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DO IT THIS WAY! METACOGNITIVE STRATEGIES IN COLLABORATIVE MATHEMATICAL PROBLEM SOLVING

ABSTRACT. Recent years have seen increasing interest in the role of metacognition in mathematical problem solving, and in the use of small group work in classroom settings. However, little is known about the nature of secondary students' metacognitive strategy use, and how these strategies are applied when students work together on problems. The study described in this paper investigated the monitoring behaviour of a pair of senior secondary school students as they worked collaboratively on problems in applied mathematics. Analysis of verbal protocols from think aloud problem solving sessions showed that, although the students generally benefited from adopting complementary metacognitive roles, unhelpful social interactions sometimes impeded progress. The findings shed some light on the nature of individual and interactive metacognitive strategy use during collaborative activity.

1. INTRODUCTION

This paper reports a study of the collaborative problem solving activity of a pair of students taking a course in applied mathematics. Group and collaborative work has been used increasingly in programs involving mathematical modelling and applications (e.g. Galbraith and Clatworthy, 1990). However, while the outcomes of group learning in such programs have been encouraging, no attribution could be made in terms of the 'anatomy' of the collaborative activity. This paper focuses on the nature and quality of the interactions between students working collaboratively on application problems.

The design of the present study distinguishes it from earlier research on the role of metacognition in the performance of mathematical tasks. Much of the latter research has used either tertiary level or primary school students as the subjects (e.g. Kroll, 1988; Venezky and Bregar, 1988); and studies which have attempted to train metacognitive strategies have tended to do so within separate 'problem solving' courses (e.g. Lester et al., 1989; Schoenfeld, 1985a), rather than treat metacognition – and problem solving itself – as a thinking process common to all branches of mathematics.

The study reported here differs from such research in two ways: the two subjects were sixteen year old secondary school students; and the problems on which they worked in this study, although challenging and unfamiliar, were similar to those they were likely to meet every day in

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their mathematics classroom. The aim of the study was to investigate the metacognitive strategies the two students used while solving these problems.

1.1. *Research Questions*

The aim of the study may be succinctly expressed through four questions:

1. Is there evidence of a characteristic structure in the subjects' problem solving attempts?
2. What metacognitive strategies does each subject use during problem solving?
3. How do the students respond to being stuck?
4. How does the presence, or absence, of metacognitive behaviour influence the outcome of problem solving?

2. SELECTIVE REVIEW OF LITERATURE

Metacognitive processes such as assessing one's knowledge, formulating a plan of attack, selecting strategies, and monitoring and evaluating progress play a central role in mathematical performance by enabling effective decisions to be made regarding the allocation of time, energy, and knowledge resources, as argued by Schoenfeld (1985a).

Various approaches to discovering factors which influence problem solving have included the expert versus novice concept (Bransford et al., 1986; Glaser, 1984; Silver, 1982); cognitive processes involving the interplay between linguistic and syntactic knowledge (Mayer, 1983; Thomas, 1988 (after Newell and Simon, 1972)); schema instantiation (Lewis, 1989; Silver, 1982; Thomas, 1988); and problem solving processes (Garofalo and Lester, 1985; Schoenfeld, 1987a).

Despite the difficulty of identifying critical factors precisely the studies have led to agreement that not only do competent problem solvers have more extensive and better organised knowledge than novices, but they also exercise better control over their problem solving behaviour. This new dimension goes beyond cognition to metacognition – thinking about thinking. While metacognition is sometimes considered an elusive concept, partly because of the difficulty in distinguishing between cognitive and metacognitive processes (Garofalo and Lester, 1985; Perkins et al., 1990) we believe that the distinction is clarified nicely in one of the earliest descriptions of metacognition (Flavell, 1976, p. 232):

Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them (...) Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these [cognitive] processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective.

Our use of the term metacognition in this paper may be taken as consistent with the above description as provided by Flavell.

In the last ten years the literature on mathematical problem solving has increasingly focused on implications for learning and instruction of metacognitive processes (Garofalo and Lester, 1985; Schoenfeld, 1985a, 1987b; Silver, 1987).

Studies concerned with improving awareness and self-regulation have included the internalisation of processes previously modelled by a teacher (Campione et al., 1989; Collins et al., 1987; Silver, 1987); studies based on promoting the transition from externalisation to internalisation of learning (e.g. Schoenfeld 1985a, 1987a, 1987b) and studies deriving from the elaboration of the concept of the Zone of Proximal Development (Vygotsky, 1978; Wertsch, 1985).

