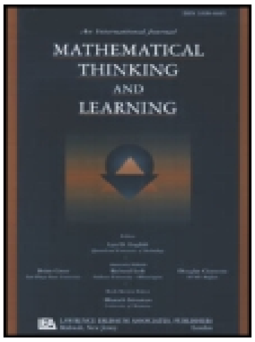

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Introducing Modeling Transition Diagrams as a Tool to Connect Mathematical Modeling to Mathematical Thinking

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ABSTRACT
This study contributes a methodological tool to reconstruct the cognitive processes and mathematical activities carried out by mathematical modelers. Represented as Modeling Transition Diagrams (MTDs), individual modeling routes were constructed for four engineering undergraduate students. Findings stress the importance and limitations of using micro-analysis to examine modeling processes. The findings and the MTDs were used to critically question the implications of using modeling cycles as a theory of mathematical modeling processes.

Over the past 25 years, the importance of mathematical modeling in the work place and science, technology, engineering, and math careers has received a steady increase in attention. Mathematical modeling has been incorporated into classrooms from elementary to postsecondary levels to teach both mathematical concepts and modeling skills (English, 2006; Hamilton, Lesh, Lester, & Brilleslyper, 2008; Mousoulides & English, 2011; Stillman, 2000). This has led to closer consideration of mathematical modeling's place in the school curriculum (Blum & Niss, 1991; Doerr & English, 2003; Lesh, Hoover, Hole, Kelly, & Post, 2000; National Governors Association Center for Best Practices and Council of Chief State School Officers (CCSSM), 2010).

Given its long history as an object of study, mathematical modeling has been conceptualized in a variety of ways. It has been investigated as a process (Blum & Leiß, 2007; Zbiek & Conner, 2006) and as a skill (Lesh, Galbraith, Haines, & Hurford, 2013; Yoon & Thompson, 2003). It has also been conceptualized as a theory of student learning (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003). Meanwhile, a wealth of companion inquiries attests to the richness of theoretical and methodological problems related to studying individuals' construction of mathematical models and defining the accompanying modeling skills (Ärlebäck, 2009; Borromeo Ferri, 2006, 2007; Doerr & Tripp, 1999; English, Fox, & Watters, 2005; Haines & Crouch, 2001; Haines, Crouch, & Davis, 2001; Kaiser, Blomhøj, & Sriraman, 2006; Lesh & Doerr, 2003; Lesh & Yoon, 2007; Lesh & Zawojewski, 2007; Maaß, 2010; Sriraman, Kaiser, & Blomhoj, 2006; Verschaffel, Greer, & de Corte, 2000; Yoon, 2006). This work has been carried out from both cognitive (e.g., Borromeo Ferri, 2007) and social perspectives (e.g., Schwarzkopf, 2007), using individual and group problem-solving protocols (Borromeo Ferri, 2007; Galbraith & Stillman, 2006; Mousoulides & English, 2011), and drawing on many theoretical frameworks (e.g., Kehle & Lester, 2003; Lesh et al., 2003; Meira, 2002).

Scholars have produced descriptive models of the phases involved in constructing a mathematical model (Blum & Leiß, 2007; Dym, 2004; Huber, 2010; Kehle & Lester, 2003; Zbiek & Conner, 2006), empirically differentiated among those phases (Borromeo Ferri, 2006), identified factors that contribute to or inhibit progress (Galbraith & Stillman, 2006; Stillman, 2000), distinguished modeling

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from problem solving (Lesh & Yoon, 2007; Lesh & Zawojewski, 2007), and developed approaches to measuring modeling competency (Haines et al., 2001; Mousoulides, Christou, & Sriraman, 2008). Thus far, the cognitive perspective on mathematical modeling has described the mathematical modeling process and the mathematical and nonmathematical thinking that support it, but some questions remain. For example, how does theory account for its functioning and development? How can mathematical modeling processes be documented?

Many descriptive models, both a priori descriptions and those based in research, characterize mathematical modeling as a cyclic, iterative process that renders a real-world problem as a mathematically well-posed problem conducive to mathematical analysis. Its solution is then interpreted in terms of real-world constraints and validated against real-world observations. Based on the validation process, the model is accepted, rejected, or revised. The revision process can lead the modeler back through the cycle. Typically, mathematical models begin as crude representations or explanations and become more detailed and sophisticated after multiple iterations of this cyclical process. This type of theoretical framework is referred to as a mathematical modeling cycle. One such example of a mathematical modeling cycle is illustrated in Figure 1.

Use of mathematical modeling cycles to describe modeling processes has diffused outside of research frameworks. Not only do mathematical modeling cycles appear in mathematics and engineering textbooks, they are featured in the Common Core State Standards (CCSSM, 2010) in two places. The CCSSM verbally described mathematical modeling in terms of a mathematical modeling cycle as a Mathematical Practice (p. 7) and offered a diagram to orient educators to mathematical modeling as a content standard (pp. 72–73). Because mathematical modeling cycles feature so prominently in literature about mathematical modeling, they are already appearing in practitioner journals as definitions of mathematical modeling (Anhalt & Cortez, 2015).

As such, understanding how we may expect students' modeling processes to align with theoretical predictions based on modeling cycles is crucial to taking advantage of them in classroom and research contexts. Some lines of inquiry around this work have suggested that individuals do not strictly follow the stages outlined by modeling cycles in order (Ärlebäck, 2009; Borromeo Ferri, 2007). Instead of being universal, individuals' modeling routes are idiosyncratic. This article offers a way to document students' mathematical thinking during mathematical modeling to better understand their modeling processes.

Paralleling descriptions of how mathematical models develop, this study tests the efficacy and robustness of Mathematical Modeling Cycles as representations of how individuals engage in mathematical modeling. To achieve this goal, this article extends work on methodological tools previously used to represent modeling processes. Modeling Transition Diagrams (MTDs) are introduced as an analytic tool for capturing and representing an individual's modeling process. The methods used to create the MTDs reconstruct the mathematical thinking that individuals exhibit while modeling by connecting macroscopic modeling processes to micro-level observable mathematical activities. In this way, analysis is grounded in empirical observations of students' mathematical thinking that are guided by theoretical descriptions of the modeling process. The net result is an elaboration of theory based on systematic examination of individuals' modeling processes.

The accompanying research questions were:

(1) How can explicitly attending to mathematical thinking broaden our understanding of mathematical modeling processes?
(2) To what extent can mathematical modeling cycles explain mathematical modeling processes?

Mathematical modeling as a process

Kaiser and Sriraman (2006) observed that "there does not exist a homogeneous understanding of modelling and its epistemological backgrounds within the international discussion on modelling" (p.

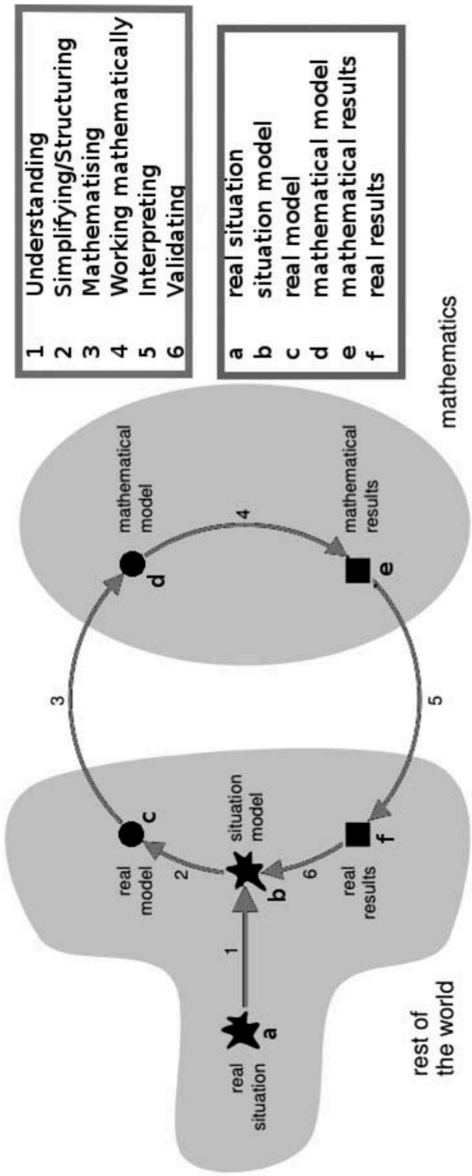

Figure 1. Research framework adapted from Blum and Leiß (2007).

302). Indeed there are many conceptualizations of mathematical modeling, each of which can be tied to the researcher's or educator's goals. This study adopted a process view of mathematical modeling because the goal was to focus on students' mathematical thinking during mathematical modeling. In this view, there is a mathematical world and the rest of the world (Pollak, 1979) or the "real" world. The modeling process bridges these two together by creating a mathematical representation of the real world. Because "mathematical model" could refer to both an external, verbal, or written expression or to an internal, conceptual mathematical structure, and each may correspond to the real-world problem situation, a careful distinction must be made. To appropriately distinguish between the two, the mathematical model's expression (e.g., graphical, symbolic, or verbal representations) were seen as distinct from how the individual coordinates mathematical structure with its external expression. In order to be sensitive to this distinction, a mathematical model was conceptualized as a tripartite entity (S, M, R), where S is the situation to be modeled, M, is the external representation in mathematical terms, and R is a relation that maps the objects and relationships found in S to the mathematical objects and relationships (Blum & Niss, 1991).

Mathematical modeling cycles explain how the relation R is formed and M is obtained. The modeling process transforms a real-world problem into a mathematically well-formulated problem. The mathematical problem can then be analyzed and interpreted in terms of real-world constraints. The interpreted solution is validated against real-world constraints.

Blum and Leiß's (2007) mathematical modeling cycle was selected as a research framework to coordinate data generation because it defines stages and transitions that appear during the modeling process. It is also sufficiently complex to subsume many other modeling cycles. Blum and Leiß's description of the mathematical modeling process as depicted in Figure 1 will be referred to as the research framework to distinguish it from the general idea of mathematical modeling cycles (of which there are many).

In Blum and Leiß's (2007) Mathematical Modeling Cycle (the research framework; Figure 1), there are six stages of model construction (stages [a]–[f]), sequentially linked by six transitions (transitions [1]–[6]). The modeling transitions are comprised of mathematical activities. The transitions were the focus of data analysis in this study, and were viewed as collections of observable mathematical activities. Table 1 and Table 2 give brief descriptions of each of the stages and transitions among the stages. The inclusion of stages [c] (the idealized problem setting) and [d] (the mathematical representation) reflect the distinction between the representation M and the relationship R. A more detailed narrative of the modeling cycle, described in terms of the mathematical activities that support it, is offered later in the article. The next section introduces the methodological tool (Modeling Transition Diagrams) and demonstrates the use of the Modeling Transition Diagram as an analytic tool to examine the mathematical modeling process in this study.

Table 1. Stages of model construction.

Stage of Model	Definition
[a] real situation	Situation, as observed in the world
[b] situation model	Conceptual model of problem
[c] real model	Idealized version of the problem (serves as basis for mathematization)
[d] mathematical model	Model in mathematical terms
[e] mathematical results	Answer to mathematical problem
[f] real results	Answer to real problem

Table 2. Transitions between stages of model construction and sample indicators.

Transition	Captures
[1] understanding	Forming an idea about what the problem is asking for
[2] simplifying and structuring	Identify critical components of the problem situation
[3] mathematizing	Represent the idealized real model mathematically
[4] working mathematically	Mathematical analysis
[5] interpreting	Re-contextualizing the mathematical result
[6] validating	Verifying results against the real-world

Origins of the modeling transition diagrams

Borromeo Ferri (2007) employed individual modeling routes to study mathematical model construction from a cognitive perspective. An individual modeling route is "the individual modeling process on an internal and external level" where "from a cognitive viewpoint, one has to speak of visible modelling routes, as one can only refer to verbal utterances or external representations for the reconstruction of the starting point and the modelling route" (p. 265). Borromeo Ferri developed individual modeling routes in order to document an individual's mathematical thinking during modeling. Her representations of individuals' modeling activities took the form of arrow diagrams imposed on the research framework along with highlights from protocol corresponding to arrows in the diagram (see Figure 2). Similarly, in the present study, only visible modeling routes are considered. However, because the arrow diagrams themselves were difficult to follow chronologically, I sought a change in representation that would still be rooted in mathematical thinking.

