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ORIGINAL ARTICLE

Eliciting mathematical reasoning during early primary problem solving

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Abstract

Mathematical reasoning, which plays a critical role in students' capacity to make sense of mathematics, is now emphasised more strongly in various curricula internationally. However, reasoning is sometimes difficult for teachers to recognise, let alone teach. This case study considers video of one teacher's implementation of a problem-solving lesson in a year 1 primary school class in Australia. It examines the opportunities this teacher provided to leverage reasoning and contributes to the body of knowledge on ways reasoning may be elicited during problem solving. The new Eliciting Mathematical Reasoning Framework arising from the analysis of the data in this study builds on and extends previous research. It provides a tool to support researchers, teacher educators, professional learning providers, and teachers in recognising and eliciting reasoning.

Keywords Eliciting reasoning · Eliciting Mathematical Reasoning Framework · Problem solving · Reasoning within problem solving · Development of reasoning · Early years primary

Introduction

Brodie (2010) asserted that mathematical reasoning (MR) was crucial in understanding mathematical concepts and flexibly using mathematical ideas and procedures to reconstruct prior mathematical knowledge. A renewed emphasis on reasoning is evident in curricula in recent years (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2017; Common Core State Standards Initiative, 2010; Department for Employment and Education (DfEE), 2014). For example, "Students are reasoning mathematically when they explain their thinking"s (ACARA, 2017); "[s]tudents

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at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments" (Common Core, 2010); "[students can d]escribe simple patterns and relationships involving numbers or shapes; decide whether examples satisfy given conditions" (DfEE, 2014); and students are using reasoning to connect new mathematical understandings with prior learning "when they adapt the known to the unknown, when they transfer learning from one context to another" (ACARA, 2017). However, Jeannotte and Kieran (2017) reported that the descriptions of MR in these curricular documents are imprecise and inconsistent, likely contributing to findings of Stylianides et al. (2007), Clarke et al. (2012), Loong et al. (2017) and Herbert et al. (2015) that many primary teachers display confusion regarding the nature of MR. It is important to support teachers' developing pedagogical practices to foster students' reasoning (Stylianides et al., 2007) and to overcome challenges teachers face when implementing lessons that build students' mathematical understandings (Lampert, 2001).

Reasoning takes place in classrooms when teachers (and researchers) use problem-solving approaches to encourage students to build mathematical understandings (e.g. Lampert, 2001; Wood et al. 2006). Problem solving, for the purposes of this study, is an activity associated with working to solve unfamiliar, challenging problems (Liljedahl, 2016). Such problem-solving lessons provide teachers with opportunities to elicit reasoning, which may not be available in other approaches to teaching mathematics (Lampert, 2001). In the problem-solving lesson that provided the data for this *Reasoning Eliciting Study*, reasoning was elicited but was not the explicit focus. The teacher's focus was instead on eliciting student communication of mathematical ideas that these students were developing. Given the identified confusion of some teachers with the nature of reasoning, and the identified existence of reasoning as part of problem-solving, this study explores the nature of the eliciting of reasoning in a context in which complex reasoning terminology was not employed. This study addresses the following research question:

> What was the nature of this teacher's eliciting of reasoning during this problem-solving lesson?

An outcome of this study is the *Eliciting Mathematical Reasoning Framework* that provides insights into different types of eliciting of reasoning and illustrates their operationalisation in an early-years classroom. It is expected that such a framework will be useful to teachers, professional learning providers, teacher educators, and researchers working to increase teacher understanding of the nature of MR. The potential for an increase in the frequency, with which reasoning is elicited in lessons through guidance provided by *Eliciting of Mathematical Reasoning Framework*, could increase teachers' familiarity with reasoning actions of students and increase teachers' understanding of MR.

Background

Mathematical reasoning

Brodie (2010) argued that reasoning arguments are developed to convince others of a claim, solve problems or bring together several ideas. Reid (2002) asserted that "[d]eveloping mathematical reasoning is central to mathematics education" (p. 5). The opportunities teachers provide for students to reason mathematically can enhance the development of their students' reasoning (Long et al., 2012; Reid, 2002; Stylianides et al., 2013). Classroom culture is influential in the development of reasoning when students are expected to clearly communicate their reasoning (Kilpatrick, et al., 2001) by expressing their ideas; explaining and justifying their thinking; and identifying flaws in others' thinking (Long et al., 2012).

In order to increase consistency between the various meanings of "mathematical reasoning", Jeannotte and Kieran (2017) conceptualised a model of MR for school mathematics. Their model structures the previously vague and sometimes contradictory descriptions of MR into two aspects: structural and process. The structural aspect which is not relevant to this study is "the form in which the reasoning is expressed, be it deductive, inductive, or abductive" (p. 9). Their process aspect of reasoning is relevant to this study. It includes exemplifying, comparing, classifying, identifying patterns, justifying, generalising, conjecturing and proving—reasoning actions relevant to primary school mathematics (Lampert, 2001).

The reasoning actions within Jeanotte and Kieran's (2017) process aspect are interrelated. Comparing and classifying cases enable identification of patterns and the formation of conjectures to be justified. Conjecturing includes a statement or collection of statements that are thought to be true but not yet known to be true (Taylor & Garnier, 2016), a reasoning action associated with *generalising*. *Justifying* includes the evaluation of conjectures (Winsler et al., 2007) using logical arguments to convince others of the validity of a claim (Jeannotte & Kieran, 2017; Mata-Pereira & da Ponte, 2017). Logical arguments can be considered proving when they "progress from empirical to deductive arguments through a dialectic between conjecturing and proving" (Stylianides & Stylianides, 2017, p. 122). *Generalising* involves moving from a few cases to making statements about a wider collection of cases or examining one case in detail to make sense of interrelationships between instances more generally (Jeanotte & Kieran, 2017; Wood et al., 2006). In Wood et al.'s study in a third-grade classroom, they report that few (3 out of 148 inquiry/arguments) involved "Constructing Synthesizing—Formulate mathematical arguments to explain discovered patterns" (p. 232), that is, the essence of generalising.