Other factors which can play a part in problem solving concern students' incapacitating beliefs about the nature of mathematics and about themselves. While these are non-cognitive in nature they have the capacity to enhance or interfere with the cognitive and metacognitive resources brought to the problem solving context. Research on the influence of attitudes, beliefs, and affects include studies by McLeod (1988) on the impact on strategic decision making; studies on students' perceptions of the nature of mathematics (Garofalo, 1989; Schoenfeld, 1988, 1989; Silver, 1987); and studies on the construction and impact of belief systems on performance (Cobb, 1986; Lave et al., 1989). Studies specific to researching the role of metacognition in relation to mathematical thinking have been conducted by Schoenfeld (1985a, 1987b), Lester (1989; Lester et al., 1989), and Kroll (1988).

Both Schoenfeld (college level), and Lester (grade 7) combined exploratory research into the nature of metacognition with training designed to improve students' awareness and self-regulation, while Kroll (college level) used the analytical techniques of the above to explore the metacognitive strategies used by three pairs of female students during a series of co-operative problem solving sessions. Schoenfeld's studies provide evidence that a classroom approach which encourages students to internalise metacognitive processes can improve problem solving performance as well as providing useful approaches to the investigation of awareness and self-control.

Lester's research encountered some difficulties attributed to the lack of cognitive resources possessed by younger students, and to classroom culture and management problems associated with teaching formats unfamiliar to the children. The outcomes of his research were less positive than reported by Schoenfeld.

Kroll's work, while based on Schoenfeld's analysis scheme, was primarily directed to gauging the effect of collaboration on problem solving, with metacognition chosen as the process to be observed. While there are some similarities with the present study, there is a major distinction in focus – we have chosen to focus on metacognition itself, with the use of collaborative problem solving a research tool for making covert metacognitive processes observable.

A summary of salient aspects of research directly relevant to the methodology of this study is provided in Table I.

3. METHODOLOGY

The research represents an intensive case study of collaborative interaction between two students over a period of ten weeks. Consequently the data gathering procedures are descriptive and qualitative, designed to portray a spectrum of individual and dyadic activity, and heavily dependent on verbal reporting.

Following a period, dominated by behaviourist psychology, when verbal methods fell into disfavour (Nisbett and Wilson, 1977), they have been firmly re-established upon a platform of information processing support (Ericsson and Simon, 1980). Their development of the dimensions of talk/think aloud, concurrent probing, and retrospective probing has been followed by similar classifications proposed, for example, by Genest and Turk (1981) and Ginsburg et al. (1983).

The Ericsson and Simon (1980) dimensions are defined as follows:

1. *talk/think aloud*: subjects report everything they are thinking. This involves concurrent verbalisation with undirected probing, and the information reported is that which engages the subject's attention during the task.
2. *concurrent probing*: subjects are instructed to report on specific aspects of processing that are of interest to the researcher. This requires intermediate processes involving scanning, filtering and generalising.
3. *retrospective probing*: subjects are prompted to recall specific actions or events. Additionally, however, the asking of general 'Why?' questions will introduce inference, generalisation, and hypothesising into the ensuing verbalisation.

TABLE I

Summary of research on metacognition and mathematical thinking

Authors	Purpose	Subjects	Methodology	Studied Metacognitive...
Schoenfeld (1985a)	Exploratory – investigate metacognitive behaviour	College students	Questionnaire – planning, organisation, problem familiarity & difficulty	Awareness
	Training – develop metacognitive skills		Uninterrupted think aloud pair protocols	Self-regulation
Lester & et al. (1989)	Exploratory – role of metacognition in problem solving	7th grade students	Written self-inventories – problem familiarity & difficulty, solution correctness	Awareness
	Training – improve awareness & self-regulation		Individual & paired clinical interviews	Self-regulation
Kroll (1988)	Exploratory – describe monitoring moves and roles	College students	Uninterrupted think aloud pair protocols	Self-regulation
			Questionnaires – problem solving ability, problem familiarity & difficulty	Awareness
			Interview on problem solving strategies	Self-regulation
			Retrospective questionnaire & discussion on strategies	Self-regulation

The extent to which the verbalisations represent mathematical quality is then for the researcher to examine.

While contemporary cognitive psychology accepts the usefulness of verbal data, and protocol analysis as a method of interpreting that data, limitations associated with the methods need to be acknowledged as follows:

1. *reactivity* – environmental influences (stress, researcher intervention, task demands) can affect cognitive processing;
2. *incompleteness* – subjects may not report the cognitive processes of interest;
3. *inconsistency* – verbal reports may not correspond to observed behaviour;
4. *idiosyncrasy* – generalisation is difficult because verbal methods are sensitive to individual differences;
5. *subjectivity* – researcher bias influences the interpretation of data (Ericsson and Simon, 1980, 1984; Genest and Turk, 1981; Ginsburg et al., 1983).