Borromeo Ferri (2007) used the individual modeling routes to trace the influence of individuals' thinking styles on their modeling decisions. The study provided evidence that empirical distinctions between different stages and transitions of model construction are possible, that micro-level analysis of modeling processes can expose modeling routes, and that modeling routes are idiosyncratic. Specifically, Borromeo Ferri argued that individual mathematical thinking styles impact the choices students make during mathematical modeling. Given these findings, examining individuals' modeling processes using micro-level analysis was appropriate.

In a different study, Ärlebäck (2009) developed Modeling Activity Diagrams (MADs) to document how mathematical models are constructed in terms of the mathematical thinking used to create it. MADs are two dimensional graphs that visually display mathematical thinking during modeling. A sample is given in Figure 3 (Ärlebäck, 2009). Ärlebäck's (2009) MADs record the length of time that the modeler was engaged in certain activities. The MADs map a group's modeling session to a set of horizontal lines, reminiscent of a musical staff, on which each line is coded for a modeling activity, and activities are recorded over time. The result is a chronological representation of mathematical activity during model construction. Ärlebäck used these representations to examine students' group work on mathematical modeling tasks in terms of certain activities—reading, making model, estimating, calculating, validating, and writing. Ärlebäck found that a particular class of problems, Fermi problems, were capable of eliciting all of the modeling activities described by mathematical modeling cycles.

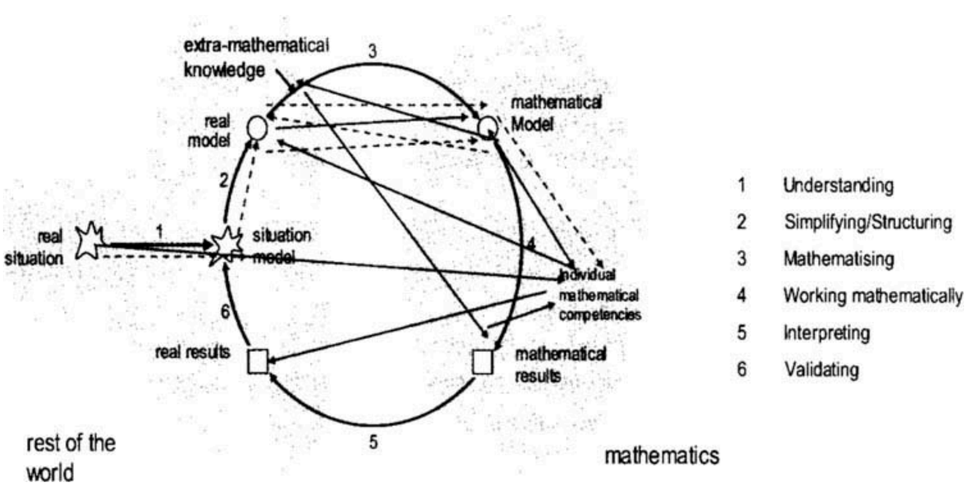

Figure 2. Individual modeling route (Borromeo Ferri, 2007).

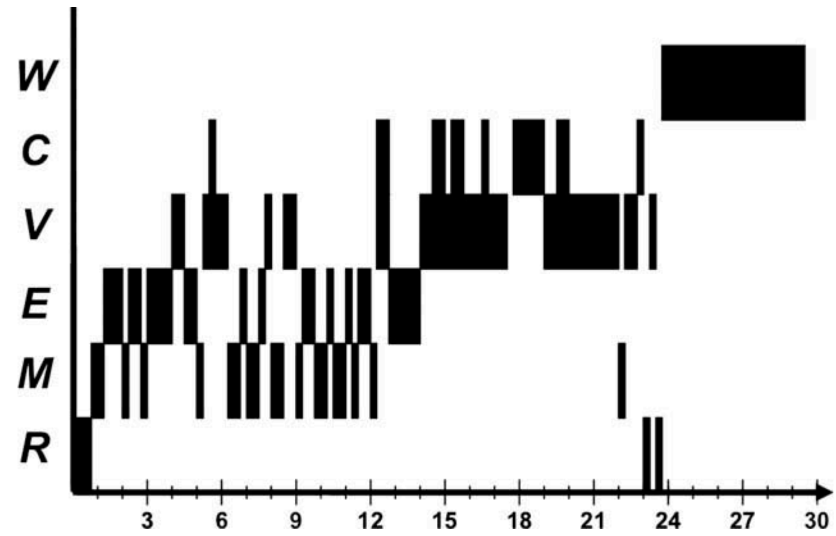

Figure 3. Sample Modeling Activity Diagram. The abbreviations along the vertical axis stand for: reading, making a model, estimating, validating, calculating, and writing (Ärlebäck, 2009).

Figure re-used with permission from B. Sriraman (Editor), *The Montana Mathematics Enthusiast*. Original figure found in Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast, 6*(3), 331–364.

The primary advantage of representing modeling processes as MADs is that they provide a chronological structure for the observations, which allows them to be more easily compared. The main drawback to using MADs is that they are highly sensitive to how interview protocol is coded. They are created via a coding procedure that depends on the grain size of analysis and also whether verbal and written evidence are considered equally. When coding, the researcher must decide when a particular transition begins or ends since only his or her observations can be recorded. Quite naturally, this can lead to measuring the length of time the individual spends carrying out the transition. But it is difficult to make sense of what the observed length of time might mean. For example, if an individual spends a long time working mathematically, it could mean that the solution had a lot of steps to it. It could also mean that the individual struggled to carry out the analysis. Or it could mean that she paused to think about something else and only appeared to be continuing the mathematical analysis. Thus it is difficult to meaningfully compare "length of time spent on a transition" meaningfully across tasks or individuals.

All of these methodological issues are important to consider when building and interpreting representations of mathematical activities. In the present study, I combined the strength of emphasizing individual modeling routes (Borromeo Ferri, 2007) with the representational power of modeling activity diagrams (Ärlebäck, 2009) to connect individuals' mathematical thinking to mathematical modeling as described by Blum and Leiß (2007). I adapted the generation of the MADs to account for difficulties reported by other scholars. To do so, I established a correspondence between Ärlebäck's (2009) activities and the transitions in Blum and Leiß (2007), which can be seen in Table 3 correspondence between Arlebäck's (2009)

Table 3. Correspondence between Arlebäck's (2009) modeling activities and the modeling transitions in Blum and Leiß's (2007) framework.

Arlebäck's Modeling Activities	Blum and Leiß's Transitions
Reading	Understanding
Estimating + model making	Simplifying/structuring + mathematizing
Calculating	Working mathematically
Validating	Interpreting + validating
Writing	

modeling activities and the modeling transitions in Blum and Leiß's (2007) framework. To aid in conceptualization and data analysis, the mathematical model (stage [c] in the research framework) was taken to mean "mathematical representation." This was necessary to make sense of the data collected in terms of separating the product of mathematical modeling (a partially conceptual, partially external construction) from the expression of it in conventional mathematical terms.

Rather than measuring the duration of each transition, I recorded the time when a particular transition was first observed. The result was the MTDs, which present visible modeling routes but preserve chronological ordering of individuals' modeling processes. Their construction and use in this study is described next.

Methodology

Because the goal of this research was to examine students' mathematical modeling processes, task-based interviews were used to elicit the transitions defined by the research framework. Each of the interview tasks was audio and video recorded and then transcribed. In this section, I provide details for the research setting, the development of an observational rubric, the construction of the MTDs, and the micro- and macro-level analyses of the students' work.

Setting and participants

The study constituted mathematical modeling processes as a single phenomenon and examined how the individuals' MTDs collectively compared to predictions made by the research framework. The sample considered for this study consisted of four engineering majors enrolled in a course on differential equations at a large, Midwestern university in the United States. Because the participants had studied engineering, science, and advanced mathematics I assumed that they would exhibit the mathematical activities relevant to mathematical modeling. Therefore, they could help generate data that would be valuable to the literature on mathematical modeling.

A calculus screening test was administered to volunteers since some of the tasks required calculus knowledge. The screening test was a combination of the Calculus Concept Inventory (Epstein, 2013), procedural calculus tasks, and a related rates modeling task. The Calculus Concept Inventory was developed and validated as a measure of individuals' conceptual knowledge of one variable differential calculus. The procedural tasks and the related rates tasks were taken from departmental final exams in single and multivariable calculus. The procedural tasks were used to ensure proficiency in calculus computations (e.g., taking a partial derivative). The related rates task was used to get a sense of students' written modeling work.

Four individuals were selected to participate in the study (Mance, Torrhen, Orys, and Trystane; all names are pseudonyms) according to a 2×2 participant selection design: high/low modeling (performance on the related rates task) and high/low calculus (performance on remaining tasks). Because introducing variety to a sample allows for generalizing in case-oriented analysis, the 2×2 selection design was chosen to ensure that there would be variety in the students' approaches to the interview tasks. For example, I expected that the more mathematically proficient students might try to use more advanced mathematical tools and concepts, whereas the comparatively weaker students might try to rely on linear or proportional reasoning. The variety among engineering majors occurred naturally and introduced different scientific knowledge bases into the sample. For example, Orys and Mance took more chemistry courses than did Torrhen and Trystane. All participants were male and they were compensated for their time.

Mance was selected to represent the low mathematics/high modeling category. He was a second-year environmental engineering major who had completed his first year engineering sequence and the calculus sequence. Torrhen was selected to represent the high mathematics/low modeling category. He was a first year honors electrical and computer engineering student intending to minor in physics. He completed the first-year engineering honors sequence and calculus. Trystane represented the low mathematics/low modeling category. Within his major, he was a second year student, but since he transferred from another university and carried AP credit hours, he was technically a senior. He was a mechanical engineering major and had completed an internship doing design work with Honda. He had completed some upper-division mechanical engineering courses. Orys represented the high mathematics/high modeling category. He was a first-year student in the honors engineering program majoring in chemical engineering. He had completed the first-year engineering course work, first-year physics, had tested out of the chemistry sequence, and completed the honors, accelerated calculus sequence.

Due to the case-oriented analysis and small sample size, it is beyond the scope of this article to report variable-oriented outcomes. Any conjectures about common patterns of modeling activity for high vs. low modelers or high vs. low calculus performance would be premature.

Instrumentation and interviews

Structure of interviews

Each subject participated in seven task based-interviews during one academic semester and one member check (Glesne, 1999) session a year later. The goal of each interview was to elicit the participants' modeling processes.

The interview methodology incorporated two principles from design research: reflection and cross-fertilization. Reflection refers to analysis conducted between interview sessions in order to generate conjectures about students' mathematical thinking and how it could be elicited via the modeling tasks and follow-up questions. Cross-fertilization is a design research principle that encourages the use of information and experiences from one interview session to inform task tweaks, follow-up questioning, and sensitivity to student responses during another session. Working with individuals with diverse mathematical proficiencies allowed for cross-fertilization (Brown, 1992) across the sessions, and opportunities for reflection between sessions were built in to the interview schedule.

Each session began with reflective and clarifying questions about the participants' thinking based on the ongoing analysis of previous interviews (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003). During each session each participant was also asked to share what he was learning in his differential equations course to establish intersession continuity and to allow for multiple points of contact with the differential equations curriculum (Nair, 2010; Tall, 1987). Because I was interested in students' mathematical thinking, and not their memory of their coursework, they were permitted to use resources and tools like reference books, the Internet, and graphing calculators.

The seventh session reserved time to pose reflective questions to the students about their experiences in the research study (Dahlberg & Housman, 1997; Lesh, Kelly, & Yoon, 2008). A final eighth session served as a follow-up and member check (Glesne, 1999). In the final session, participants were asked to directly comment on the researcher's account of their modeling processes. In total, 60 hours of interviews were conducted, 51 events were analyzed, of which 17 events (approximately 8 hours) are presented here.