"Mathematical situations often present an overabundance of information, visual cues, and possible patterns, making it impossible to process everything at once" (Lobato et al., 2013, p. 809). Krutetskii (1976) identified the "mental activity" of *analysing*, describing it as a preliminary process of exploring a problem by breaking it down into parts, "to generalise mathematical relations one must first dismember them" (p. 228). Krutetskii further identified the simultaneous consideration of several elements as a more complex type of analysis. Thus, Krutetskii provides actions (breaking into parts, considering simultaneously) additional to Jeanotte and Keiran's (2017) analytical processes of exemplifying, comparing and classifying to explore a problem. Exemplifying supports generalising from conjectures and validating (Jeanotte & Keiran, 2017). Comparing and classifying involve consideration of the examples generated in exploring a problem (exemplifying) to identify relationships (Mason, 1982) between the examples to produce conjectures. Students may search purposefully and systematically with the intention of verifying their conjectures. Krutetskii (1976) described capable students' approaches to solving a problem as progressing from *analysing* to *generalising*: "they isolate different elements in its [the problems'] structure [*analysing*], assess them differently, systematize them, … [and] seek out mathematical relationships [*generalising*]" (p. 227). Krutetskii identified even more complex processes, associated with simultaneous consideration of several elements, that were employed for purposes of judgement [*validating*].

For the purposes of this study, Jeannotte and Kieran's (2017) process aspects of MR and Krutetskii's (1976) analysing actions have been grouped into three broad types of reasoning: *analysing*, *validating* and *generalising* (see Table 1) to form the theoretical framework of reasoning processes employed.

The MR processes and actions and description displayed in Table 1 have been used to categorise the different types of reasoning elicited, to inform the coding of the data.

Eliciting reasoning during mathematical problem solving

The focus of this study is the eliciting of reasoning during problem solving, where problem solving is the process of grappling with new and unfamiliar tasks where the means of solution are unknown and/or there is not one specific solution (open task) (Silver, 1997). Silver described *Eliciting* as "the set of teaching actions that serve the function of drawing out students' mathematical ideas" (p. 111). Although many researchers have referred to reasoning terminology (e.g. explain, justify; make decisions, conjecture) within problem-solving activity in schools (Clarke & Clarke, 2003; English & Gainsburg, 2015; Lesh et al., 2000; Lithner, 2017; Schoenfeld, 1992), very few (e.g. Lampert, 2001) explicitly and consistently identify these as reasoning actions during the implementing of problem solving. Terms like "Conjecture: Reasoning" (p. 369) are part of Lampert's usual classroom language included on worksheets and used in discussions during problem solving, to explain students' reasoning.

Eliciting of students' mathematical ideas occurs during Williams' (2014) *Engaged to Learn* (E2L) approach which was developed to increase students' mathematical understandings through problem solving. The teacher does not provide mathematical input during students' activity, but rather employs open questioning, and drawing of attention, with an absence of hinting, telling, affirming, and querying (see Williams, 2020). Within E2L, students work in small groups on non-routine, complex but accessible tasks that provide opportunities for explorations in which groups select which

Table 1 Theoretical framework of reasoning processes (informed by Jeannotte and Kieran (2017) and Krutetskii (1976))

MR process	MR actions	Description
Analysing		Involves exploring a problem by deconstructing it into essential parts (analysing), then simultaneous analysing of these parts (synthetic-analysis) (Krutetskii, 1976; Williams, 2007)
	Exemplifying	"Supports other MR processes by inferring examples" (Jeannotte & Kieran, 2017, p. 14)
	Comparing	"Infers, by the search for similarities and differences, a narrative about mathematical objects or relations" (Jeannotte & Kieran, 2017, p. 11)
Validating	Justifying	"Aims at changing the epistemic value (i.e., the likelihood or the truth) of a mathematical narrative" (Jeannotte & Kieran, 2017, p. 1) "Involves searching for data, warrant, and backing to allow for modifying the epistemic value of a narrative" (Jeannotte & Kieran, 2017, p. 12)
	Proving	"Modifies the epistemic value of a narrative from likely to true" (Jeannotte & Kieran, 2017, p.12)
Generalising		"Infers narratives about a set of mathematical objects or a relation between objects of the set from a subset of this set" (Jeannotte & Kieran, 2017, p. 9)
	Classifying	"Infers, by the search for similarities and differences between mathematical objects, a narrative about a class of objects based on mathematical properties and definitions" (Jeannotte & Kieran, 2017, p. 11)
	Identifying a pattern	"Infers a narrative about a recursive relation between mathematical objects or relations" (Jeannotte & Kieran, 2017, p.10)
	Conjecturing	"Infers a narrative about some regularity with a likely or probable epistemic value and that has the potential for mathematical theorization" (Jeannotte & Kieran, 2017, p. 10)

Table 2 Thought processes during mathematical problem solving (Woods et al., 2006)

Mathematical thinking	Examples of cognitive activity
Recognizing comprehending	+ Understand concepts behind taught idea or known strategy
Recognizing applying	+ Know when to use a known mathematical idea
Building-with analyzing	+ Apply known mathematical procedures in a new context + Solve using a problem with a slight twist + Familiarise self with problem using specific numerical examples + Systematise the numerical results and search for patterns
Building-with synthetic-analyzing	+ In contrast and comparison of two methods for the difference + Interconnect various representations, operations and assumptions + Use more than one pathway to solve a problem + Produce an independent generalisation—"small discovery" + Analyse one case or form a guiding principle to formulate a new rule
Building-with evaluative-analyzing	+ Interconnect solution pathways for the purpose of identifying flaws and strengthening arguments + Pull together ideas for making a judgement + Evaluate whether a method or result is reasonable, efficient or elegant
Constructing synthesizing	+ Formulate mathematical arguments to explain discovered patterns + Explore the problem from many perspectives rather than just work towards a solution + Integrate concepts to create new thoughts or ideas (new insight). Could vary in: + Number of concepts involved + Diversity of the domains that concepts were drawn from + Size of the conceptual leap + Spontaneity with which the process is undertaken + Progressively explore the problem to continually develop new insights
Constructing evaluating	+ Progressively reflect on the situation as a whole for the purpose of recognizing inconsistent information and/or finding a more elegant solution pathway + Reflect on the process of problem solution for the purpose of recognizing its limitations and its application to other contexts + Reflect on the solution pathway developed and its possible contribution to generic mathematical processes to employ in the future

mathematics, and which representations, to use. Through group reports and subsequent whole class discussion, a variety of mathematical ideas and representations are linked. The eliciting of mathematical thinking during the E2L approach is theoretically framed by *Thought processes during problem solving* (Wood et al., 2006) adapted from Williams (2002), which builds upon Dreyfus et al. (2001) and Krutetskii (1976) (see Table 2).