However, it could be argued that the last two limitations are unimportant. Whether the purpose of research is to generate or test hypotheses, verbal methods provide the rich descriptions which link theory to the idiosyncratic individual from whom the data are collected. Similarly, in any form of research the significance of data must always be judged relative to the researcher's explicit or implicit theories and assumptions (Ginsburg et al., 1983).

The three remaining limitations are analysed in Table II, in terms of the previously proposed dimensions for classifying verbal methods.

In summary it can be argued that *concurrent* verbalisation methods with the instruction to *report* provide the most accurate description of cognitive processes during task performance because they do not require subjects to use inferential processes or retrieve information from long term memory. Questions of reactivity and incompleteness may be addressed using Schoenfeld's (1985a, 1985b) pair protocol method for investigating metacognitive behaviour. Schoenfeld (1985a) argues there are two reasons why pair protocols are more likely to capture students' typical thinking than single student protocols. First, two students working together produce more verbalisation than one because both must explain and defend the decisions they make (this reduces the problem of *incompleteness*); and second, the reassurance of mutual ignorance alleviates some of the pressure of working under observation (this reduces the problem of *reactivity*). In addition, two students are likely to begin their examination of a problem from differing perspectives, and the requirement to produce a

TABLE II

Limitations of verbal methods

Limitation	Variables for Classifying Verbal Methods		
	Time of Verbalisation	Intervention	Instructions
1. Reactivity	Applies to concurrent methods only. Requirement to verbalise while performing task may affect cognitive processing	Applies to probed methods only. Probing directs attention to information that would not otherwise be heeded, and can induce subjects to change thinking strategies to those thought to be of interest to the researcher.	If instructed to explain, subjects must use inferential processes not otherwise required to perform task.
2. Incompleteness	For concurrent methods, requirement to verbalise while performing task may cause verbalisation to cease or become sketchy; some contents of STM may be obliterated and so unable to be reported.	Applies particularly to unprobed methods. Subjects may not report all processes of interest to the researcher.	If instructed to report, subjects may not verbalise processes of interest.

mutually acceptable solution could provoke considerable reflection on, and monitoring of, their own and each other's thinking. Thus, paired problem solving stimulates metacognitive activity, as well as making such activity observable.

Nevertheless, additional measures may be necessary to further address the limitations of incompleteness and reactivity. For example, Schoenfeld (1985b) suggests that incomplete reports from the initial exploration of metacognitive behaviour could be followed up in more detail by conduct-

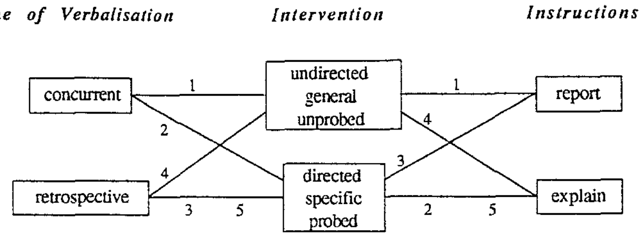
TABLE II

(continued)

Limitation	Variables for Classifying Verbal Methods		
	Time of Verbalisation	Intervention	Instructions
	For retrospective methods complete retrieval from LTM is unlikely, especially if there is a time lag between task completion and verbalisation.	However, probing may still fail to uncover processes of interest if subjects have no access to them, or are unable to describe them.	Even if instructed to explain, subjects may not be capable of describing salient mental processes.
3. Inconsistency with observed behaviour	Applies to retrospective methods only. Because information must be retrieved from LTM, subjects may produce a rationale rather than an accurate report.	General probes may cause errors in retrieval from LTM.	If instructed to explain, subjects may try to infer missing information. Even if instructed to report, subjects may produce a rationale or interpretation.

LTM = long term memory
STM = short term memory

ing what could be described as a *retrospective clinical interview* in which classification variables are changed to *retrospective* verbalisation, *specific* probing, and the instruction to *explain*. Although this procedure would not produce as accurate a picture of subjects' thinking as concurrent methods, some clues as to their beliefs and knowledge structures may still be gleaned.



