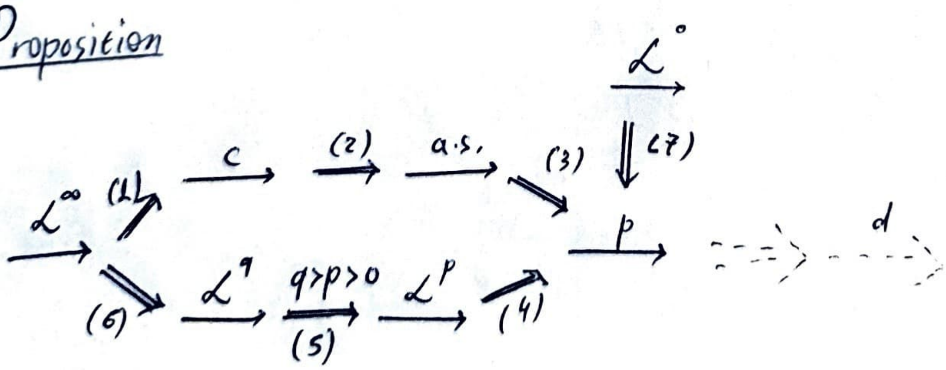
$$= \lim_n P(\sup |X_k - X| \geq \varepsilon)$$

(3) Let $Y_n = \sup_{k\geq n} |X_k - X|$, Then equivalently $Y_n \xrightarrow{P} 0$

(4) $\sup_{k\geq n} |X_k(\omega) - X(\omega)| \geq \varepsilon \Leftrightarrow \omega \in \bigcup_{k\geq n} \{|X_k - X| \geq \varepsilon\}$. So

$\{\sup_{k\geq n} |X_k - X| \geq \varepsilon\} = \bigcup_{k\geq n} \{|X_k - X| \geq \varepsilon\}$, so (2) $\Leftrightarrow$ (4).

<u>Proposition</u>



Additionally, no implication holds as an equivalence. Without supplementary conditions.

Proof

(1) $\xrightarrow{\mathcal{L}^\infty} \to \xrightarrow{c}$

Let $\varepsilon>0$. Then

$$\exists n_0(\varepsilon): \forall n \geq n_0, \quad \|X_n - X\|_\infty \leq \varepsilon.$$

Also, $|X_n - X| \overset{a.s.}{\leq} \|X_n - X\|_\infty$, so

$$\forall n \geq n_0 \quad |X_n - X| \overset{a.s.}{\leq} \varepsilon \to P(|X_n - X| > \varepsilon) = 0 \quad \forall n \geq n_0.$$

We deduce that:

$$\sum_{k \geq n_0} P(|X_n - X| > \varepsilon) = 0, \quad \forall n \geq n_0.$$

$$\Rightarrow \lim_{n\to\infty} \sum_{k \geq n} P(|X_k - X| > \varepsilon) = 0 \Rightarrow \sum_{k \geq n} P(|X_k - X| > \varepsilon) \overset{<+\infty}{\Rightarrow} X_n \xrightarrow{c} X.$$

(2) $\left(\xrightarrow{c} \to \xrightarrow{a.s}\right)$

Reminder: $X_n \xrightarrow{a.s} X \Leftrightarrow \forall \varepsilon > 0 \; P\left(\bigcup_{k \geq n} |X_k - X| > \varepsilon\right) \to 0.$

By assumption,

$$X_n \xrightarrow{c} X \to \overset{\forall \varepsilon > 0}{\lim_{n}} \sum_{k \geq n} P(|X_k - X| > \varepsilon) = 0.$$

But

$$P\left(\bigcup_{k\geq n} |X_k - X| > \varepsilon\right) \underset{\text{additivity}}{\overset{\sigma\text{-sub}}{\leq}} \sum_{k\geq n} P(|X_k - X| > \varepsilon) \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow X_n \xrightarrow{a.s.} X.$

(3) Let $\varepsilon > 0$.

$$P(|X_n - X| > \varepsilon) \leq P\left(\sup_{k\geq n} |X_k - X| > \varepsilon\right) \xrightarrow[n\to\infty]{(*)} 0$$

(*) by assumption and the equiv. characterisation $X_n \xrightarrow{a.s.} X \Leftrightarrow \sup_{k\geq n} |X_k - X| \xrightarrow{P} 0$.

($0 \leq R_n \overset{a.s.}{\leq} S_n$ και $S_n \xrightarrow{P} 0$, Then: $R_n \xrightarrow{P} 0$

$0 \leq |X_n - X| \leq \sup_{k\geq n} |X_k - X| \xrightarrow{P} 0$)

(4) ($\xrightarrow{\mathcal{L}^p} \Rightarrow \xrightarrow{P}$)

Let $\varepsilon > 0$, $0 < p < +\infty$

$$P(|X_n - X| > \varepsilon) = P(|X_n - X|^p > \varepsilon^p) \overset{\text{Markov}}{\leq} \frac{E|X_n - X|^p}{\varepsilon^p} \xrightarrow[n\to\infty]{\text{by assumption}} 0$$

$\Rightarrow X_n \xrightarrow{P} X.$

(5) ($\xrightarrow{\mathcal{L}^q} \overset{q>p>0}{\Longrightarrow} \xrightarrow{\mathcal{L}^p}$)

If $X_n \xrightarrow{\mathcal{L}^q} X \Rightarrow E|X_n - X|^q \longrightarrow 0 \Rightarrow \|X_n - X\|_q \longrightarrow 0 \xRightarrow[\substack{\|Y\|_p \leq \|Y\|_q \\ (0<p<q<+\infty)}]{\text{Lyapunov}}$

$\|X_n - X\|_p \to 0 \Rightarrow E|X_n - X|^p \to 0.$

So $X_n \xrightarrow{\mathcal{L}^p} P$ (τα ανήκουν σε αυτόν τον χώρο αφού έχουν σύγκλιση, έχουν δηλαδή πεπερασμένες ροπές κλπ. Θέλω σίγουρα, τελικά η ακολουθία να βρίσκεται μέσα).