Strong synergies are evident between reasoning processes (Table 1) and mathematical thinking employed during problem solving (Table 2). The reasoning process of *analysing* is synonymous with *Building-with* actions *of analyzing*, and *synthetic-analyzing*, whilst *validating* is consistent with the *Building-with* action of *evaluative-analyzing* and *generalising* and *proving* align with *Constructing* (*synthesizing* and *evaluating*).

Methodology

In this *Reasoning Eliciting Study*, "we feel that we might get insight into the [research] question by studying a particular case" (Stake, 1995, p. 3, 4). The "case" is the bounded system (Stake, 1995) of the teacher's (Earl's[1]) interactions with his year 1 class of boys and girls (in a small rural government primary school in Australia) during one 80-min problem-solving lesson (hitherto referred to as the "case" lesson). This *Reasoning Eliciting Study* sits within a broader study of a 6-year, whole-school professional learning program (PLP)—that included Earl's school—where teachers experimented with the *Engage to Learn* (E2L) approach to problem solving under the guidance of the PLP leader, Williams.

Herbert selected this lesson for this *Reasoning Eliciting Study* because, on inspection of lesson videos of the broader study, Williams identified frequent eliciting of reasoning in this lesson even though the term "reasoning" was not included in the talk of the teacher or the students during the lesson.

The findings provide insights into types of reasoning eliciting activity undertaken by this one teacher, in one lesson (Stake, 1995), rather than a complete list of all possible types of eliciting of reasoning actions. A case study approach is "open to the use of theory or conceptual categories that guide the research and analysis of data" (Meyer, 2001, p. 331). This case study is theoretically framed by literature related to reasoning (see Table 1) and thought processes during problem solving (see Table 2).

Earl and his class

Earl, with over 5 years teaching experience, participated in the PLP about eliciting mathematical thinking rather than reasoning. He developed his own refinements of E2L, informed by his previous observing, trialing, discussing and reflecting on problem-solving lessons implemented and observed by teachers during the PLP. The case lesson was one such lesson. Students worked in groups of two to four selected by the teacher with roles allocated by Earl. Roles included recorder, reporter, encourager and timekeeper. The class, year 1 students (21 students aged 5–7 years) from two composite classes (foundation/prep, year 1), undertook problem solving with Earl with once a week as part of the usual school program.

Note: Letters have been used to identify students, e.g. Student A.

Content of "case" lesson

Prior to the "case" lesson Earl had read the first page of the storybook "The Doorbell Rang" (Hutchins & Keating, 1986): "I've made some [12] cookies for tea," said

[1] Pseudonym.

Ma. “Good,” said Victoria and Sam. “We’re starving.” “Share them between yourselves,” said Ma. “I made plenty.” (p. 1).

He asked “how many whole biscuits will Sam and Victoria have each?”.

This single solution question focused students’ attention on the 12 cookies, how the cookies might be shared and types of language and representations they might decide to use in communicating to the class what they had done.

“Case” lesson

The “case” lesson followed this introductory lesson with the same storybook employed as a stimulus for students’ thinking. In the “case” lesson, Earl read the first three pages of the book, finishing with “‘No one makes cookies like Grandma,’ said Ma [and were just about to eat the 12 cookies] as the doorbell rang”. Earl set the book aside and set the groups to work on the task “with your group you’ll need to choose how many people are at the door and how Sam and Victoria will work out how everyone can get the same amount of whole cookies”.

Data collection

The lesson was video-recorded since “participant verbal reports [alone] of conversations, behaviors and events distort and fail to include details necessary for deep understanding of the processes under study” (Woodside, 2010, p. 9). Four video cameras were employed to capture the classroom interactions. The data for this article was from the video focused on the teacher that captured the reporter at the board, the class sitting on the floor and the teacher. The video images were sufficiently clear for identification of inscriptions, such as drawings, symbols and writing, which group reporters attended to on group worksheets as they reported to the class. The audio from the video was transcribed verbatim and used in conjunction with the video images to identify instances of eliciting of reasoning.

Data analysis

For the purpose of this study, “eliciting of mathematical reasoning” was considered to occur where teacher actions could result in student reasoning (whether or not it did). It was identified through the video data, supplemented by the transcript because “[s]peech and gesture together can often provide a clearer and more accurate picture” (Kelly et al., 2002, p. 22).

Our fine-grained analysis is presented in a similar structure to Powell et al. (2003), but with the primary focus of the analysis on the eliciting of reasoning rather than the development of mathematical thinking. The phases of analysis were:

- Herbert presented the research focus and video illustrations to the broader project team.
- The broader project team, including researchers with a background in reasoning, discussed incidences of eliciting of reasoning seen in the video.

- Herbert repeatedly viewed the video (Akerlind et al., 2005) to identify eliciting of reasoning and code as types of reasoning (Table 1), referring to the transcript as necessary (Herbert & Pierce, 2013; Herbert et al., 2015).
- Williams (informed by class observation, repeated viewing of the video, discussion session with broader team, and the lesson transcript) reviewed Williams' coding.
- Herbert and Williams discussed and came to consensus where there were discrepancies in their code allocations.
- Herbert selected representative illustrations of each code category to include in the paper that were clear and concise.
- Williams identified and coded the types of mathematical thinking (Table 2) that might occur in response to the eliciting activity (Williams, 2007)
- Williams and Herbert discussed and where necessary refined the coding of mathematical thinking.

Like Pea (2006), this analysis provides more "complete records of complex phenomena than earlier methods" (p. 1325).

Results

This section presents the findings from the analysis of the interactions during the reporting session recorded in the video data. Where the data is sufficient to make inferences { } brackets are used to enclose the researchers' inferences. In response to the task (see "Methodology") different groups chose various numbers of people at the door including 4, 6, 8, 10, 12, and 17.

Student D: We think there were 10 people at the door.
Student E: there are four people [at the door]
Student F: We thought 17 was how much people were at the door.

Earl asked the students to "work out how many whole cookies each person should get to be fair". Working with their chosen number of people and 12 cookies, groups investigated the problem in whatever way they chose. Earl employed the E2L approach as groups explored his question, concluding the lesson with a reporting session. The reporter from each group successively attaching their group's A3 worksheet to the board for the purpose of explaining their group's thinking to the class. Earl asked questions of each reporter and then encouraged other class members to ask questions. Figure 1 shows an image taken from the video illustrating one group's work on this problem which includes diagrammatic, numerical and verbal representations.