Path 1: Talk/Think Aloud (Ericsson & Simon, 1980; Genest & Turk, 1981; Ginsburg et al., 1983); *Pair Protocols* (Schoenfeld, 1985a, 1985b)

Path 2: Clinical Interview (Ginsburg et al., 1983); *Concurrent Probing* (Ericsson & Simon, 1980)

Path 3: Retrospective Probing (Ericsson & Simon, 1980); *Reconstructive Procedures* (Genest & Turk, 1981)

Path 4: Retrospective General Report (Ericsson & Simon, 1980)

Path 5: Retrospective Clinical Interview (Schoenfeld, 1985b)

Fig. 1. A taxonomy of verbal methods

Reactivity is much more difficult to eliminate because – no matter how much attention is given to putting the pair of subjects at ease – the research setting is essentially atypical for them. For this reason, and because the interaction between subjects now produces a different kind of reactivity in that each can influence the thinking of the other, the verbalised thoughts of pairs of students may not represent the problem solving approach each would have taken if working alone, silently and unobserved. Therefore, inferences drawn from verbal protocols need to be confirmed by data from other sources, such as informal classroom observations of problem solving, interviews, or students' performance on similar problems in the classroom or in examinations (Ginsburg et al., 1983).

After the limitations of each of the methods classified in Figure 1 were evaluated Schoenfeld's (1985a, 1985b) variation of Path 1 (pair protocols) was selected as the primary method of gathering verbal data on the students' self-regulatory strategies, supplemented by retrospective interviews of the type described by Paths 3 (probing) and 5 (clinical interview) (Figure 1).

4. EXPERIMENTAL CONTEXT

4.1. *Setting*

Since the interest was student collaboration in applying mathematics the study was located within a group studying mechanics in the penultimate year of high school. The course contains a component of mathematical modelling (Galbraith and Clatworthy, 1990) in which students are used to working (and arguing) in small group situations, so the experience of interpersonal collaboration is a familiar one.

4.2. *Problem Selection*

Problem selection was guided by three main criteria:

1. the questions had to be relevant to the students' classroom experience if the protocols were to provide insights into thinking processes typical of the subject area;
2. the questions had to be challenging enough to require, and elicit, metacognitive behaviour, while simultaneously being within the capacity of the subjects to solve with existing knowledge;
3. the questions needed to contain a blend of genuine 'problems' and routine exercises, so that initial success on the latter would help put the students at ease at the start of each think-aloud session.

In all four different questions were used (reproduced below). Of these CRICKET and MASCOT represented genuine problems, while PULLEY and GOLF were standard applications.

GOLF

A golfer hits a ball from a point on a level fairway, and 2 seconds later it hits the fairway 50 m away.

Find:

(a) the velocity and angle of projection of the golf ball
(b) the maximum height of the ball above the fairway

CRICKET

A batsman hits a cricket ball 'off his toes' towards a fieldsman who is 65 m away. The ball reaches a maximum height of 4.9 m and the horizontal component of its velocity is 28 m/s. Find the constant speed with which the fieldsman must run forward, starting at the instant the ball is hit, in order to catch the ball at a height of 1.3 m above the ground. (Use $g = 9.8$)

PULLEY

Two bodies of mass 4 kg and 3 kg are at rest on two smooth inclined planes placed back to back. The bodies are connected by a string passing over a smooth pulley at the top of the planes. If the 4 kg mass rests on a plane inclined at 35° to the horizontal, find the inclination of the other plane.

MASCOT

A mascot suspended from a car's rear view mirror hangs vertically when the car is moving with uniform velocity of 80 km/hr along a straight level road. The brakes are applied so that the car is stopped with uniform retardation. Find the angle through which the mascot is deflected if the car comes to rest 137 m after the brakes are applied.

4.3. *Subject Selection*

The class was observed initially for a period of three hours spread over two weeks, and on the basis of observation (confirmed by the class teacher) two students were invited to participate in the study. These students demonstrated a capacity to verbalise and to reflect on their thinking, that is, they possessed the two characteristics deemed essential to a study of metacognitive processes: they possessed metacognitive awareness, and they were able to make their strategy use overt so that it could be observed and analysed. The following descriptions of the students have been prepared from data obtained from three sources: extended classroom observation, questionnaires probing metacognitive awareness, and discussions with the students' teacher. (Further information on the first two data gathering procedures is given in the next section.)

David (pseudonym) is a clever student, good at mathematics. He perceives (questionnaire data) that his most common errors are mechanical (slips due to carelessness, poor setting out, or lack of practice), also confirmed by his class teacher. His strategic awareness is not as well developed as his knowledge of personal limitations, for example he was unable to describe in detail (questionnaire) his actions when stuck on a problem. He was able (questionnaire) to give an accurate assessment of the difficulty and level of familiarity of test questions and a sound estimate of the correctness of his solutions.