Interview task design: The process

Task selection began with a survey of the literature on mathematical modeling. I examined the research and educational literature on mathematical modeling along two strands: (i) principles for designing tasks that would elicit modeling transitions and (ii) aspects of mathematical thinking related to modeling. This coordinated, multitiered review helped to articulate objectives for task design. The tasks needed to fall between the depth of a classroom project (e.g., recommendations from Lesh et al., 2000) and multiple-choice competency assessments (Haines et al., 2001) to be suitable for one-on-one cognitive interviews. Sources included the Lively problems (Arney, 1997), community-maintained collections (Community of Ordinary Differential Equations Educators, 2012), the biennial International Community of Teachers of Mathematical Modelling and Applications (ICTMA) publications, special issues of *ZDM–The International Journal on*

Mathematics, mathematics and engineering education research journals, differential equations textbooks, modeling textbooks designed for engineers (e.g., Edwards & Hamson, 2007), and teacher resources (Mason & Davis, 1991).

The task pool was created and revised iteratively. A panel of mathematics educators, mathematicians, and instructors of differential equations were asked to assess the face and content validity of the tasks. Based on the panel's input, tasks were field tested with a second panel of mathematics educators and engineering undergraduates. Construct validity was assessed by examining the results of the field tests for whether the modeling transitions were elicited. Unsuitable tasks (e.g., problem statement was incomprehensible; task did not elicit transitions) were modified or removed from the pool. The tasks reported here elicited all modeling transitions and drew on a variety of mathematical and nonmathematical contexts.

Some of the tasks used concepts from differential equations. Although other scholars have studied learners' mathematical thinking within the applied differential equations context, the tasks used in those studies did not meet the goals of the current work, as described next.

Several scholars have examined students' knowledge construction in differential equations within an inquiry-oriented social setting. In particular, the work of Rasmussen and colleagues (e.g., Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014) demonstrated that students are capable of reinventing solutions to and analyzing graphically and numerically solutions of differential equations. The tasks they selected were therefore oriented toward recreation of mathematical concepts, which was not a focus of the present study. Additionally, the researchers made simplifying assumptions for the student or provided instructions for which representations (graphical, numerical symbolic) to use. For example, the canonical predator-prey problem used by Tabach and colleagues (2014) yields a system of linear differential equations in which students are guided to generate (and justify the interpretations of) multiple representations of solutions to those equations. Students were told to assume that there was a population of ten rabbits which produced continuously, had no predators, and had unlimited resources. Students were then asked to produce a graph that would describe the rabbit population. Thus, most of the modeling work—the modeling transitions simplifying/structuring and mathematizing—had already been carried out by the task-writers.

For the present study, tasks were selected according to whether they meet the design principles of modeling contexts (Lesh et al., 2000; Maaß, 2010) rather than for their capacity to reveal student knowledge construction in differential equations. The tasks are described next.

Interview task design: The tasks

Students' work on six modeling tasks served as the basis for analysis in this report. I chose this collection for their theoretical capacity on both mathematical and pedagogical levels. Collectively, tasks make available a range of mathematical and nonmathematical contexts. They utilize concepts from arithmetic to differential equations and involve everyday contexts as well as those that resemble school mathematics. Each task was sufficiently complex so I had opportunities to discuss and question aspects of the students' modeling decisions. The tasks, and their backgrounds in the literature, are described in the following section.

Tropical FIsh Tank problem

One canonical ordinary differential equations problem is called a "mixing problem." In a mixing problem, a contaminant is introduced to a (possibly) changing volume of liquid. The goal is to model the amount of contaminant in the tank at any time t. The modeler has to coordinate multiple conceptions of change in a quantity: absolute change, average rate of change, and instantaneous rate of change. The result is a first-order, linear, nonhomogeneous, differential equation for the instantaneous rate of change of contaminant in the tank. Along with information about the initial conditions, the equation can be solved for the quantity of contaminant. The mathematical content of this task has been discussed at length elsewhere (Czocher, Tague, & Baker, 2013) and student difficulties with coordinating quantity with rate-of-change of that quantity have been documented (Rowland, 2006).

To regulate the pH balance in a 300 L tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. The strength of the buffering solution varies according to $1 - e^{-\frac{t}{60}}$ *grams per liter. The buffering solution enters the tank at a rate of 5 liters per minute. How much buffering agent is in the tank at any point in time?*

Fermi problems

These problems are named after physicist Enrico Fermi who famously asked his undergraduate physics class "How many piano tuners are there in Chicago?" They are commonly used in physics or science classes to teach dimensional analysis and estimation skills. With only modestly accurate estimations for input parameters, and typically using only arithmetic operations, the results of Fermi problems are surprisingly accurate. The tasks are pedagogically useful because they force the modeler to clarify assumptions and conditions that arise from making educated guesses about the circumstances surrounding the problem (Sriraman & Lesh, 2006).

Fermi problems have several distinguishing characteristics. They (a) are accessible to different educational levels; (b) allow for increasing levels of complexity; (c) are realistic, since they are directly related to the daily environment; (d) are open so that the modeler must explicitly engage in simplifying and structuring; (e) require the modeler to make reasonable estimates for identified parameters; and (f) promote discussion because the modeler is naturally inclined to validate and justify his or her assumptions and estimate. For these qualities, three Fermi problems were used and are reported in this study: the Piano Tuners Problem, the Cell Problem (Schoenfeld, 1982), and the Empire State Building Problem (Ärlebäck, 2009):

The Piano Tuners Problem: *How many piano tuners are there in New York City?*

The Cell Problem: *Estimate how many cells there are in the average adult human body.*

The Empire State Building Problem: *There is an information desk on the street level in the Empire State Building. The two most frequently asked questions to the staff are: (i) How long does the tourist elevator take to the top floor observatory? (ii) If one instead decides to walk the stairs, how long does this take? Your task is to provide answers to the staff at the information desk, including the assumptions on which you base your reasoning.*

The Baker's Yeast problem

The Baker's Yeast problem can be tackled using discrete mathematical model (geometric progression) as well as a continuous model (exponential growth). The question was phrased so as to obtain a concrete numerical answer and then encourage the students to rely on other methods, other than direct observation, to validate their predictions.

Baker's yeast is a type of fungus that reproduces through budding. Each cell reproduces once every 30 minutes. To grow yeast for baking bread, you have to proof it first—allow it to form a colony—in a bowl of warm water. Suppose that in a particular bowl, after 6 hours, the surface of the water is covered in yeast cells. When was one half of the surface covered?

The Falling Body problem

The Falling Body problem is a typical dynamics task used in calculus and first year mechanics courses. It is based on a differential equations task where a body is falling with no constant acceleration. In such a case, air resistance is nonzero. The problem can be solved with reasonable accuracy using only kinematics under the assumption that acceleration is constant by treating the body as though it is in free fall, which is standard in first year mechanics courses.

On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin, TX died from injuries sustained after jumping from a second floor window to escape a fire at his home. What was his impact speed?

Analytic methods

Since the goal of this study was to examine students' modeling processes, one student working on one task (defined as an *event*) was taken to be the unit of analysis. An event is a set of activities linked together into a larger pattern (Spradley, 1980), such as the transitions defined by the research framework. Viewing the student's work on a task as a stream of observable behaviors that seem distinct allowed for the development of the coding rubric in terms of mathematical activities that support each modeling transition.

Each of the interviews was audio and video recorded and underwent micro- and macro-level analysis. The events were transcribed and parsed into complete student ideas and interviewer interjections. Micro-level analysis treated utterances as the unit of analysis, which were analyzed according to the observational rubric (described next) using the method of constant comparison (Glaser & Strauss, 1967). Micro-level analysis yielded the production of the MTDs. Macro-level analysis treated each event as a unit of analysis and the collection of MTDs was analyzed via cross-event analysis (Katz, 1983) and compared to predictions from the research framework to develop a typology (Goetz & de Lecompte, 1981).

It is important to note that because that the purpose of the study was to examine the mathematical modeling process, it is beyond the scope of this study to compare students' modeling competence. Analysis traced the mathematical thinking that supported the modeling transitions rather than differentiating performance by their mathematical characteristics (e.g., high vs. low math category).

The rubric: Indicators of mathematical activity

In order to ground the study of mathematical modeling processes in mathematical thinking, an observational rubric of mathematical activity was developed inductively through analysis of the interviews. This technique is in line with the method of constant comparison (Glaser & Strauss, 1967). At the outset, a rubric of expected mathematical and epistemic activities (written and verbal) was created based on a literature review. Each of these activities was then linked to specific indicators (see Table 4) observable during the interviews.

A table of anticipated mathematical activities (both written and verbal) was created based on the review of modeling cycles. For example, understanding captures the process of forming an idea about for what the problem is asking. Some indicators for this transition were reading the task, returning to elements in the statement of the task, clarifying what needs to be accomplished, and taking stock of data/information given. Protocol that could be described as the modeler "taking stock of data/information given" would be assigned this indicator and then coded as understanding. When a segment of modeling activity could not be coded using existing indicators, a new indicator was added to the rubric. For example, working mathematically was initially conceptualized as carrying out symbolic or verbal operations (e.g., solving an algebraic equation, taking a derivative). Orys was observed "using deductive reasoning to make mathematical inferences about his solution to the differential equation he derived for the Tropical Fish Tank problem. This was interpreted as working mathematically. The indicator making inferences or deductions without reference to nonmathematical knowledge was added to the rubric. In such cases, all previously analyzed protocols were compared against the new rubric. In this way the final rubric (Table 4) was created as a typology for identifying modeling transitions. A detailed example of how the rubric was applied in data analysis is provided in the following section.

Data processing

A spreadsheet was created for each event to organize data analysis. Each student or interviewer utterance was assigned to a row. The rows had the following columns: time, description of the interviewer's or students' conduct, transcript segment, memos about protocol content, an indicator

Table 4. Observational rubric with indicators for transitions in the modeling cycle.

Activity	Trying to Capture	Indicators
Understanding the problem	Forming an initial idea about what the problem is asking for	Reading the task Returning to elements in the statement of the task Comparing the problem to problems seen before Clarifying what needs to be accomplished Taking stock of data/information given
Simplifying/ structuring	Identify critical components of the mathematical model; create an idealized view of the problem	Making assumptions to "simplify" the problem [note: not all assumptions may simplify] "Classifying" the problem (e.g., stating that the problem is a "maximization" problem) Listing assumptions Referring to assumptions Listing variables, parameters, constants Mentioning variables, parameters, constants Specifying conditions Operationalizing quantities or relationships (e.g., interpreting "best" as "largest") Using data/information from the problem statement to help with any of the above Introducing outside knowledge to help with any of the above Carrying out an "experiment" to observe stated or implied conditions, parameters, constants, conditions, relationships "Running out" of conditions, assumptions, variables, parameters Estimating a parameter Drawing or labeling sketches that correspond to stated or implied conditions/assumptions/variables/parameters
Mathematizing	Represent the real-model mathematically	Writing mathematical representations of ideas (e.g., symbols, equations, graphs, tables, etc.) Speaking in terms of symbols, operations, or relationships Cataloguing/searching for existing equations that relate given variables Using dimensional analysis in order to incorporate a parameter, variable, constant
Working mathematically	Mathematical analysis	Explicit algebraic manipulations Speaking about algebraic manipulations Making inferences and deductions without reference to nonmathematical knowledge Changing mathematical representation Explicit mathematical operations that may not be arithmetic/algebraic (e.g., comparing, rounding, partitioning)
Interpreting	Re-contextualizing the mathematical result	Referring to units Stating the answer as an answer to the contextual question, not just the mathematical question Considering whether the result answers the question posed Interpreting meaning from an equation or its elements; or from a mathematical representation Speaking about the result in context of the problem Referring to conditions/variables/parameters from "simplifying/structuring"
Validating	Verifying results against constraints	Implicit or explicit statements about the reasonableness of the answer/model Checking extreme cases (of variables, parameters, relationships, etc.) Checking special cases (of variables, parameters, relationships, etc.) Comparing an answer to a known (theoretical or practical) result Checking/comparing behavior of elements of the model Indication that he will solve the problem a different way Estimating an appropriate result Adding limitations of the model or result (note: this is like simplifying/structuring but is done on the basis of results/ finalized model) Talk about ideal results Comparing merits of different models Dimensional analysis

serving as evidence for a modeling transition, the modeling transition observed, and memos about analysis (including links to other tasks, participants, and literature). In this way, the events were reduced to individual modeling routes (Borromeo Ferri, 2007).