The results are presented in the order in which MR processes are displayed in Table 1 (*analysing*, *validating* and *generalising*) with associated thought processes (Table 2) indicated in { }. Excerpts of interactions are included to provide evidence of these eliciting of reasoning actions. The quotes, which illustrate the types of comments and questions Earl employed, are not neces-

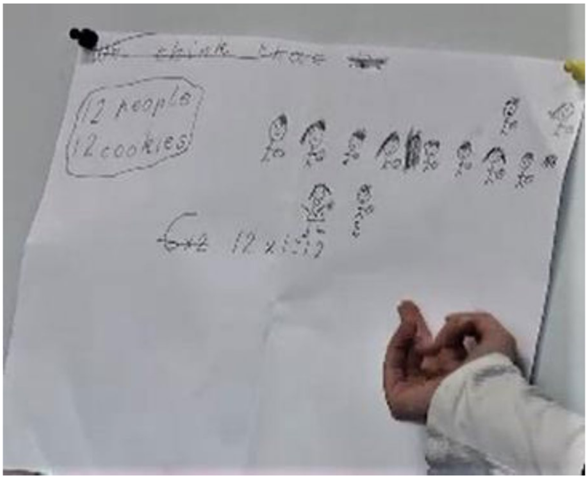

Fig. 1 Reporter linking numerical and diagrammatic representations utilised by her group

sarily sequential so "*" is used to separate the quotes. Where the quotes are sequential, they are grouped together with a single asterisk and the line of space between these quotes has been omitted.

Eliciting actions for mathematical reasoning

Crucial to the eliciting of types of reasoning was Earl's encouraging of students' contributions which resulted in their communication, both oral and other forms, of their mathematical thinking. Through the process of coding outlined above, these three main eliciting categories emerged:

- Eliciting types of reasoning (with subcategories analysing, validating and generalising)
- Eliciting communication of reasoning (with subcategories oral and other representations)
- Eliciting reasoning through encouraging student contributions (with subcategories share thoughts with class, value contribution, share ideas in easy to understand ways, understand ideas of others)

Eliciting types reasoning

Data illustrating each of the three subcategories *analysing*, *validating* and *generalising* are now presented.

Analysing

At the beginning of the lesson, students were encouraged to explore the problem posed by Earl.

** Earl: With your group this morning, you need to choose how many people are at the door so that everyone can get the same amount of whole cookies.*

During the reporting session, students were encouraged to listen and try to understand the thinking of others, their solution processes and their solutions. Opportunities for reasoning were further encouraged by Earl communicating that class members asking questions of reporters helped to deepen the thinking that occurred.

> * Earl: Now yesterday whilst people were reporting, the questions that were being asked really helped us to understand the maths and really really encouraged the reporter to think a little bit deeper about what they had done, so I'm going to encourage that today.

The following excerpt illustrates how Earl's eliciting encouraged students to *explain* their analysis of the problem more fully.

> * Student D: We think there is 10 people at the door.
> *Earl: What did that mean?* {pose question to elicit mathematical meaning about group focus}
> Student D: That means they get one cookie each. There's 2 people inside and 10 people outside [holding up 2 fingers] which makes 12 people {elicit connections between cookies and people}.
> Earl: Is there a sum that you know that would show that? {elicit links between representations}
> Student D: [Reads $10+2=12$ from worksheet] Ten plus two equals 12
> *Earl: 12 cookies altogether so what does that mean for the people?* {elicit link between cookies and people in numeric representation}
> Student D: They all get one each.

This exchange, where Earl's questions elicited Student D's explanation, is one of the many instances that occurred in the reporting stage of the lesson where a student responded to Earl's question with further explanation of their group's analysis of the problem, thus demonstrating Earl's eliciting of reasoning from the students in his class in several ways.

The following excerpt illustrates Earl's eliciting of *analysing* (inspecting component parts) to find out more through simultaneously thinking about symbolic and physical representations (synthetic-analysis) and his encouragement of class members to ask questions to assist them in understanding more.

* [Student F's inscription seen in the video]: 3 + 3 + 3 + 3 + 3 + 2 = 17.

Earl: I'm wondering if some people might like to think about that. There're some questions already happening about this work. {Elicit thinking from others about what they do or do not understand for the purpose of asking questions to understand more}.

Validating

Also evident in the data were instances where Earl's eliciting encouraged his students to justify (validate) their solutions. Illustrations of eliciting of validating are given next.

Earl expected other students to consider groups' findings and test the reasonableness of the work presented. The following exchange demonstrates how Earl's eliciting led to a student justifying their group's solution processes and solution, indicating Earl's success in eliciting this reasoning process.

* *Earl: What have you done and why?* {eliciting of justification of mathematics used}.
Student E: I've done that because there are four people and this is my box [pointing to one circle] and this is Jacob's box this is Victoria's box and this is Sam's box and they've each got three each and it shares the twelve.
Earl: Can you show us how it is twelve? {further eliciting of justification of mathematics used}.
Student E: I just did it before 3, 6, 9, 12 [showing one finger for each count].
Earl: Are you counting by something?
Student E: Yes counting by threes.
Earl: I saw you write something there and it's not something we do all the time but I think it's pretty important for the rest of these guys to see it. As you were drawing these circles you were writing some numbers there. What were those numbers? {Eliciting justification of representation employed}
Student E: Those were just counting numbers like 1,2,3,4,5,6 … 12 [pointing to each number inside the circles].

This interaction demonstrates eliciting communication of *Recognizing* an appropriate selection of number of people, and appropriate mathematics in various representations, and of *Build-With* through connecting representations to justify. Earl requested explanation "what have you done?" and justification "why?" (first line of this exchange). When the student provided a partial justification, indicated by "because", Earl elicited further explanation about the step not explained (*Can you show us how it is twelve?*). Earl's request for further elaboration (*Are you counting by something?*) contributed to the explanation by drawing attention to the counting the student had done, but not mentioned. Earl's question "what were those numbers?" was intended to elicit connections between diagrammatic and symbolic representations (*synthetic-analysis*), but this question did not explicitly elicit that connecting since the student responded with only names symbolic representation but not its meaning in this context. That said, the student had connected cookies and people in the first line of the transcript, indicating awareness of the meanings of this mathematics in the context under study.

Generalising

Whilst there are many examples in the data of Earl eliciting justifying, there are very few where he was identified to be eliciting generalising. Groups chose a variety of numbers of people at the door so there was some potential for Earl to elicit evidence of various generalisations (e.g. if there are more than 12 people it is not possibly to give out whole cookies fairly; sometimes there are cookies left). In the following exchange with a reporter, Student B, Earl attempted to draw out mathematical thinking that had the potential to lead to conjecture then generalisation.