In summary the person and task components of David's metacognitive knowledge (Garofalo and Lester, 1985) are fairly well developed, as he has a reasonable appreciation of his abilities and weaknesses and some understanding of the reasons. However, his conscious knowledge of his own repertoire of strategies, and the usefulness of strategies for dealing with particular tasks, is not as extensive.

Rick (pseudonym) is also a capable student, slightly behind David in his quickness to grasp new topics in the judgment of his teacher. In class his extroverted behaviour results in a constant stream of chatter, interjections, and argument. By contrast in one to one situations he has demonstrated a capacity to be both courteous and thoughtful. He dislikes writing down ideas and calculations, and is able to solve many difficult text-book examples almost entirely in his head, estimating the answer before confirming it on his calculator. Rick knows that a major source of his errors arises from the translation of his mental images into the written word. He was aware of the usefulness and relevance of strategies for organising information and knew that impulsiveness created barriers to solving problems. Nevertheless he tended not to act on such knowledge.

Rick did not show the same level of perception as David in judging the familiarity level of questions or in estimating the quality of his solutions.

The collaborative work of David and Rick exhibited a variety of surface features. Frequently they worked independently on text-book exercises unless one requested help from the other. When working on problems their interaction was more collaborative, but it also had an adversarial flavour as each sought to establish intellectual sovereignty and/or superiority. The contrast between Rick's persuasive, but often incorrect, reasoning and David's clearer understanding of the problems they attacked together, produced rich verbalisation of both students' thinking – precisely the kind of verbalisation that was needed to enable think aloud problem solving to reveal their metacognitive processes.

5. DATA GATHERING PROCEDURES

The main method of data collection was through two videotaped think aloud sessions over a four week period. The subjects attempted two problems in each session. Retrospective interviews were used for supplementary data gathering when the pair protocols were found difficult to interpret through incompleteness.

In addition two questionnaires were used to obtain information about the level of metacognitive awareness of all students in the class prior to the selection of the target students. The first questionnaire focussed mainly on person and strategy factors across an unspecified range of tasks. It contained items drawn from Schoenfeld's (1989) instrument for exploring students' beliefs and behaviours; and from Garofalo's (1987) suggestions for questions which teachers might use to help students to develop awareness.

The second questionnaire was more closely linked with a set of specific tasks – questions on the end of term mathematics test. When combined with the students' written work on the test this questionnaire fulfilled three purposes.

1. It collected data about awareness (task and person factors) by asking students to rate each problem's familiarity and difficulty together with their confidence in the correctness of their solution.
2. It provided further data on the subjects' problem solving ability, enabling inferences drawn from the think aloud protocols to be checked.
3. It enabled a judgment to be made as to the typicality of the subjects' think aloud problem solving behaviour in the observation context, where the presence of the researcher might create atypical pressures.

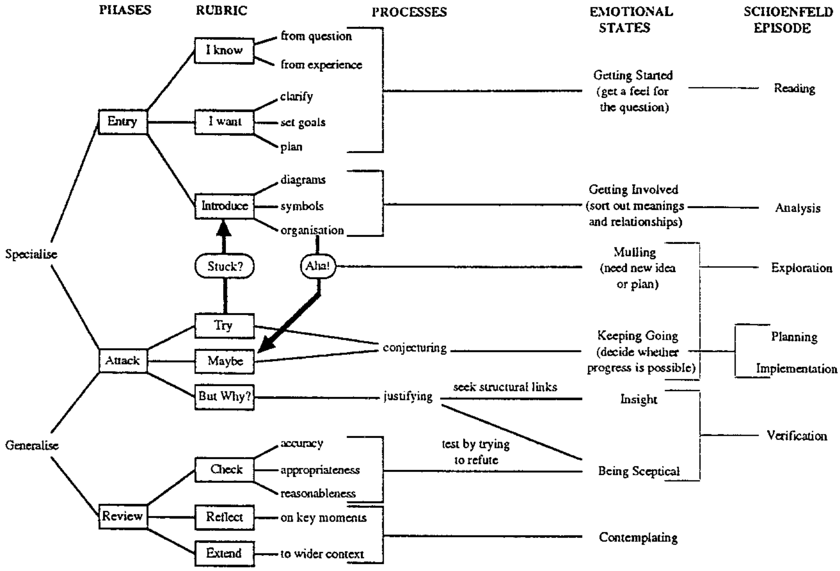


Fig. 2. Integration of Mason, Burton & Stacey's (1985) problem solving model with Schoenfeld's (1985a) episode analysis.