To represent the individual modeling routes as MTDs, I used MATLAB. Each route was represented as an $n \times 2$ matrix. The first column was a vector of timestamps (in seconds). The second column was a vector of integers between 1 and 6, corresponding to a specific transition in the research framework. The matrix was plotted as a block at a y-value associated with a transition number in the research framework matched to a t-value along the time axis. Each MTD is a visual summary of the verbal, textual, and mathematical activities that the student exhibited during his modeling process.

An ideal MTD was created to represent the hypothetical modeling route where the individual continually cycled through the transitions of the research framework. The hypothetical modeling route was processed and graphed according to the analytic methods. The result is the saw tooth pattern depicted in Figure 4.

The spreadsheets, interview recordings, MTDs, and students' written work were triangulated to coordinate macro-level analysis. I considered each event as a constellation of personal experiences, mathematical knowledge, and nonmathematical knowledge used in the course of modeling phases (Stillman, 2000) and searched the MTDs for patterns and anomalies via cross-event analysis (Katz, 1983). Cross-event analysis proceeded task-by-task because comparisons across students were not the focus. The students' MTDs were compared to the hypothetical modeling route to generate conjectures about the patterns among data. I used writing to process my evaluation of the conjectures in relation to the data sources. By maintaining close contact between the evolving conjectures and the data sources, I began to search for ways in which the data confirmed or refuted previously reported findings (Becker, 1998, 2007). The conjectures were continually refined in light of the new data analyzed, the theoretical framework, and prior research. Detailed records were kept of all analytic decisions, notes on related literature, and analytic memos. Member checks (Glesne, 1999) were used to verify the accuracy of analysis during a follow-up session one year after the close of data collection.

Reading the MTDs

An MTD is a two-dimensional graphical representation of a mathematical modeling event. The horizontal axis marks time and the vertical axis marks the transitions in the research framework, where each transition corresponds to a number between 1 and 6. A mark on the graph represents that the corresponding transition was observed at that time. Only the time when a particular transition was first observed was marked. The graphs should be read from left to right for an overview of the individual's modeling route over time and from bottom-to-top to get a sense of the sequencing of transitions. Since each box represents a transition between two stages of model construction, one can imagine that on either side of that box the student was mentally or mathematically operating on two different stages of the model. For example, mathematizing (transition [3]) requires operations on the idealized version of the problem situation (stage [c]) and a mathematical representation (stage [d]).

The markings are intentionally scaled to be large so that they are visible when printed. Elongated boxes are artifacts of the scale and not an indication of the length of time that an individual was engaged in an activity. Since it was possible to code an utterance with more than one transition, it is possible that there is more than one box associated with a time.

Sample coding

This section offers an example of how the transcripts were coded using the observational rubric. The excerpt in Table 6 is from Mance's work on the Tropical Fish Tank problem. The first column is the time in minutes and seconds, the second column summarizes the action, the third column is the

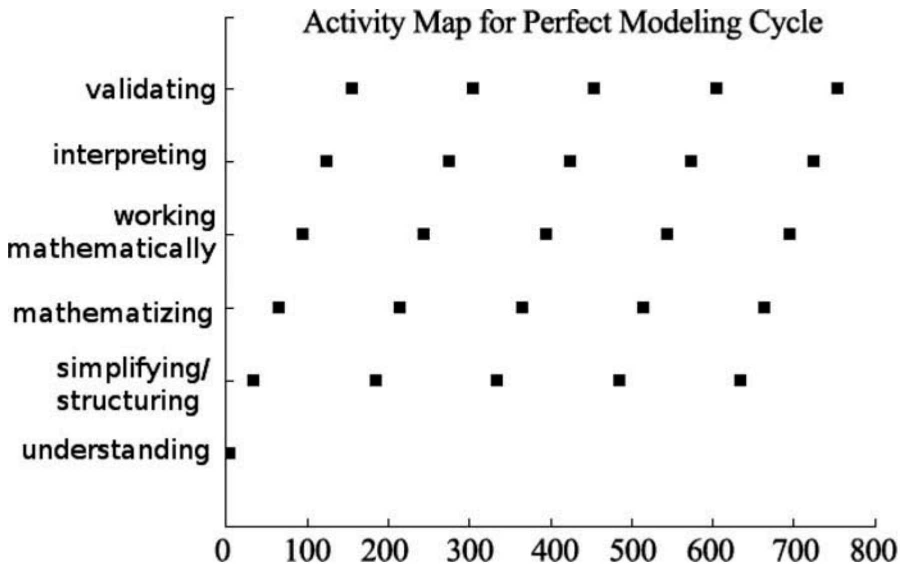

Figure 4. Idealized modeling transition diagram.

verbatim transcript, the fourth column is the appropriate indicator from the observational rubric, and the final column is the transition from the research framework indicated by the observational rubric. The associated MTD is in Figure 10 and the written work is shown in Figure 5.

In the first row of Table 6, beginning at 0 seconds into the task, Mance read the problem silently to himself. The indicator exhibited here was reading the task (column 4) which is one way that understanding the problem was operationalized. Therefore column 5 has the transition understanding the problem. A mark was placed at $(0, 1)$, time 0 seconds and height 1 corresponding to the transition understanding, on the MTD. At 15 seconds, Mance began writing $c_{1,f} = 500$ g and $c_2 = 0$ g/L. The indicators exhibited here were taking stock of data/information given and listing/mentioning variables, parameters, or constants. These activities corresponded to the transitions understanding and simplifying/structuring. In the MTD, two marks were placed. One at $(15, 1)$ and another at $(15, 2)$, corresponding to understanding and simplifying/structuring being observed at 15 seconds. Later, at time $2:53$, Mance explained his work, “If you just multiply those two [expressions] together, you’ll have five times the buffering strength entering and that’d give you grams per minute.” He also wrote $5\left(1 - e^{-\frac{t}{60}}\right)$ g/min to accompany this statement. The statement and accompanying written work were coded with the indicators speaking in terms of operations and dimensional analysis which correspond to the transitions mathematizing and validating, respectively. In the MTD, two marks were placed. One at $(173, 3)$ corresponding to mathematizing occurring at time $t = 173$ seconds and another at $(173, 6)$ corresponding to validating occurring at time $t = 173$. In this way the entire transcript was coded and processed into a MTD. This procedure was carried out for all of the events. Results of the cross-case comparison are presented next.

Results

In this section, I present the cross-case analysis of the students’ modeling processes by describing the ways in which the MTDs did and did not conform to the idealized modeling cycle represented in Figure 4. An overview of the MTDs (Figure 6–Figure 10) reveals that individuals do not only move “forward” in the cycle, but return to visit previous stages of model construction. In the idealized version of a modeling cycle, the modeler transitions from one stage of model construction to the next (real situation $\xrightarrow{understanding}$ conceptual model

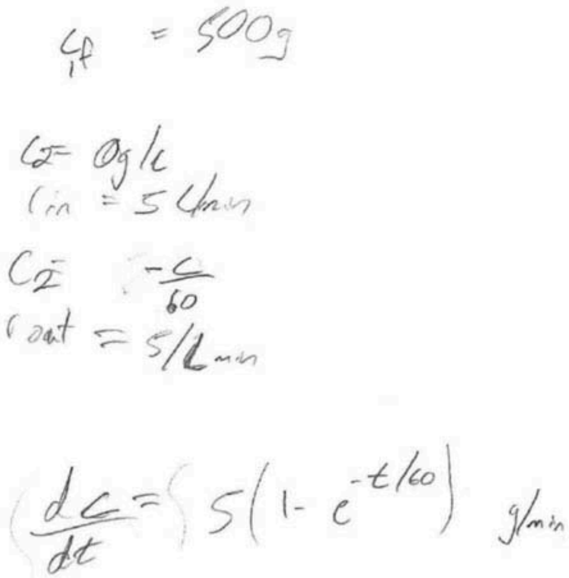

Figure 5. Mance's written work on the Tropical Fish Tank problem.

$\xrightarrow{simplifying/structuring}$ real model $\xrightarrow{mathematizing}$ mathematical representation $\xrightarrow{working mathematically}$ mathematical results $\xrightarrow{interpreting}$ real results $\xrightarrow{validating}$ real situation) cycling through this sequence of transitions until a satisfactory model is constructed. However in any single MTD, there are transitions that appear out-of-order. This was largely because three of the modeling transitions (understanding, simplifying/structuring, and validating) appeared early and often throughout the students' modeling processes. Because each student was given as much time as needed to explore the tasks, not all students got to try all of the planned tasks. In particular, the Baker's Yeast Problem was attempted by Mance and Trystane and Mance did not attempt the Piano Tuner's Problem. Table 5 summarizes the tasks, who tried each one, and the duration of each event.

Macrostructure: Signal and noise

A signal is defined as "any nonverbal action or gesture that encodes a message" and as "an electric quantity … whose modulation represents coded information about the source from which it comes" (Princeton University, 2006). I introduce the signal metaphor to describe the saw tooth pattern visible in Figure 4. If the "message encoded" is information about the individuals' mathematical thinking during mathematical modeling, then the signal is his passage through the activities prescribed by the research framework. The signal is graphically represented as a quantity against time. Here, it is the expression of the progression of modeling transitions $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6$. Noise is the deviations which do not conform to the signal pattern. With this metaphor in place, analysis can be presented as a characterization of the noise and its possible sources.

As an example of noise, consider the first 200 seconds of Torrhen's work on the Empire State Building (Figure 7), we see the sequence understanding, understanding, simplifying/structuring, understanding, simplifying/structuring, and simplifying/structuring as he revisited the problem statement and made assumptions about the situation. Beginning at 500 seconds, we see a jump from validating back to understanding and then to mathematizing as he adjusted his mathematical representation in light of his re-reading of the problem. The source of the noise in this case is

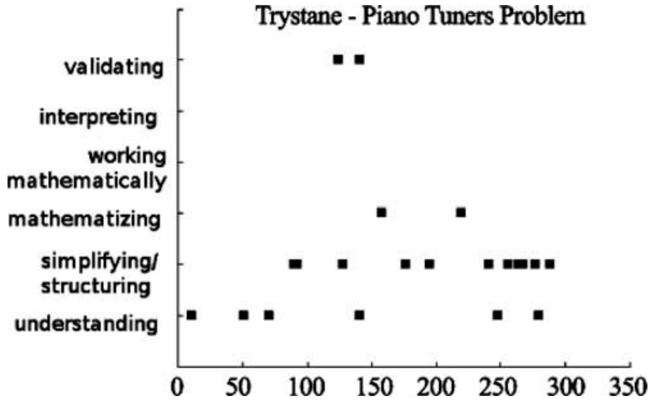

Figure 6. Trystane's MTD for the cell problem.

Table 5. Mean time on task and students' attempted tasks.

Task	Attempted by	Mean Time on Task
Cell Problem	All	17:41
Empire State Building Problem	All	13:48
Piano Tuners Problem	Torrhen, Trystane, Orys	16:34
Falling Body Problem	All	26:26
Tropical Fish Tank Problem	All	39:12
Baker's Yeast Problem	Mance, Trystane	8:54

Torrhen's adjustment of his mathematical representation (mathematizing), which he does without revising his assumptions (simplifying/structuring). As another example, the excerpt of Mance's work on the falling body problem in Table 6 shows how transitions may be out of order but the individual continues to progress in the modeling process.