* Earl: How many people at the door. 17 that's a lot of people at the door so what's going to happen with the cookies? {eliciting Building-With: analyzing and synthetic-analyzing (simultaneously considering cookies and people)}.

Student B: They are all going to get one each {it can be inferred that synthetic-analysis has not occurred for this student}.

Earl: One each? How many cookies are there?

Student B: 17

Earl: Why, is my question! Why did your group decide that 17 was not going to work out? Why?

The inferred intent of this eliciting activity was to encourage connections between number of people and the number of cookies (synthetic-analysis), for the purpose of highlighting that 17 does not work. Finding why it does not work could open up an opportunity to conjecture, test, and finally make a generalisation such as "if there are more than 12 people, whole cookies cannot be shared fairly". Student B was unclear about the thinking about this that occurred in his group.

Eliciting communication of reasoning

Earl accepted communication through any representation, for example, oral, gestural, symbolic or diagrammatic. Data illustrating each of the subcategories of *communication* are now presented (*oral communication* and *communicating through other representations*).

Oral communication

Integral to Earl's eliciting of reasoning was his expectation of sharing thinking with other group members, and communicating of group thinking to the whole class, with students contributing comments and questions to elicit further elaboration from reporters. The following quotes demonstrate Earl's belief in the importance of the students communicating ideas clearly. He encouraged students to listen to each other and to try to understand their ideas. These foci illustrate Earl's eliciting of students' mathematical thinking through the sharing of each group's thinking and his encouragement of students' questions as crucial to the class developing understandings.

* Earl: Okay, lots of great ideas coming out there [Student A] so let's just go really quite slowly so that [Student B] and [Student C] can get up to speed as much as they can be- [Student B], do you understand all the maths that is going on here? {expecting students to communicate in inclusive ways}
* Earl: Listen, it's important to work together. … I think you'll need to describe. You'll need to be really clear for this guy … yesterday whilst people were reporting the questions that were being asked really helped us to understand the maths {Valuing contributions from class members}.

Earl's requests for further explanations and for reasons why certain actions were taken were interspersed as appropriate.

Earl encouraged class members to ask the reporter questions to clarify their explanations. He also encouraged the reporters to elaborate their answers to these questions.

* Earl: There're some questions already happening about this work so Student F, people with their hands up [indicating Child F should choose]

Student F: [indicating which student would respond]: Student E.

Student E: What's the 1 + 1 + 1 + for?

Earl: Oh you're seeing some of the maths crossed out. Student F, this stuff up here [pointing to work sample] what was that?

Student F: People, then crossed it out.

Earl: So these were the people and you [your group] crossed them out. How many people were you working out with this?

Student F: 17.

Earl: What did you discover? Why did you cross it out? {further eliciting of thinking about 17}.

Earl then used this opportunity to elicit explaining of why 17 people did not work.

Throughout the entire lesson, Earl demonstrated the expectation that students should communicate their reasons. This quote illustrates that expectation.

* Earl: Which bit are you going to talk about champ? Okay, tell us about why that one? [Eliciting of justification of selection of number of people].

The classroom norms Earl established emphasised the importance of students developing logical understanding (Skemp, 1979): communicating their mathematical thinking in their group and to the class as a whole.

Communicating through other representations

Pictorial representations were common. Groups used either drawings only, combinations of drawings, words and number symbols or explicitly linked pictorial representations with numeric representations. Representing of reasoning in diverse ways that enabled eliciting of connections between these representations was evident. Diversity of representations was encouraged and added to the richness of the reporting of groups' solution processes.

The following eliciting from Earl demonstrated his valuing of a wide variety of representations.

* Earl: Student C—the group work was pretty much normal and they had all the working out and their maths ideas were all recorded on sheets and from these [worksheets] and from the important maths thinking.
* Earl: Can you show us that [on the diagram] somehow?
* Earl: If you've changed your ideas. If you've tried something but you weren't quite sure if it worked, please don't scratch it out. That might actually help another group or that might help you later on to see some of the thinking you were doing.
* Earl: Sometimes we draw different diagrams so that we can get thinking about our work.

These exchanges illustrate that Earl is prepared to accept a diagram as a form of *communicating* of reasoning, and written forms of "working out". His statements indicate that all working towards a solution is acceptable and valued, which is emphasising the importance of interim thinking.

The structure of Earl's lesson and the evidence of his classroom norms show that Earl is accepting verbal responses to questions and that these may be connected to other representations. Earl's question "anything else you can tell us?" uses "tell" more broadly than only a verbal response. Diagrammatic and symbolic responses were also used to communicate solution paths.

* Earl: You have been experimenting with that symbol and that operation a little bit.
* Earl: And you've got something else on your sheet that I think that you might be able to share with the group. What's all that down the bottom there?

Student D: Twelve times one equals twelve [pointing to $12 \times 1 = 12$ on work sample].

Earl: Could you explain 12 times $1 = 12$ what does that mean?

These statements and questions suggest that Earl has noticed and accepted symbolic representations as a suitable way to communicate students' thinking. When considered in conjunction with other questions and comments Earl made during the lesson, it is most likely that he is trying to help the class to connect verbal, diagrammatic and symbolic representations to increase students' understandings of symbolic representations. Earl's eliciting of reasoning about why particular mathematics was used and the form in which it was communicated (*Recognizing*) go beyond explanation of known mathematical procedures. Students select and combine mathematics in unfamiliar ways as they progress towards curtailing their mathematical thinking (synthesis) to communicate their new ideas through connections between representations.

Eliciting reasoning through encouraging student contributions

This eliciting of reasoning process occurred through action types which encouraged students to participate in reasoning actions*: Sharing ideas in easy to understand ways*, *Valuing contributions*, and *Understanding ideas of others.*

Share thoughts with class:

* Earl: Now yesterday … the questions that were being asked really helped us to understand the maths … so I'm going to encourage that today.

Value contribution:

* Earl: Okay, lots of great ideas coming out there [Student A].

Share ideas in easy to understand ways:

* Earl: So let's just get really quite slowly so that [Student B] and [Student C] can get up to speed as much as they can be.

Understand ideas of others:

* Earl: [Student B], do you understand all the maths that is going on here?