Further, observation and participation in the mathematics classes by the first named author continued over a period of ten weeks. This regular presence in the classroom allowed extended observation of the subjects' problem solving behaviour in its natural setting, and allowed the inference that the think aloud sessions were indeed typical episodes in relation to problem solving activity on the part of these students.

5.1. *Pair Protocols*

A scheme of analysis was devised that represents a selective extension of Schoenfeld's (1985a) episode analysis synthesised with aspects of Mason, Burton, and Stacey's (1985) problem solving model (Figure 2). Schoenfeld's scheme aims to highlight major strategic decisions, suggest when they should have been made (if absent) and assess the quality of the decisions per se. The protocol is parsed into macroscopic episodes, representing periods of time during which the subjects are engaged in distinctive types of problem solving behaviour. Those types for which 'ideal' characteristics are described in Schoenfeld (1985a) are *reading*, *analysis*, *exploration*, *planning*, *implementation* and *verification*.

Mason et al. (1985) encourage the use of key words such as 'STUCK' or 'BUT WHY?' to capture the emotion as well as the mathematics of problem solving endeavours. They nominate specialising and generalising as two fundamental processes that span the problem solving phases of Entry, Attack, and Review. These phases of work correspond to qualities of experience in dealing with a problem, and are structured by characteristic activities. The *Entry* phase involves self-interrogation by the three questions What do I know? What do I want? What can I introduce? The *Attack* phase consists of two processes: conjecturing (a cyclic process of articulation, testing, and modification), and justifying (which involves testing conjectures against the original data as if to convince a sceptic). The *Review* phase involves checking calculations and arguments, reflecting on key ideas, and seeking to extend the solution to a wider context. The reflecting and extending processes do not have counterparts in Schoenfeld's framework, as they are intended to develop problem solving expertise rather than help in solving a particular problem.

The first two research questions were addressed through a modified version of Schoenfeld's episodic analysis which calls for three classes of metacognitive decision points to be identified:

1. transitions between episodes, which signal major strategic decisions where the direction of problem solving changed significantly;
2. points where new information was recognised, or local assessments of specific aspects of the solution were made;
3. times when an overall review and evaluation of progress was warranted.

In practice, the third type of decision point is difficult to identify (as Schoenfeld, 1985a, acknowledges) and is not pursued in detail in the present study. Instances of the first type of decision point, transitions between episodes, were identified as periods of activity separating two clearly defined episodes, during which the students paused to review their progress or consider their next move. If such activity was absent, no transition was coded.

Modifications to the analysis of the second class of decision points were made because of the purpose of the project – to investigate the unique contributions and interpersonal strategies employed by two students working collaboratively over time rather than following the metacognitive behaviour of many students in more general terms. The precise methodological amendments are described below.

1. New information points were subdivided into two types:
   - points where previously overlooked or unrecognised *information* came to light (abbreviated as NI)
   - points where the possibility of using a new *procedure* was mentioned (abbreviated as NP).

The NI/NP's were classified further according to:

- who initiated the NI/NP
- how relevant the NI/NP was to the task
- the nature of the response to the NI/NP (ignore, reject, accept)
- how appropriate the response was in context.

2. Local Assessments (LA's) of a particular aspect of a solution were classified according to who made the assessment, and the function of the assessment:
   - knowledge (assessing what is known/not known)
   - task difficulty
   - procedure (checking accuracy of execution, assessing relevance or usefulness)
   - result (assessing accuracy or reasonableness).
3. Global Assessments (GA's) of the general state of the solution were also made.

The third research question was addressed from the perspective of Mason et al. (1985). The first response to being 'stuck' (return to Entry phase to re-assess what is known or wanted, or to reformulate the problem) marks a return to *reading* or *analysis* (Figure 2). Other responses involving specialising, generalising, mulling, and distilling are included in episodes of *exploration* (Figure 2).