(6) $\left(\xrightarrow{\mathcal{L}^\infty} \Rightarrow \xrightarrow{\mathcal{L}^q} \right)$

$0 \le \|X_n - X\|_q \le \|X_n - X\|_\infty .$

If $X_n \xrightarrow{\mathcal{L}^\infty} X \Rightarrow \|X_n - X\|_\infty \to 0 \xRightarrow{\text{ineq}} \|X_n - X\|_q \to 0 \Rightarrow X_n \xrightarrow{\mathcal{L}^q} X.$

(7) $\left(\xrightarrow{\mathcal{L}^0} \to \xrightarrow{P} \right).$

Let $\varepsilon > 0$.

$0 \le \sout{P(X_n \ne x)}\, P(|X_n - X| > \varepsilon) \le P(X_n \ne x) \xrightarrow{\text{by assumpt}} 0 \Rightarrow X_n \xrightarrow{P} X$

Counterexamples

(1) $\xrightarrow{\mathcal{L}^p} \not\Rightarrow \xrightarrow{a.s.}$, $p \in [0, +\infty)$, so

$\xrightarrow{P} \not\Rightarrow \xrightarrow{a.s.}$ $\left(\begin{array}{l} \xrightarrow{\mathcal{L}^p} \Rightarrow \xrightarrow{P} \\ \text{if } \xrightarrow{P} \Rightarrow \xrightarrow{a.s.} \end{array} \right\} \text{we showed this} \Rightarrow \xrightarrow{\mathcal{L}^p} \Rightarrow \xrightarrow{a.s.} \Big)$

Let $(\Omega, \mathcal{A}, P) = ((0,1), \mathcal{B}(0,1), \lambda)$ and set

$X_n = \mathbb{1}_{A_n}$

where (A_n) is the "triangular" sequence of partitions of $(0,1)$ with successively decreasing length $\frac{1}{2}$.

We will show that

$$X_n \xrightarrow{\mathcal{L}^p} X \Leftrightarrow E|X_n - X|^p \to 0, \text{ where } X = 0.$$

We have:

$$X_n \xrightarrow{\mathcal{L}^p} 0 \Leftrightarrow E|X_n|^p \to 0$$

, και είναι το λ μέτρο σε χώρο

shown.

$$E|X_n|^p = E(\mathbb{1}_{A_n}^p) = E(\mathbb{1}_{A_n}) \overset{P(A_n)}{=} \lambda(A_n) \xrightarrow[n\to\infty]{} 0.$$

<u>But</u>, $X_n \overset{a.s.}{\not\to} 0$, because $\limsup X_n = \limsup_n \mathbb{1}_{A_n} = 1 \ \forall \omega \in (0,1)$

($\mathbb{1}_{A_n}(\omega) = 1$ for infinite n).

Indeed $X_n \overset{a.s.}{\not\to} 0$.

(2) $\xrightarrow{a.s.} \not\Rightarrow \xrightarrow{\mathcal{L}^p}$, $0 < p \leq +\infty$ and $\xrightarrow{\mathcal{L}^0} \not\Rightarrow \xrightarrow{\mathcal{L}^p}$.

In the space of (1),

$$X_n = n^{1/p} \mathbb{1}_{(0,\frac{1}{n})}.$$

Obviously, $X_n(\omega) \xrightarrow[n\to\infty]{} 0, \ \forall \omega \in (0,1) \Rightarrow X_n \xrightarrow{a.s.} 0$.

But

$$E(|X_n - 0|^p) = E(X_n^p) = E(n \cdot \mathbb{1}_{(0,\frac{1}{n})}) = n \cdot \lambda((0,\tfrac{1}{n})) = n \cdot \frac{1}{n} = 1 \underset{n\to\infty}{\not\to} 0$$

$$\Rightarrow X_n \overset{\mathcal{L}^p}{\not\to} 0$$

But also: $P(X_n \neq 0) = \lambda(0, \frac{1}{n}) = \frac{1}{n} \to 0.$

So, $X_n \xrightarrow{\mathcal{L}^0} 0.$

(3) $\mathcal{L}^\infty \not\Rightarrow \mathcal{L}^0$, so $\underset{(\neq \mathcal{L}^0)}{\xrightarrow{\cdot}} \not\Rightarrow \xrightarrow{\mathcal{L}^0}$.

($\cdot$ → whatever convergence)

Take $X_n(\omega) = \frac{1}{n}, \forall \omega \in \Omega.$

$$\|X_n - X\|_\infty \leq \sup_{\omega \in \Omega} |X_n(\omega) - 0| = \frac{1}{n} \to 0 \Rightarrow X_n \xrightarrow{\mathcal{L}^\infty} X$$

($\|\cdot\|_\infty$ = ess sup)

$$P(X_n \neq 0) = P(\tfrac{1}{n} \neq 0) = 1 \not\to 0.$$

So $X_n \overset{\mathcal{L}^0}{\not\to} X$

Assume $\underset{\neq \mathcal{L}^0}{\xrightarrow{\cdot}} \Rightarrow \xrightarrow{\mathcal{L}^0}$. Then $\xrightarrow{\mathcal{L}^\infty} \Rightarrow \underset{(\neq \mathcal{L}^0)}{\xrightarrow{\cdot}}$. So

$\xrightarrow{\mathcal{L}^\infty} \Rightarrow \xrightarrow{\mathcal{L}^0}$ impossible contradiction.