Such multiple actions to establish and sustain the classroom culture were employed before, during and after teacher eliciting of student reasoning. Earl's activity elicits mathematical thinking as he encourages students to:

- share their thoughts ["*lots of great ideas coming up*"] rather than only completed ideas;
- think more deeply about what they have found *["[your] questions really helped us to understand"*];
- share their own ideas in easy to understand ways *["[go] quite slowly so … [others] can get up to speed"*]; and
- understand the ideas of others ["*do you understand all the maths that is going on here"*].

He demonstrated that he considered it important that all students have opportunity to understand, and that he was monitoring this.

By valuing all contributions, it appears that Earl has increased the likelihood that groups put forward tentative ideas. This should progress the development of ideas faster. In doing so, there is potential for students to develop a stronger understanding of mathematics relevant to the task, and further understanding of explorations undertaken by their group and other groups. Encouraging elaborations that include the connecting of representations (*Building-with: synthetic-analysis*) and the justifying of ideas reported (*Building-with: evaluative-analysis*) should increase class members' understandings of ideas. A question that remains unanswered is whether *Constructing*, *synthesis*, would occur in later lessons in this problem-solving task when students have generated further instances of different numbers of people at the door to support *generalising* from patterns generated.

Summary of Analysis

Illustrations of Earl's eliciting of mathematical reasoning actions have been mapped against the categories: *Eliciting types of reasoning*, *Eliciting communication of reasoning* and *Encouraging student contribution* to communicating mathematical ideas and sustaining these communications, which could be considered classroom norms that set the scene for reasoning. This encouraging of student contributions was crucial to eliciting of reasoning because it valued inclusivity, encouraged interim thinking including partially developed ideas and made transparent the expectation that the class value the same. This task enabled group choice of numbers of people at the door, solution processes employing multiple representations (including mathematical arguments), autonomous selection of modes of communication and opportunities for both teacher and students to elicit reasoning. As such, it is potentially useful for others interested in illustrations of eliciting of reasoning, especially in early years, and as a tool to experiment with to develop such skills.

Earl's strategies for progressing mathematical understanding included elicited student responses that (a) shared, explained and discussed what they had done (*analysing*); (b) gave reasons for the mathematics they used (*validating*); (c) encouraged *conjecturing* (that could lead to *generalising*); (d) employed communication through different *representations* (and connected representations); and (e) contributed by *sharing*. Eliciting of reasoning was embedded in Earl's eliciting of mathematical thinking. In addition, Earl explicitly sustained classroom norms that encouraged students to reason by valuing their ideas, providing opportunities to learn and encouraging them to take on eliciting roles themselves. Earl highlighted the usefulness of such activity throughout the lesson. Earl's faith in his students was demonstrated by his statements indicating that all students would be able to understand if ideas were clearly presented.

Discussion

Findings from this study add to the growing literature connecting reasoning and problem solving by emphasising the role of the eliciting of reasoning in developing new (to the students) mathematical understandings during problem-solving activity. Previous research connecting problem solving and reasoning has focused differently to this study, by not specifically focusing on the *eliciting* of reasoning (e.g. Yackel & Cobb, 1996; English & Gainsburg, 2015; Schoenfeld, 1992; Lithner, 2017; Lesh et al., 2000; Lampert, 2001).

The Eliciting Mathematical Reasoning Framework

The main contribution of this study is the *Eliciting Mathematical Reasoning Framework* that has emerged, which brings together and extends previous research on MR to include the eliciting of MR during problem solving in the early years.

Table 3 Williams and Herbert's (2021) Eliciting Mathematical Reasoning Framework

Category of eliciting actions	Purposes of teacher's eliciting of reasoning	Illustration of teacher's actions and associated mathematical thinking elicited (see Key below)
1. Types of reasoning: analysing, validating or generalysing	• Enable student autonomy in the selection of the mathematics they explore	• Choose how many people are at the door so that everyone can get the same amount of whole cookies (R)
	• Elicit explanations clarifying reasoning	• What did that mean? (R). … So would you be able to explain what the numbers were about and how it will all work? (BWsa)
	• Elicit steps during student-controlled solution processes	• What happened next? What did your maths show you? (Ra, BWsa)
	• Elicit student focus, why mathematics selected, interpretations of mathematics generated, why these interpretations	• 12 cookies altogether so what does that mean for the people? [$10+2=12$ on worksheet] (BWsa)
	• Elicit justification of choices, and suitability of mathematical approach	• What have you done and why? (BWea)
	• Elicit further justification	• Can you show us how it is twelve?
	• Elicit explanations to increase likelihood of class conjecturing/hypothesising	• Okay, tell us about why that one? (BWea)
	• Provide opportunities to start to form conjectures	• You thought about 17. … What did you discover? (BWea)
	• Elicit a response to a counter example	• 17 that's a lot of people at the door so what's going to happen with the cookies?
2. Eliciting communication of reasoning: oral and other representations Intended to increase the quality of communication, and reasoning, by encouraging reporters to elaborate further Valuing different representations in explanations	• Elicit from reporter communication previous discussion between small group and teacher	• I noticed on your page you had this sum …. Do you know why this sum? (BWsa and/or BWea)
	• Elicit thinking/questions from others about the finding of one group	• Does anyone have any questions about that? (Ra, BWea)
	• Valuing students' independent thinking	• I like what you're thinking
	• Share thoughts	• Does anyone have any questions about that? (Ra, BWea)
	• Expecting clear communication of strategies and solution processes from reporters and other students • Eliciting links between representations	• Listen, it's important to work together. … I think you'll need to describe. You'll need to be really clear for this guy Sometimes we draw different diagrams so that we can get thinking about our work … And you've got something else on your sheet that I think that you might be able to share with the group. What's all that [diagram] down the bottom there? (BWsa)

Table 3 (continued)

Category of eliciting actions	Purposes of teacher's eliciting of reasoning	Illustration of teacher's actions and associated mathematical thinking elicited (see Key below)
3. Eliciting reasoning through encouraging student contributions increases the likelihood that students articulate their reasoning	• Establish and sustain classroom norms	• We ask questions and have lots of discussion, that's the maths stuff we want to record
	• Explicit statements of expectations that class share ideas in easy to understand ways, and understand ideas of others	• Your words and your explaining would be [about] something that your group found really helpful today
	• Value all contributions	• Yesterday whilst people were reporting … questions … being asked really helped us to understand the maths … [and] encouraged the reporter to think a little bit deeper
	• Structure of problem-solving lesson indicates expectation that students' thinking will be shared	• Is there a way you can really show that? When you share is there a way that you will be able to explain your maths thinking? (Ra and/or BWa)
	• Eliciting to facilitate that one student's explanation could be understood by other students	• So let's just get really quite slowly so that [Student B] and [Student C] can get up to speed as much as possible
	• Eliciting responses to check a student's understanding of another student's ideas	• Do you understand all the maths that is going on here? (Ra and/or BWa)
	• Valuing contributions from individuals and other class members	• Okay, lots of great ideas coming out there