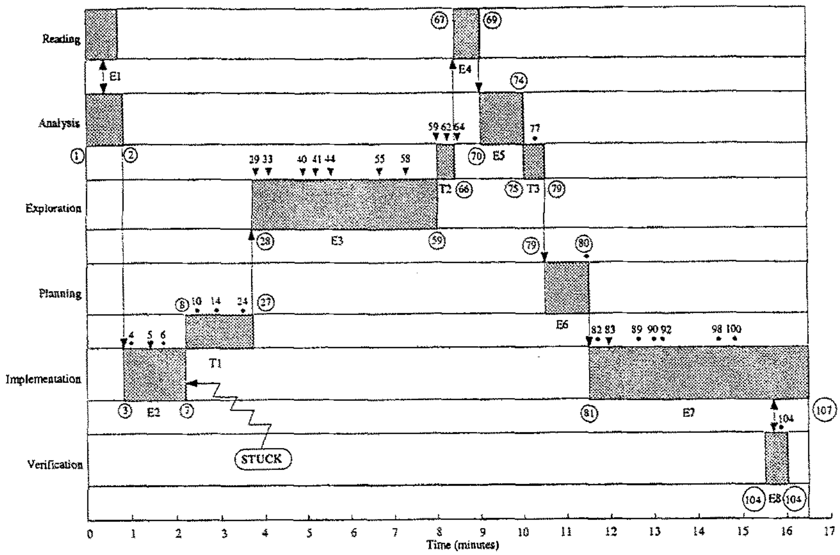
The final research question relating the presence or absence of metacognitive behaviour to the outcome of problem solving was addressed by analysing the students' responses to being stuck. This analysis provided evidence for two types of control decision 'discontinue inappropriate strategy', and 'exploit knowledge and procedural resources', particularly through the incidence and quality of Local/Global Assessments and New Information/New Procedure points respectively.

5.2. *Retrospective Interviews*

The information from these interviews was related to the clarification of pair protocol data, and was incorporated into the episode parsing narratives for the corresponding contexts.

6. DATA ANALYSIS

The following section shows the analysis procedures applied to one of the four problem solving protocols. Analysis of the CRICKET problem has been chosen because it best illustrates the students' typical metacognitive



Key to Symbols
E1, E2, etc = Episode 1, Episode 2, etc; T1, T2, etc = Transition 1, Transition 2, etc
Other numbers refer to Moves in the protocol. Circled numbers show beginning and end of episode.
▼ indicates New Information/New Procedure point (NI/NP)
• indicates Local Assessment (LA)

Fig. 3. Time line representation – CRICKET protocol.

behaviours and collaborative style. The problem context is given below. While couched in 'cricket' terms the problem could equally be set within the context of baseball or softball.

CRICKET

A batsman hits a ball 'off his toes' towards a fieldsman who is 65 m away. The ball reaches a maximum height of 4.9 m and the horizontal component of its velocity is 28 m/s. Find the constant speed with which the fieldsman must run forward, starting at the instant the ball is hit, in order to catch the ball at a height of 1.3 m above the ground. (Use $g = 9.8$)

Although this context bears some resemblance to textbook questions, it has the added complication of the separate motion of the fieldsman to catch the ball, and this imbues the question with problem status as far as the students are concerned. A successful solution requires careful identification of the goal, and working backwards to establish subgoals and suitable strategies for achieving them.

6.1. *Episode Parsing*

In transcribing the protocols the following conventions were observed:

1. Complete turns at speaking are numbered sequentially and referred to as *Moves*.
2. The symbol [...] indicates that part of the transcript has been omitted.