The noise visible in the MTDs did not have a single character from event to event, across events, across tasks, or across individuals. For example, in the ambiguous Fermi type problems, the most common type of noise was a notable presence of the simplifying/structuring activity. For the non-Fermi Falling Body problem, validating was pervasive in only some of the events. Orys (a mathematically strong student) tended to exhibit validating transitions more consistently within and across tasks, but Torrhen (also a mathematically strong student) exhibited validating transitions more often in any single event. Working mathematically was the least exhibited activity among all participants and across all events, but was most visible in the Tropical Fish Tank problem where the analysis of a differential equation was required. Taken together, these observations suggest that the modeling process is idiosyncratic and dependent upon the knowledge, strategies, and techniques that the individual brings into the task and also on task characteristics.

The following results present and describe the common sources of noise: a substantial presence of understanding, simplifying/structuring, and validating activities (as compared to the idealized MTD in Figure 4). The pervasiveness of these transitions within any single event points to a heightened interaction between the tasks, characteristics of the individual, and his knowledge bases for which has yet to be accounted.

Macrostructure: Banding

Data indicated a correspondence between the plurality of bands in the MTD and the modeler's approach to the task. A single band corresponds to the student applying a single scheme to solve a well-known problem type. Multiple bands corresponds to the student making multiple passes

Table 6. Excerpt from the event map for Mance's work on the Tropical Fish Tank problem.

Time (mm:ss)	Action	Description	Indicator	Transition
0	S reads problem silently		Reading the task	Understanding
0:15	S begins writing	((working silently)) [[writes $c_{1f} = 500$g; $c_2 = 0$ g/L]]	Taking stock of data/information given; listing variables, parameters, etc.	Understanding; simplifying
1:15	S traces words in task S works silently		Returning to elements in the statement of the task	Understanding
1:58	S explains work	It's uh, it gave me the initial values, so um, the strength, of the buffering solution varies according to time according to $1 - e^{-\frac{t}{60}}$ grams per liter.	Specifying conditions;	Simplifying/ structuring
2:10	S explains work	You're given a rate in which is the buffering solution enters at 5 L /min [[writes $r_{in} = 5$ L/ min]]	Listing parameter	Simplifying/ structuring
2:15	S explains work	so if you multiply those two together you're gonna get an answer in grams per minute and that's how fast it's gonna be changing	Speaking in terms of mathematical operations; interpreting meaning	Working mathematically; mathematizing
2:28	S explains work	You know your fish tank is 300 liters. [[writes $V_i = 300$ L]]	Stating parameter	Simplifying/ structuring
2:33	S explains work	In order for the fish tank to not over flow I'm gonna assume that rate in is also your rate out.	Making assumption to simplify the task	Simplifying/ structuring
2:39	S explains work	So the rate out is 5 L per minute. [[writes $r_{out} = 5$ /L min]]	Assigning value to parameter	Simplifying/ structuring
2:46	S explains work	You can make that [[the rate out]] negative depending on how you wanna look at it.	Operationalizing quantities or relationships	Simplifying/ structuring
2:53	S explains work	If you just multiply those two together you'll have 5 times the buffering strength entering and that'd give you grams per minute. [[writes $5\left(1 - e^{-\frac{t}{60}}\right)$ g/min]]	Speaking in terms of operations; dimensional analysis	Mathematizing; validating
3:02	S explains work	It's asking for how much buffering is in it at any point in time	Clarifying what needs to be accomplished	Understanding
3:07	S explains work	If you were to plug in a time for that you'd be multiplying for a minute rate.	Speaking about algebraic manipulations; considering whether result answers question posed	Working mathematically; interpreting
3:13	S explains work	That would give you the amount of buffering agent in the tank at that given time.	Speaking about result in context of problem; considering whether result answers question	Interpreting
3:28	S explains work	So, I think.	Implicit statement about reasonableness of answer	Validating
3:32	S reads problem aloud	strength of buffering solution	Returning to elements in statement of the task	Understanding
3:36	S affirms	Yeah, that'd be right.	Implicit statement about reasonableness of answer	Validating
3:50	S justifies	So I think it's, C is equal to 5 times that. Because if you plug in time you're gonna get an answer in grams and that's what you want.	Speaking in terms of symbols, operations, or relationships; dimensional analysis	Mathematizing; validating

through the transitions in the research framework as he dealt with the decision-making required during mathematical modeling.

Some of the MTDs exhibited only one band from the saw tooth pattern (the sequence understanding simplifying/structuring mathematizing working mathematically interpreting validating) whereas others exhibited multiple bands. Each band corresponds to one pass through the transitions of the research framework. For example, though Mance's MTD from the Falling Body problem (Figure 9) is noisy, there was a definite upward rightward tendency to the graph. This tendency corresponded to his selection of one situation model and real model and sticking with them throughout the task. He did not revise his mathematical representation nor his conceptualization of the real situation.

In other events, such as Torrhen on the Piano Tuners problem (Figure 8), there were multiple bands. In this event, there were two groupings of bands (0–1000 seconds and 1000–2000 seconds), which corresponded to his change in approach to the task. Another example is Trystane's work on the Falling Body problem (Figure 9), where each band corresponded to a change in assumptions (e.g., introducing an additional variable) or to a change in the mathematics he used to address the task (e.g., switching from algebraic to differential equations).

Pervasiveness of understanding

The understanding transition was not limited only to the beginning of an event, but could be found interspersed among other transitions. The most common indicator for this activity was the student returning to the problem statement by reading it (or part of it) aloud. This suggested that individuals were regulating their modeling processes (in the sense of Schoenfeld, 1992) by monitoring how their immediate goals or subgoals related to the problem statement. In other instances the student returned to the statement to find or justify information for the simplifying/structuring transition. Although the pervasiveness of the understanding transition is not explicitly accounted for by the research framework, it can be explained by considering an individual's understanding of what needs to be done as a moving target. The modeler forms an initial interpretation of the problem (Doerr & Tripp, 1999; Lesh et al., 2003), which is a consequence of his understanding of the problem statement. The mathematical model and the individual's conceptualization of how mathematics could be used evolve in tandem (Kehle & Lester, 2003; Schwarzkopf, 2007; Verschaffel et al., 2000). The individual's interpretation of the task therefore impacts the individual modeling route and subsequent mathematical thinking. Revisions in the individual's conceptualization appear as understanding without necessarily being followed by the rest of the modeling transitions. Thus, attending to understanding suggests a way to study the interaction between task and individual.

Pervasiveness of the simplifying/structuring transition

The simplifying/structuring transition was observed throughout the events, but was especially pervasive on the Fermi problems or during tasks when the individual frequently considered the real-world context of the task (e.g., Trystane in the Falling Body problem). During the Fermi problems, students repeatedly considered the real-world context in order to identify and account for quantities that would impact the predicted quantity, detect sources of variation in those quantities, and consider qualitative relationships among them. The participants were more willing to visibly reason about assumptions and estimates in these tasks in order to connect the real world situation to mathematics. These tasks were not necessarily more authentic, but they facilitated entry to the problem because they used more familiar mathematical structures somehow bringing the situations "closer" to reality. Fermi tasks encouraged this type of activity independently of the individual or his progress through the transitions of the research framework, confirming that they in particular are valuable resources for evoking students' assumption-making and estimating.

Pervasiveness of the validating transition

Validating occurred early and often throughout the students' work, with the exception of two of the MTDs (Trystane on the Empire State Building problem, Figure 7, and Mance on the Falling Body problem, Figure 9). This observation differs from the predictions of the research framework, which places the validating transition at the end of each full cycle. Validating often occurred at sites where there were no real results to verify, and so the students must have been justifying whether to accept or reject some other aspects of their models. For instance, in Table 6, Mance's work, validating occurs at 3:28 and 3:36. In both cases, Mance confirmed the reasonableness of the mathematical model by commenting directly on the representation ("That would give you the amount of buffering agent in the tank at that given time. *So I think*") and comparing it to what was requested in the problem statement ("…strength of buffering solution. … *Yeah, that'd be right*"). Both instances of validating were coded as such because they fit the indicators, but neither compared the outcome of mathematical analysis to a real-world setting. In this case, Mance was validating that the model satisfied the request of the problem statement.

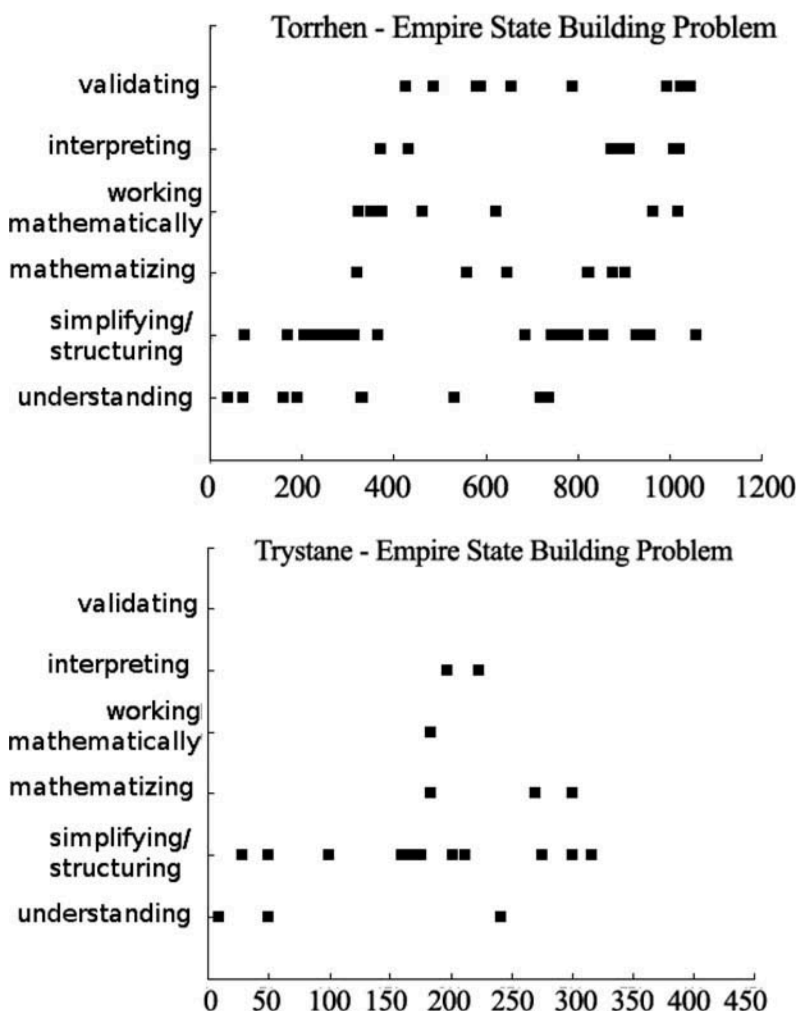

Figure 7. Modeling transition diagrams for the Empire State Building problem (Top: Torrhen; Bottom: Trystane).

Across all tasks, Mance tended to rely primarily on checking that his analysis of the model was correct (i.e., checking his computations). He tacitly assumed that the mathematical model was accurate once he had recorded a symbolic representation for it. His work on the Falling Body problem exemplifies this trend—he checked and corrected his manipulations of the kinematics equations he introduced but not whether it represented the situation (it did not). Torrhen and Orys also checked their computations but also regularly performed dimensional analysis and checked whether their models were representative of the situation in the problem context.

Trystane's MTD for the Empire State Building problem misses validating transition altogether. Of the four participants, Trystane was the only one who had never visited a tall building and therefore he did not have any lived experiences riding a tourist elevator. He made no observable attempt to judge whether the real result was appropriate or whether the variables he selected were relevant to the model (beyond assuming that they were).

Anomalies

Three of Trystane's MTDs were anomalous when compared to the idealized MTD and to the other events in the data sample. They were anomalous because they were missing one or more modeling transitions. The Empire State Building problem (Figure 7) had no validating transition and the Piano Tuners (Figure 8) and Cell (Figure 6) problems had no working mathematically or interpreting activities. These MTDs can be explained by examining the interaction between the Trystane's knowledge and the task.