Key: R (Recognizing; "a" analysing to identify relevant mathematics); BW (Building-With, Analysis; "a" analysis involving breaking into component parts, "sa" synthetic-analysis, "ea" evaluative-analysis); C (Constructing, includes synthesis "s", evaluation "e")

The *Eliciting Mathematical Reasoning Framework* (Table 3) was formulated through the simultaneous analysis of eliciting of reasoning actions (Table 1), thought processes during problem solving (Table 2) and additional actions identified during the analyses of data. This framework synthesises findings arising from answering the research question: “What was the nature of this teacher’s eliciting of reasoning during this problem-solving lesson?” It classifies the nature of the eliciting of reasoning that occurred into three categories: *Eliciting of types of Reasoning*, *Eliciting communication of reasoning* and *Eliciting reasoning through encouraging student contributions* (Table 3, column 1); documents the purposes for which the teacher employed this eliciting of reasoning (column 2); and illustrates the teacher’s eliciting of reasoning activity and the mathematical purposes such eliciting of reasoning addressed (column 3). *Eliciting communication of reasoning* is consistent with the emphasis on communication of reasoning advocated in curriculum documents (ACARA, 2017; Common Core State Standards Initiative, 2010; Department for Employment and Education (DfEE), 2014) and thus adds support to the usefulness of the framework for teachers applying curriculum requirements.

The first two categories in this framework (*Eliciting types of reasoning* and *Eliciting communication of reasoning*) explicitly refer to the eliciting reasoning. The third category (*Eliciting reasoning through encouraging student contributions*) (Table 3, row 3) is a catalyst to this eliciting of reasoning. It builds a classroom culture in which students feel safe to participate, share their interim thinking and explain their reasoning (Inoue et al., 2019).

Category 1: eliciting types of reasoning The eliciting of reasoning occurred throughout the lesson, for example, when Earl commented “12 cookies altogether” then asked “so what does that mean for the people?” he drew attention to the need for students to give reasons (Table 3, *Eliciting types of reasoning*: *analysing*) for what was on their sheet and the need to make connections between the physical situation and mathematics within it (Table 3, *Eliciting types of reasoning*: *validating*). Like Yackel (2001), Earl expected his students “to clarify aspects of their mathematical thinking that they think might not be readily apparent to others” (Yackel, 2001, p. 13) and justify their mathematical claims “So what did that mean?” These expectations are consistent with Kilpatrick et al.’s (2001) statement “[c]lassroom norms can be established in which students are expected to justify their mathematical claims and make them clear to others” (p. 130). It appears that Earl endeavored to elicit a conjecture from the group that chose 17 “‘Why?’ is my question. Why did your group decide that 17 was not going to work out? Why?” (see Results: “Generalising”). The reporter did not respond with reasoning, most likely because he had not fully understood the group discussion at that time. Since forming conjectures frequently involves identifying commonalities across cases (Lannin et al., 2011), this group may not have been exposed to sufficient cases (numbers of people at the door) at this time or may need more time to undertake such thinking (Clarke & Clarke, 2003).

Category 2: eliciting communication of reasoning Skemp (1979) described being able to communicate mathematical ideas developed as an additional type of mathematical understanding—logical understanding (beyond relational understanding—the communicating of relational understandings developed). Illustrations of eliciting of reasoning associated with communicating this reasoning are thus contributing to opportunities for students to develop deeper mathematical understanding. Other researchers have emphasised the importance of students communicating reasoning to convince others through opportunities to express their ideas and build mathematical understanding (Brodie, 2010; Long et al., 2012; Reid, 2002; Stylianides, et al., 2013). Earl (as in Williams, 2005) elicited mathematical ideas in common language from the students (as illustrated through this student response "There's 2 people inside and 10 people outside [holding up 2 fingers] which makes 12 people") and provided mathematical terms when the majority of the class appeared to understand a conceptual idea and were expressing it in common language "You have been experimenting with that symbol and that operation a little bit."

This category includes both oral communication of reasoning and communication of reasoning through the use of other representations (e.g. see student work in Fig. 1 where verbal, diagrammatic and symbolic representations were employed). Reasoning may be expressed and conveyed through a variety of representations (Bakker & Hoffmann, 2005) as evident in Earl's interactions with his students "sometimes we draw different diagrams so that we can get thinking about our work". Earl also accepted students' gestures as their responses when they pointed to various places on the worksheet and, like Lampert (2001), progressed student work by his introduction of language. Earl's focus was on encouraging students to connect mathematical representations and explain to others how they made those connections. In eliciting clearer communication of thought processes employed, simple *analysis* (Wood et al., 2006) was encouraged when Earl's eliciting of reasoning drew attention to parts of a symbolic representation (e.g. "Could you explain 12 times 1 = 12 what does that mean?") and *synthetic-analysis* (see Table 2) when students were asked to make connections between different representations (e.g. "… these circles … and … [those] numbers there. What were those numbers?"). *Evaluative-analysis* (see Table 2) in conjunction with *synthetic-analysis* was encouraged through Earl's eliciting of justifying of mathematics generated and supporting its reasonableness by drawing on another representation "…on your page you had this sum …. Do you know why this sum?" Language associated with reasoning emerged as he pursued this purpose.

Category 3: eliciting reasoning through encouraging student contributions This includes teacher valuing of students' explanations, and discussions, and teacher asking of questions whilst encouraging students to participate in such questioning as well, to increased opportunities to learn. The attention the teacher paid to the pace and clarity of the reports "so let's just get really quite slowly so … [they] can get up to speed …", and to whether learning was occurring for individual students "do you understand all the maths that is going on here?", demonstrated a valuing of all class members. Such actions made explicit the classroom norms and the socio-mathematical norms (Yackel & Cobb, 1996; Yackel, 2001). Lampert (2001), like Earl, "involved other students in supporting and furthering [student's] thinking" (p. 145), which further contributed to the culture developed. Earl's "handing over questioning to the class" was a strategy he appeared to employ when insufficient reasoning had been elicited by his own actions. Earl's valuing of students' ideas (echoed in the work of Funahashi and Hino (2014), Smith and Stein (2011) and Kilpatrick et al. (2001)) encouraged them to communicate more of their thinking (Table 3, *Value contribution*). Inoue et al. (2019) found that highly effective teachers of inquiry-based learning, like Earl, were adaptable and inclusive. They, like Earl, addressed "'students' whole person development and creat[ed] a collaborative and inclusive learning community" (p. 376). The present study contributes to the body of research on ways to develop classroom cultures in which reasoning can flourish by *illustrating* a diversity of this teacher's actions that contributed to this development.

Generic eliciting

It has become apparent from the data presented that there is a general type of eliciting of reasoning that could be applicable to a variety of problem-solving tasks. These include generic non-task specific questions and questions that can be adapted variables in other tasks. Such question types may assist teacher who are not yet familiar with the nature of reasoning and/or how to elicit it.

Non-task specific questions Earl frequently employed generic eliciting of reasoning actions (see Table 3, column 3). For example: "What was your thinking today?" [*Analysing*]; *"Why?"* [*Justifying*]; "so let's just get really quite slowly so … these students … can get up to speed …" [*Share ideas in easy to understand ways*]. Such generic action types could elicit reasoning when implementing other problem-solving tasks (e.g. Wood et al., 2006).

Questions changing context variables In addition, Earl employed eliciting of reasoning actions that could be adapted to other problem-solving situations. For example, a change in the variables (cookies and people) in say "12 cookies altogether so what does that mean for the people?" (Table 3, Category 1) might in another task be "12 *lions* altogether so what does that mean for the *area of the cage*?" Some of the illustrations in Table 3 have thus provided insights into how eliciting of reasoning may be transferrable from one task to another. In summary, this study illustrates and categorises generic types of eliciting of reasoning that may facilitate students' reasoning.

Contribution of the eliciting of mathematical reasoning study

The Eliciting Mathematical Reasoning Framework benefits researchers, educators, professional learning leaders, and teachers. It provides an analysis tool for researchers, guidance for professional learning leaders and educators, and supports early years teachers building their pedagogical expertise in eliciting of reasoning. The framework thus has the potential to support researchers and teacher educators as they address reported difficulties many teachers have with identifying and eliciting reasoning (Jacobs et al., 2010; Clarke et al., 2012; Herbert et al., 2015; Long et al., 2017). This research further contributes to existing knowledge about reasoning by:

- highlighting what mathematical reasoning during problem-solving can look like in early years
- illustrating teacher questions and classroom norms that can elicit mathematical reasoning
- highlighting potential for eliciting reasoning in other problem-solving contexts
- illustrating task design that offers opportunities for frequent eliciting of mathematical reasoning

Importantly, this study suggests that reasoning could be elicited by teachers who do not yet have secure knowledge of reasoning including reasoning terms, without this absence hindering their experimentation within problem-solving activity.

Fruitful areas for further research arising from this study also include: "How could this framework be employed as a useful analysis tool for teacher professional learning?" "How would other teachers, using the same problem-solving task, operationalise the eliciting of reasoning?" "Would subsequent parts of the task lead to the eliciting of types of reasoning not elicited so far?" "Are the types of reasoning elicited dependent upon the problem-solving task employed?", "Can this task be employed to elicit reasoning at other year levels?" The robustness of the framework can be investigated through further research in other contexts: early years classrooms, other year levels and diverse subject domains.

Conclusion

The *Eliciting Mathematical Reasoning Framework* captures the intertwined nature of the eliciting of reasoning and progressing mathematical thinking during problem solving. Eliciting of reasoning, before and during student communicating of their mathematical thinking, was an engine that drove further progress in such thinking. The three categories of the *Eliciting Mathematical Reasoning Framework* help make transparent the roles the teacher played in *setting up situations* in which reasoning was more likely to occur and eliciting *reasoning* through the eliciting of communicating of mathematical ideas represented on group developed worksheets. The nature of the task, and the *E2L* Approach employed, provided opportunities for the teacher to undertake these roles. Within a safe and collaborative classroom environment, the nature of the task encouraged idiosyncratic group reasoning and the approach encouraged and stimulated *communication of that reasoning* at the group and the whole class level.

Although these findings from one case, of one teacher interacting with one class for one lesson, are not generalizable, they have provided insights into categories of eliciting actions. Further research could examine whether each category is necessary and whether these three categories are sufficient. The applicability of the *Eliciting Mathematical Reasoning Framework* to eliciting of reasoning in other classrooms with other teachers and other tasks is signaled by the generic *Teachers' Eliciting of Reasoning Actions* identified.

The interrogation of videos of classroom interactions in an early years primary mathematics problem-solving lesson demonstrates the usefulness of this approach for researching teachers' actions for eliciting MR. This methodological approach with microanalysis of classroom video to identify teachers' eliciting of reasoning actions could be employed in problem-solving lessons in other early year contexts, other year levels and may be adaptable to lessons in other areas of the curriculum. The theoretical lenses employed to identify MR and mathematical thinking, integral to the development of the *Eliciting Mathematical Reasoning Framework*, should help to inform its use as an analysis tool for researchers.

The *Eliciting Mathematical Reasoning Framework* contributes to research on eliciting MR by identification and illustration of types of eliciting of reasoning actions that occurred. It provides guidance for researchers, teacher educators and teachers intending to leverage the power of reasoning to build students' understandings of mathematical ideas through progressive connections between them.

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Availability of data and material Data is stored on a secure university password protected drive. Data is kept 6 years after final publication as stipulated by Deakin University HREC.

Code availability Studiocode was used to assist in coding.

Declarations

Ethics approval Ethic approval was gained from the Deakin University Human Ethics Committee, and permission to research was granted by the Victorian Dept. of Education & Training.

Consent to participate The researchers explained the teachers' and students' part in the research after the principal had agreed to the research. The school was sent a Plain Language Statement and consent form for the participating school principal to sign. Interested teachers were given a Plain Language Statement and consent form which described the research project and their involvement. The research was explained to the students by the researchers. Students were given their own simplified PLSs. The parent/s or guardians of students in the teachers' primary class were forwarded the PLS for parents and the consent form for parent/s and students. This occurred after the teachers of these classes had agreed to take part in the study.

Consent for publication The plain language statements provided details of participation and use of data in publications.

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