Trystane was the only of the four students to never have ridden the tourist elevator in a tall building. Although he constructed a mathematical model, analyzed it, and interpreted it, his work was based on his lived experiences in regular elevators. He lacked relevant encyclopedic knowledge of the world (Stillman, 2000), which negatively impacted his model building capacity. For example, he considered the number of stops made by the elevator and built this factor into his model. Although he was able to analyze the model, he did not outwardly reflect on whether the variables he was selecting were indeed relevant or whether the final answer was reasonable. Either of these would have been tagged as validating transitions and their absence explains the lack of validating transitions in the MTD.

In the Cell problem and Piano Tuners problem, Trystane did not exhibit working mathematically or interpreting activities. In both cases, he recorded mathematical representations that included parameters he could not estimate. He was left with models that could not be analyzed even though they represented his interpretation of the situation. In the Cell problem, Trystane adjusted his variables and parameters multiple times as he searched for a set that was compatible with information (parameter estimates) he found using a Google search. When he realized that an accurate answer would require considering the sizes of cells from different organs, he alluded to a weighted sum model using verbal and diagrammatic descriptions. A symbolic representation never materialized and so he could not move forward with mathematical analysis.

In the Piano Tuners problem, Trystane set up a proportion to relate the number of piano tuners in New York to the number of piano tuners in Chicago. He was unable to use proportional reasoning to analyze the model because he did not have enough information about either city. On multiple occasions, however, Trystane assessed his choices for variables to include in the model and how they might be related, demonstrating his intention to validate many components of the model, not just its predictions.

These anomalies demonstrate that although the MTDs are highly idiosyncratic, the MTDs make it possible to make inferences about the mathematical modeling process. Taken together, these examples provide evidence that the validating transition is broader and more functional than checking whether a mathematically derived prediction is in line with the real world. Trystane's work on the Empire State Building problem demonstrates that carrying out the validating transition is related to his ability to

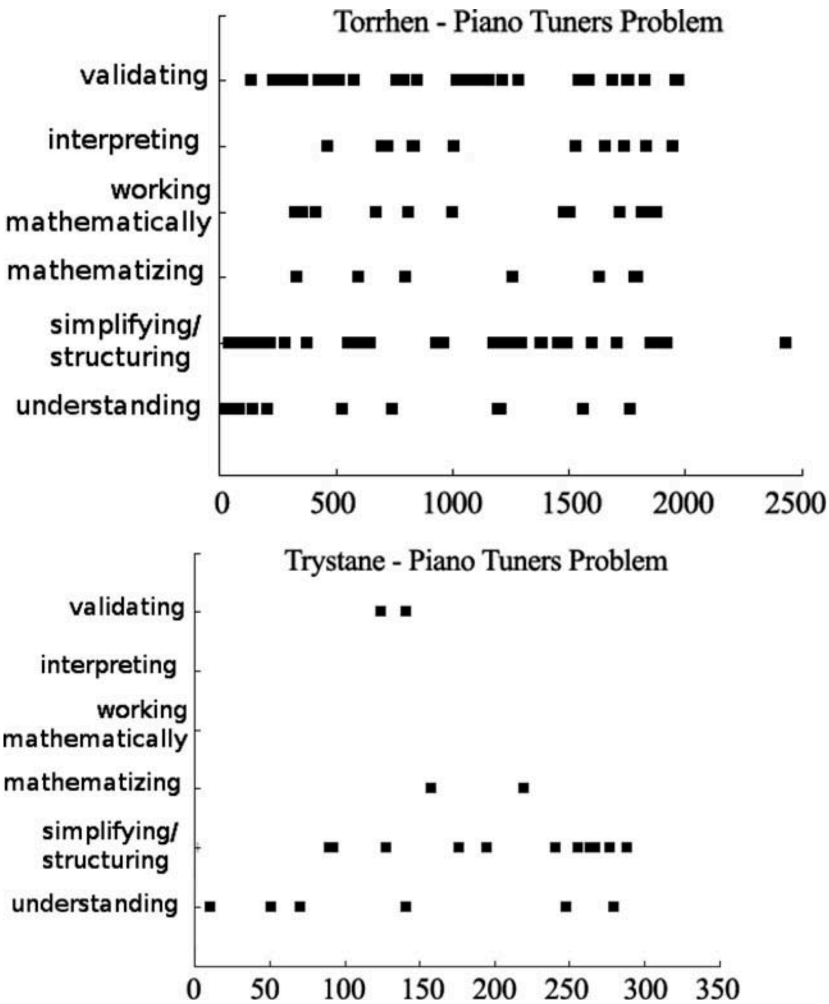

Figure 8. Modeling transition diagrams for the Piano Tuners problem (From the top: Torrhen, Trystane).

develop expectations about the situation. His work on the other two tasks demonstrate that validating can occur even when working mathematically and interpreting are not present. Since all three anomalies occur for Trystane on Fermi problems, they may be task and individual dependent. However, in general, the MTDs show visually that modeling transitions are not strictly ordered.

Discussion

Blum and Leiß (2007) mathematical modeling cycle (the research framework) provides a view of the mathematical modeling process as a cyclic flow of validating and revising models. The flow moves through various transitions as the modeler progresses in developing the model. Drawing on this work, the current study captured students' modeling routes by graphically representing them as MTDs in an attempt to document how individuals carried out the modeling process in connection to mathematical thinking. In parallel with the mathematical modeling process itself, the remainder of the article discusses the interpretation and validation of the results to reflect on both the model and the theory that produced it.

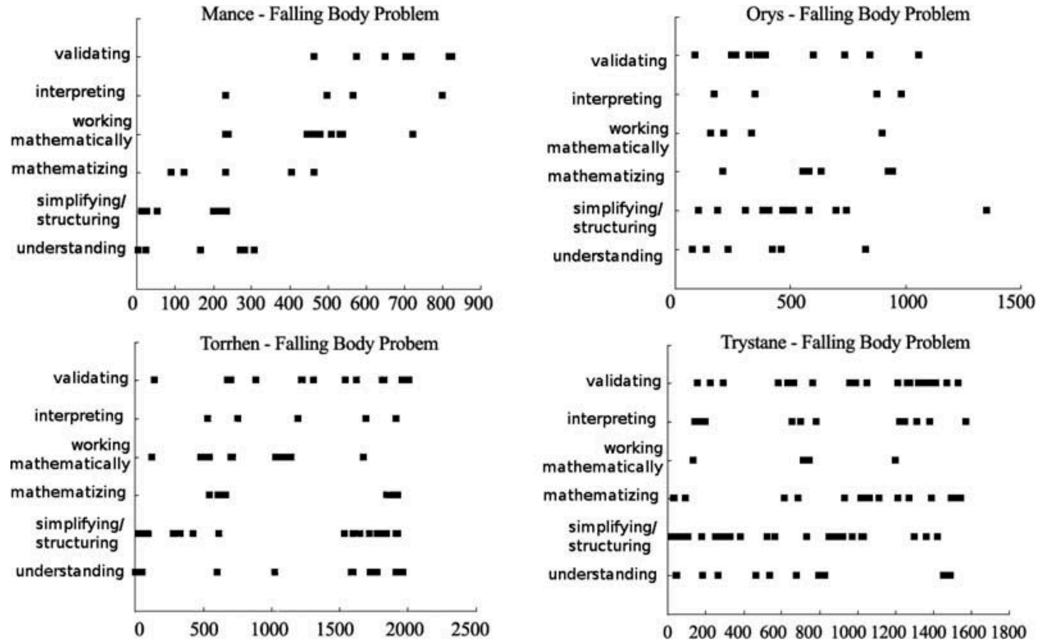

Figure 9. Modeling transition diagrams for the Falling Body problem (From the top: Torrhen, Orys, Trystane).

Prevalence of the validating transition was perhaps the most surprising observation since it was predicted to potentially occur prominently at the end of a modeling cycle. Yet successful problem solvers do spend more time thinking about the problem and tend to go back and reflect (Schoenfeld, 1985). The MTDs from the two students in the high math category tended to exhibit more instances of understanding and validating than the two students from the low math category. This suggests some commonalities between successful problem solving and progress in mathematical modeling through analogous activities. Future research could examine the extent to which encouraging validating at various stages of model construction may facilitate the students' progress and the extent to which looking back and validating overlap.

Because it is difficult to measure divergence between the idealized modeling process and the empirical data, this study qualitatively described how students' modeling processes intersected with the different phases depicted in the research framework. Analysis of the MTDs provided further support that the modeling process is complex and often does not match a sequential progression towards model construction as described in various mathematical modeling cycles. In particular, micro-level analysis of students' work demonstrated that the mathematical modeling process could be directly connected to observable mathematical activities. Macro-level analysis across events revealed that individuals engage in understanding, simplifying/structuring, and validating transitions throughout the modeling process, not only at their respective places in the research framework. Indeed, modeling processes may be better described as "haphazard jumps between different stages and activities" (Ärlebäck, 2009, p. 353). Micro-level analysis revealed that students' progress in mathematical modeling is highly dependent on their own mathematical thinking and nonmathematical knowledge. Therefore, analysis confirms earlier hypotheses that the modeling process is idiosyncratic (Ärlebäck, 2009; Borromeo Ferri, 2007; Niss, Blum, & Galbraith, 2007).

Although the sampling categories did not drive analysis, it is still possible to consider the variation in individuals' modeling routes represented in the MADs participant-by-participant or task-by-task. In this work, no universal patterns could be identified. When relevant sources of

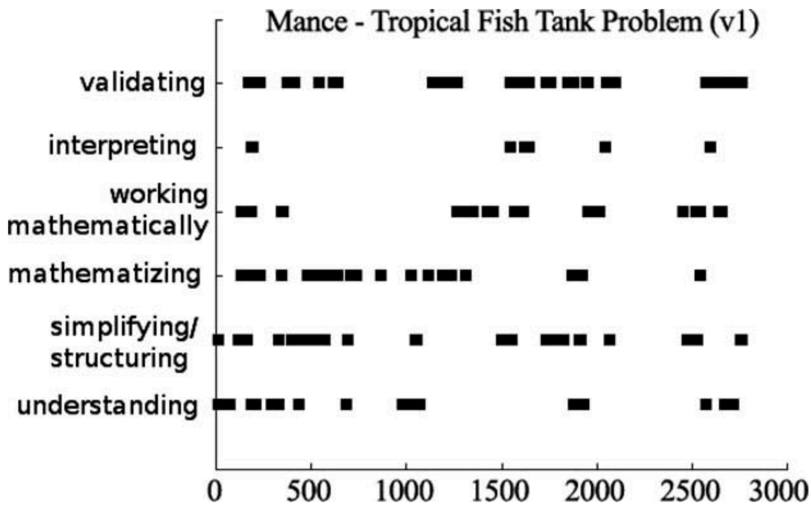

Figure 10. Mance's modeling transition diagram for the Tropical Fish Tank problem.

knowledge were either missing or inaccessible, some modeling transitions seemed to vanish and subsequently halt progress. At other times, the modeler continued on without hindrance. For example, Trystane's work on the Empire State Building problem provides evidence that without personal knowledge of the situation, a validated mathematical model may not be constructed. Mance's work on the Falling Body Problem exemplifies which are familiar both mathematically and contextually produce MTDs that more closely resemble that corresponding to the ideal modeling cycle. Trystane's work on the Empire State Building problem had no validating transition, and Orys's work could be characterized by a strong presence of validating transitions. In particular, the simplifying/structuring and validating transitions were pervasive in his work on the Piano Tuners. In contrast, Trystane's work on the Piano Tuner's problem was also missing some transitions. However Mance's work on the Falling Body Problem and Torrhen's work on the Empire State Building problem very closely resemble the predictions from the research framework.

Some scholars have developed alternative frames for analyzing students' modeling patterns in order to address the naturally arising question of how modeling processes depend on prior knowledge. For example, Stillman (2000) offered a tripartite framework to characterize the knowledge sources brought to bear on a task: academic knowledge (knowledge learned in a classroom environment), encyclopedic (general knowledge about the world), and episodic (lived experiences). Episodic prior knowledge was found to be more likely to have a positive influence on progress than the other two types of knowledge. These results echo earlier findings that primary grade students were more successful in solving word problems when the contexts they encountered involved familiar events, people, or activities (Verschaffel et al., 2000). Galbraith and Stillman (2006) and Stillman, Brown, and Galbraith (2013) used this framework to elaborate task-specific knowledge elements that could hinder student progress in executing transitions in a modeling cycle.

The absence of discernable task-by-task or participant-by-participant patterns in the MTDs, coupled with previous findings, suggest two inferences. First, well-developed prior knowledge bases are important to modeling success. Second, future research may examine interactions among knowledge bases, rather than documenting whether the student brought that knowledge to bear. This would aid in theorizing how each knowledge base contributes to successful modeling. For example, one might consider how personal, lived experience guides the student's identification and prioritization of related variables, parameters, assumptions, conditions, and constraints (e.g., simplifying/structuring during the Empire State Building task) or whether previous scholastic experience with a problem context makes that task routine (e.g., the Falling Body Problem). Examples from physics education research (e.g., Wittmann, 2006) may be well-adapted to suit these purposes.

It is intuitively sensible to suggest the MTDs did not resemble the research framework because students are not yet "expert modelers." If that is the case, then the implications and recommendations would be to design a set of interventions for educators to carry out that would help reduce differences between the MTDs produced by students and the illustration in Figure 4. Such a conclusion, however, fails to provide convincing explanation for why students should be coached to produce MTDs that resemble those of "expert modelers."

First, it is not clear what is meant by an "expert" modeler. Applied mathematicians, engineers, statisticians, life and physical scientists, those who teach mathematical modeling, and classroom mathematics teachers may all be considered modeling "experts." Their individual modeling routes may vary substantially depending on the context to be modeled, the mathematics required to do so, and the individual's familiarity with the mathematical and nonmathematical domain. That is, expert modeling may not yield "perfect" MTDs. By extension, the modeling processes of each type of expert may vary and it is therefore unclear which "experts" we would want our students' growth to imitate.

Second, there is an implicit assumption that "perfect" MTDs would indicate that expert-level modeling is occurring. This is not necessarily the case. The MTDs do not reveal whether the mathematical model is accurate nor whether it serves its purpose. For example, Mance's work on the Falling Body Problem resembled the ideal MTD but his model was incorrect, was analyzed with errors, and produced an inaccurate result. In contrast, Orys's work on the Piano Tuners Problem yielded a reasonable prediction but his modeling route involved a lot more simplifying/structuring and validating than predicted by the modeling cycle. Likewise, Torrhen's work on the Piano Tuners Problem did not resemble the ideal MTD but he too obtained a reasonable estimate. The MTDs are limited to revealing only the observable modeling transitions. Thus, setting the ideal MTD as a goal for students' modeling processes may not necessarily result in successful modeling.

It is possible that "perfect" MTDs may not exist. It is also possible that they may only be elicited when the real problem has already been solved by the modeler and in effect he or she knows what to do. In science and industry, mathematical models are developed based on aspects of a real-world system that are understood by the modeler. They are used to measure, make decisions, replicate systems, predict outcomes, explain outcomes, or manipulate the system (Thompson & Yoon, 2007). Toward a given purpose, the modeler infers properties or consequences of aspects of the system that are not understood through analysis of the mathematical model. In terms of problem solving: they use givens to work out unknowns. Many aspects of the mathematical models are implicit and are used as interpretational systems by the modeler rather than as explicit objects of reasoning (Borromeo Ferri & Lesh, 2013). In the present study, the students were simultaneously deriving the mathematical model and communicating it to the interviewer. Thus the variation among MTDs could be representative of the difficulty in moving from an implicit model to an explicit model. It may be that already-explicit models conformed to the research framework and that "perfect" MTDs are a consequence of experience. New lenses on modeling processes are necessary in order to examine ways in which implicit models are used implicitly and how they are (if ever) made explicit.

These considerations caution us about interpreting modeling cycles as the key to instruction about how to carry out mathematical modeling processes. Theories that describe processes like learning, modeling, or problem solving (e.g., Action, Process, Object, Schema [APOS] theory, van Hiele levels, and Polya's problem solving steps) help researchers describe global construction of an entity such as a mathematical structure or a problem solution. As has been pointed out, conceptual development within an individual happens in fits and starts with fallbacks and regressions (Battista, 2004; Pirie & Kieren, 1994). Global models are not always helpful in describing an individual's route and may even serve to impede understanding an individuals' thinking.

Likewise modeling cycles, including the research framework or that presented in the CCSSM (2010), are protocols that serve to organize student activity into large-scale structure of how a mathematical model is constructed. Such overviews do not necessarily capture the nuances of the winding and idiosyncratic paths which depend on student knowledge of the problem context, the mathematical knowledge available and seen as relevant to him, nor his capacity to recognize when a model must be adjusted in light of either of those knowledge bases. That is, like presentations of Polya's characterization of problem solving, modeling cycles are descriptive rather than prescriptive.

Because modeling cycles function as a tool to explain the modeling process to teachers and to teacher educators, it should be emphasized that modeling cycles themselves are models of a process and not the reality of that process. Such an emphasis would provide for a more inclusive view of students' work when it does not conform to the ideal modeling cycle. It would also support moving "beyond the notion of trying to directly teach what experts apparently 'do' to novices" (p. 769), and beyond transforming descriptive information about experts' behavior into (potentially unhelpful) prescriptive lists for students to learn (Lesh & Zawojewski, 2007).

The three transitions understanding, simplifying/structuring, and validating were pervasive in the MTDs because problems where real-world givens and goals must be interpreted mathematically are not straightforward. They seem integral to progress in mathematical modeling because the primary goal of modeling is to adapt or create a mathematical structure that can be used to interpret the real-world situation. For problems that do not reference the real world, the modeler adapts or creates a mathematical structure that can be used to interpret another mathematical situation or structure. Specifically problems requiring a proof demand interpretation and manipulation of axioms to formulate arguments that may be entirely independent of empirical reasoning. During procedural tasks, the solver may need to use notation or conventional representations to access and manipulate mathematical structure. Again, the solution may be entirely independent of empirical reasoning. Modeling, in contrast, requires empirical reasoning to carry out validating since it compares the mathematical interpretation of the real-world situation against the real world. Thus, acceptable solutions to these different classes of problems derive from differing epistemologies and might induce different cognitive demands. It is also possible that understanding, simplifying/structuring, and validating are modeling-specific instantiations of more general problem solving activities that could be observed during other types of problem solving. It would be valuable to investigate the extent to which each of the modeling transitions has analogues in other kinds of problems solving.

The Fermi tasks were useful for challenging the engineering students to reveal their mathematical thinking about variables and constraints related to the problem contexts. In the MTDs this is visible through repeated appearance of the simplifying/structuring transition. Engineering students may benefit from exposure to problems whose contexts are amenable to multiple valid mathematical models so that they can practice validating their models and selecting a "best" model according to particular criteria. Tasks that require both analysis and synthesis, rather than those which have already been analyzed in terms of domains of knowledge, would help students strengthen their thinking within each of the modeling transitions.

I note here that four methodological decisions could have impacted the results reported here. First, the grain size selected for micro-level analysis impacted the appearance of transitions in the research framework. The grain size was the shortest meaningful utterance that could be tagged with an indicator from the rubric and each tagged utterance received a timestamp. Since the coding procedure traced each transition, there were more of them graphed in the MTDs than in previous representations of individual modeling routes. The choice addressed previously reported limitations in analysis via MADs and helped to organize the modeling routes chronologically. However the grain size made it difficult to discern macroscopic structure in the MTDs. Thus analysis was capable of identifying sources of noise but it was incapable of characterizing it.

A related point is that all of the MTDs began with the understanding transition, which is predicted by the research framework. None began with validating or interpreting. This is a consequence of the coding rubric and the nature of written tasks. The student must begin by reading the task, which was classified as understanding. A student could not generally begin with validating because there is nothing to validate. Even in the absence of written tasks, it is unlikely that mathematical modeling could begin with any transition other than understanding because one needs to understand that there is a problem to begin searching for its structure.

A second point is that the small number of events sampled makes generalization to other tasks or to the population difficult. Although I was careful in analysis to describe macroscopic patterns in the collective group of MTDs and to avoid task-by-task or participant-by-participant analysis, variation in the MTDs may potentially be the result of observing error rather than an underlying phenomenon. Future research should work toward examining the sources of noise in terms of individual or task characteristics. Specifically, it should address which individual modeling routes may be more probable and under which circumstances they might be expected. Knowing how modeling routes are linked to task, individual, or even interviewer characteristics would aid in the development of tasks and learning environments that could contribute to the growth of students' modeling skills.

Third, in this study the students' modeling process was prioritized over the models' accuracy. Correcting students' work was not a part of this study's interview methodology, because neither teaching mathematics nor teaching modeling were goals of this study. Yet I recognize that an invalid model is of limited use. One of the major unresolved issues in developing modeling skills is how to encourage students to validate their models and how to get them to spontaneously recognize errors in representation or analysis. Researchers and teachers often default to pointing out students' errors so that they may be corrected. However, this interrupts the students' modeling process and the students' concept of the situation may not be compatible with that of the teacher or researcher. The final related issue is that in the current study I did strategically interact with the students' work by posing reflective questions which challenged the students' models and assumptions. These interactions may ultimately be responsible for the pervasiveness of the understanding, simplifying/structuring, and validating transitions. Therefore the impact of the interview environment and interviewer interventions should be more closely examined in order to understand what elements could be borrowed as principles for nurturing students' abilities to construct accurate, valid models.

Finally, because the participants in this study were engineering majors and as such, it is likely that they had more training in dealing with variables and constraints as they arise during simplifying/structuring and validating than would be expected from other populations. Their differential equations course was amenable to mathematical modeling and their experiences in engineering and science classes may have positively impacted their sensitivity to carrying out the modeling transitions. The students were aware of the need to carefully consider potential variables and assumptions, reevaluate problem contexts, and build in check points. The associated understanding, simplifying/structuring, and validating transitions may be supported or even explicitly demanded by the engineering curriculum. If so, research and teaching in mathematical modeling would benefit from identifying and examining these aspects of the engineering curriculum and incorporating them into the students' mathematical education. Such a move would also serve to better align the mathematics teaching and learning with the rest of the science, technology, engineering, and math curricula.

Conclusions

Any model of a human process is "good" insofar as it is useful. The modeling cycle used here (Blum & Leiß, 2007) was useful as an overview of the modeling process, for task design, and for data processing. In general, modeling cycles are a powerful way to explain modeling tasks to teachers and teacher educators and they are already being used to introduce modeling to these audiences (e.g., Anhalt & Cortez, 2015; Bal & Doğanay, 2014; Clarke, Roche, & Mitchell, 2015; CCSSM, 2010). Because researchers are engaged in a model development process that parallels the students' mathematical model construction process (Lesh & Carmona, 2003), it is important to recognize that our early-iteration models of mathematical modeling are marked by inadequacies analogous to those of the mathematical models produced by students. As suggested by the mathematical modeling research forum at IGPME38, "modeling cycles unintentionally hide much of the real work of mathematical modeling" (Cai et al., 2014). Indeed, the MTDs revealed that the modeling process looks messy and haphazard in real time lending support to the idea that modeling cycles are highly idealized and simplified (Ärlebäck, 2009). For example, when the modeling task is challenging or the individual does not bring to bear the anticipated domain or contextual knowledge, the MTD may not be recognizable as a modeling cycle. Alternatively, when a task has become routine for an individual, the modeling route for that task may better resemble the modeling cycle overview.

The MTDs are a useful tool for documenting individuals' movement through the modeling process and provide a vehicle for systematically examining our models of mathematical modeling. Yet there are limits to relying solely on the MTDs as a source of information about how individuals are thinking mathematically, how they understand the modeling task, or how they validate their models. In this regard, it may be more beneficial to idealize the qualities and skills we want students to develop and then use modeling tasks to foster those, rather than popularizing modeling cycles as an ideal to imitate (regardless of how closely it reflects experts' mathematical modeling processes). Thus, the way forward is not to obtain a correct model of individuals' modeling processes—no model is correct. The way forward is to go beyond the schematics to understand how individuals synthesize their mathematical and nonmathematical knowledge.

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References

Anhalt, C. O., & Cortez, R. (2015). Mathematical modeling: A structured process. *The Mathematics Teacher*, *108*(6), 446–452. doi:10.5951/mathteacher.108.6.0446

Ärlebäck, J. B. (2009). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast*, *6*(3), 331–364.

Arney, D. C. (Ed.). (1997). *Interdisciplinary lively application projects*. Washington, DC: The Mathematical Association of America.

Bal, A. P., & Doğanay, A. (2014). Improving primary school prospective teachers' understanding of the mathematics modeling process. *Educational Sciences: Theory & Practice*, *14*(4), 1375–1385.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning*, *6*(2), 185–204. doi:10.1207/s15327833mtl0602_6

Becker, H. (1998). *Tricks of the trade*. Chicago, IL: University of Chicago Press.

Becker, H. (2007). *Writing for social scientists*. Chicago, IL: University of Chicago Press.

Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling: Education, engineering, and economics* (pp. 222–231). Chichester, UK: Horwood.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37–68. doi:10.1007/BF00302716

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM—International Journal on Mathematics Education*, *38*(2), 86–95. doi:10.1007/BF02655883

Borromeo Ferri, R. (2007). Modelling problems from a cognitive perspective. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modeling: Education, engineering, and economics* (pp. 260–270). Cambridge, UK: Woodhead Publishing Limited.

Borromeo Ferri, R., & Lesh, R. (2013). Should interpretation systems be considered mathematical models if they only function implicitly. In G. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 57–65). Dordrecht, The Netherlands: Springer.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, *2*(2), 141–178. doi:10.1207/s15327809jls0202_2

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Geiger, V., Stillman, G., & Kwon, O. (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional, and teacher education

perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th meeting of the international group for the psychology of mathematics education* (pp. 145–172). Vancouver, Canada: IGPME.
Clarke, D. M., Roche, A., & Mitchell, A. (2015). Mathematical modeling and pure mathematics. *Mathematics Teaching in the Middle School*, *20*(703), 476–482. doi:10.5951/mathteacmiddscho.20.8.0476
Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13. doi:10.3102/0013189X032001009
Community of Ordinary Differential Equations Educators. (2012). *CODEE digital library*. Retrieved from http://www.codee.org/
Czocher, J. A., Tague, J., & Baker, G. (2013). Where does the calculus go? An investigation of how calculus ideas are used in later coursework. *The International Journal of Mathematical Education in Science and Technology*, *44*, 673–684. doi:10.1080/0020739X.2013.780215
Dahlberg, R. P., & Housman, D. L. (1997). Facilitating learning events through example generation. *Educational Studies in Mathematics*, *33*(3), 283–299. doi:10.1023/A:1002999415887
Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, *34*(2), 110–136. doi:10.2307/30034902
Doerr, H. M., & Tripp, J. S. (1999). Understanding how students develop mathematical models. *Mathematical Thinking and Learning*, *1*(3), 231–254. doi:10.1207/s15327833mtl0103_3
Dym, C. (2004). *Principles of mathematical modeling*. Burlington, MA: Elsevier.
Edwards, D., & Hamson, M. (2007). *Guide to mathematical modeling* (2nd ed.). New York, NY: Industrial Press.
English, L. D. (2006). Mathematical modeling in the primary school: Children's construction of a consumer guide. *Educational Studies in Mathematics*, *63*(3), 303–323. doi:10.1007/s10649-005-9013-1
English, L. D., Fox, J. L., & Watters, J. J. (2005). Problem posing and solving with mathematical modeling. *Teaching Children Mathematics*, *12*(3), 156–163.
Epstein, J. (2013). The calculus concept inventory–Measurement of the effect of teaching methodology in mathematics. *Notices of the AMS*, *60*(8), 1018–1026.
Galbraith, P., & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modelling process. *Zentralblatt Für Didaktik Der Mathematik*, *38*(2), 143–162. doi:10.1007/BF02655886
Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory*. New Brunswick, NJ: Aldine Transaction.
Glesne, C. (1999). *Becoming qualitative researchers*. New York, NY: Longman.
Goetz, J. P., & de Lecompte, M. (1981). Research and the problem of data reduction. *Anthropology & Education Quarterly*, *12*(1), 51–70. doi:10.1525/aeq.1981.12.1.05x1283i
Haines, C., & Crouch, R. (2001). Recognizing constructs within mathematical modelling. *Teaching Mathematics and Its Applications*, *20*(3), 129–139. doi:10.1093/teamat/20.3.129
Haines, C., Crouch, R., & Davis, J. (2001). Understanding students' modeling skills. In J. F. Matos, K. Houston, W. Blum, & S. P. Carreira (Eds.), *Modelling and mathematics education* (pp. 366–380). Chichester, UK: Woodhead Publishing.
Hamilton, E., Lesh, R., Lester, F., & Brilleslyper, M. (2008). Model-Eliciting Activities (MEAs) as a bridge between engineering education research and mathematics education research. *Advances in Engineering Education*, *1*(2), 1–25.
Huber, M. (2010). *Teaching differential equations with modeling and visualizations*. Retrieved from http://www.codee.org/library
Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *Zentralblatt Für Didaktik Der Mathematik*, *38*(2), 82–85. doi:10.1007/BF02655882
Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *Zentralblatt Für Didaktik Der Mathematik*, *38*(3), 302–310. doi:10.1007/BF02652813
Katz, J. (1983). A theory of qualitative methodology: The social system of analytic field work. In R. M. Emerson (Ed.), *Contemporary field research: Perspectives and formulations* (pp. 127–148). Boston, MA: Little, Brown and Company.
Kehle, P. E., & Lester, F. K. (2003). A semiotic look at modeling behavior. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 97–122). Mahwah, NJ: Routledge.
Lesh, R., & Carmona, G. (2003). Piagetian conceptual systems and models for mathematizing everyday experiences. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 71–96). Mahwah, NJ: Routledge.
Lesh, R., Cramer, K., Doerr, H. M., Post, T., & Zawojewski, J. S. (2003). Model development sequences. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 35–58). Mahwah, NJ: Routledge.
Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning and problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 3–33). Mahwah, NJ: Routledge.
Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003). Beyond constructivism. *Mathematical Thinking and Learning*, *5*(2–3), 211–233. doi:10.1080/10986065.2003.9680000

Lesh, R., Galbraith, P. L., Haines, C., & Hurford, A. (2013). *Modeling students' mathematical modeling competencies*. (R. Lesh, P. Galbraith, C. Haines, & A. Hurford, Eds.). Boston, MA: Springer US.
Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 591–645). Mahwah, NJ: Lawrence Erlbaum Associates.
Lesh, R., Kelly, A. E., & Yoon, C. (2008). Multitiered design experiments in mathematics, science, and technology education. In A. E. Kelly, R. A. Lesh, & J. Y. Baek (Eds.), *Handbook of design research methods in mathematics education* (pp. 131–148). New York, NY: Routledge.
Lesh, R., & Yoon, C. (2007). What is distinctive in (our views about) models & modelling perspectives. In P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 161–170). New York, NY: Springer.
Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763–804). Charlotte, NC: Information Age.
Maaß, K. (2010). Classification scheme for modeling tasks. *Journal Für Mathematik-Didaktik*, *31*(2), 285–311. doi:10.1007/s13138-010-0010-2
Mason, J., & Davis, J. (1991). *Modelling with mathematics in primary and secondary schools*. Sydney, Australia: UNSW Press.
Meira, L. (2002). Mathematical representations as systems of notations-in-use. In K. Gravemeijer, R. Leher, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 87–104). Dordrecht, The Netherlands: Kluwer Academic.
Mousoulides, N. G., Christou, C., & Sriraman, B. (2008). A modeling perspective on the teaching and learning of mathematical problem solving. *Mathematical Thinking and Learning*, *10*, 293–304. doi:10.1080/10986060802218132
Mousoulides, N. G., & English, L. D. (2011). Engineering model eliciting activities for elementary school students. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modeling* (pp. 221–230). New York, NY: Springer.
Nair, G. (2010). *College students' concept images of asymptotes, limits, and continuity of rational functions*. Columbus, OH: The Ohio State University.
National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices, Council of Chief State School Officers.
Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3–32). New York, NY: Springer.
Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: how can we characterize it and how can we represent It? *Educational Studies in Mathematics*, *26*, 165–190. doi:10.1007/BF01273662
Pollak, H. O. (1979). The interaction between mathematics and other school subjects. In UNESCO (Ed.), *New trends in mathematics teaching* (Vol. IV, pp. 232–248). Paris, France: UNESCO.
Princeton University. (2006). Signal. In *Wordnet 3.0*. Princeton, NJ: Author. Retrieved from http://wordnetweb.princeton.edu/perl/webwn?s=valid
Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, *26*(3), 195–210. doi:10.1016/j.jmathb.2007.09.004
Rasmussen, C., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(2), 161–172. doi:10.1080/002073900287219
Rowland, D. R. (2006). Student difficulties with units in differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology*, *37*(5), 553–558. doi:10.1080/00207390600597690
Schoenfeld, A. H. (1982, March). On the analysis of two-person problem solving protocols. In *Proceedings of the 66th annual meeting of the American Educational Research Association*, New York, NY, National Science Foundation.
Schoenfeld, A. H. (1985). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361–379). Hillsdale, NJ: Lawrence Erlbaum Associates.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan Publishing Company.
Schwarzkopf, R. (2007). Elementary modelling in mathematics lessons: The interplay between "real world" knowledge and "mathematical structures." In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics* (pp. 209–216). New York, NY: Springer.
Spradley, J. (1980). *Participant observation*. New York, NY: Holt, Rinehart and Winston.
Sriraman, B., Kaiser, G., & Blomhoj, M. (2006). A brief survey of the state of mathematical modeling around the world. *Zentralblatt Für Didaktik Der Mathematik*, *38*(3), 212–213. doi:10.1007/BF02652805

Sriraman, B., & Lesh, R. (2006). Estimation is natural: A case for fermi problems. *Zentralblatt Für Didaktik Der Mathematik*, *38*(3), 247–254. doi:10.1007/BF02652808

Stillman, G. (2000). Impact of prior knowledge of task context on approaches to applications tasks. *The Journal of Mathematical Behavior*, *19*(3), 333–361. doi:10.1016/S0732-3123(00)00049-3

Stillman, G., Brown, J., & Galbraith, P. (2013). Identifying challenges within transition phases of mathematical modeling activities at year 9. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 385–398). New York, NY: Springer.

Tabach, M., Hershkowitz, R., Rasmussen, C., & Dreyfus, T. (2014). Knowledge shifts and knowledge agents in the classroom. *The Journal of Mathematical Behavior*, *33*, 192–208. doi:10.1016/j.jmathb.2013.12.001

Tall, D. (1987, July). Constructing the concept image of a tangent. In *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 3, pp. 69–75). Montreal, Canada.

Thompson, M., & Yoon, C. (2007). Why build a mathematical model? A taxonomy of situations that create the need for a model to be developed. In D. K. Lyn, & D. English (Ed.), *Handbook of international research in mathematics education* (pp. 193–200). Mahwah, NJ: Routledge.

Verschaffel, L., Greer, B., & de Corte, E. (2000). *Making sense of word problems*. Lisse, Netherlands: Taylor & Francis.

Wittmann, M. (2006). Using resource graphs to represent conceptual change. *Physical Review Special Topics - Physics Education Research*, *2*(2), 020105. doi:10.1103/PhysRevSTPER.2.020105

Yoon, C. (2006). *A conceptual analysis of the models and modeling characterization of model-eliciting activities as "thought-revealing activities."* Bloomington, IN: Indiana University.

Yoon, C., & Thompson, M. (2003). Cultivating modeling abilities. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 201–209). Mahwah, NJ: Routledge.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics*, *63*(1), 89–112. doi:10.1007/s10649-005-9002-4