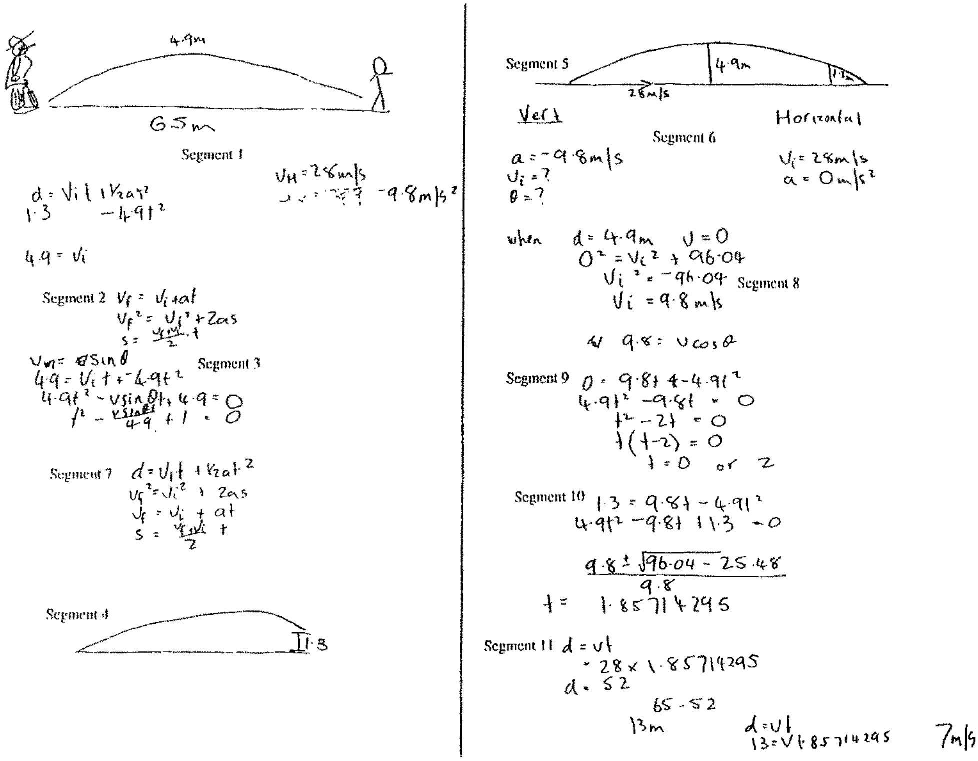


Fig. 4. Rick and David's written work for the CRICKET problem.

3. The symbol ... signifies a pause when verbalisation temporarily ceased.
4. The initials R, D and MG refer to Rick, David and the first named author.

The overall structure of the solution analysis for the CRICKET protocol is shown in Figure 3. This involves 107 moves by the collaborating students. Their written work is presented in Figure 4.

6.2. *Episode 1 – Reading/Analysis*

Rick began to read the problem statement, pausing occasionally to interpret its meaning and draw a diagram.

1. R: (Reading) A batsman hits a cricket ball off his toes – that means it starts at ground level – towards a fieldsman who is 65 metres away. Right – diagram. (He takes a sheet of paper and starts to draw.)

David continued reading aloud, adding details to the diagram (marked as Segment 1 in Figure 4) and writing down some of the given information.

Although this episode has been double coded as Reading and Analysis, there was only minimal analytical behaviour. The conditions of the problem were noted quite carelessly; for example, the diagram incorrectly showed the fieldsman standing at the point where the ball would have hit the ground if it had not been caught. The vital information that the ball was caught at a height of 1.3 metres was not marked on the diagram at all. The goal was not clarified, and there was no consideration of how the 'givens' might be used to obtain the goal. Because Rick and David failed to establish exactly what they knew and needed to find, their solution attempts later ran into difficulties. There was no assessment of the state of their knowledge before they immediately jumped into Implementation.

6.3. *Episode 2 – Implementation*

During this episode Rick and David worked separately on finding the length of time the ball was in the air. This was an appropriate subgoal, but the students appeared to differ in its interpretation. David correctly substituted the catching height of 1.3 metres into the equation $d = v_i t + \frac{1}{2}at^2$, but then realised that this approach would not work because initial vertical velocity was also unknown. Rick, however, used a formula remembered from Physics lessons for calculating the time of flight of a projectile allowed to hit the ground: $t = \frac{v_f - v_i}{a}$, where v_f and v_i refer to vertical velocities. David's monitoring skill was evident here as he pointed out to Rick that this formula was useless because initial vertical velocity was unknown. Unfortunately, Rick ignored his partner's warning and set off on a wild goose chase by incorrectly using the initial *horizontal* velocity of 28 m/s as the value for initial vertical velocity.

6.4. *Transition 1*

The first Transition marks a change of strategy from 'find the time of flight', to 'find the initial vertical velocity', a necessary reallocation of resources as the pair realised that their progress was blocked. In this exchange David made many Local Assessments of Rick's flawed procedure for finding t.

8. D: What are you doing?
9. R: Look v_i is 28, v_f is negative 28. OK, multiply by 2, divide by 9.8 to get the time [...]
10. D: No ... first we've got to find out how long it's in the air.
11. R: Yeah, I'm telling you!
12. D: That's ... the horizontal component –
13. R: Oh is it? ... (to MG) It's in the air for ... 5.71 minutes? Seconds?

14. D: I don't think so.

24. D: [...] you don't know the velocity horizontally – I mean vertically.
25. R: That's why I'm using this one –
26. D: Yeah, well there's no acceleration horizontally so why do you need a ?
27. R: Oh I see, I see.

During this Transition David effectively played the role of a sceptical enemy to whom Rick had to justify his procedure. Fortunately, David prevailed and Rick accepted that his formula was inappropriate.

6.5. *Episode 3 – Exploration*

This episode contains several points where new information was noticed and new approaches for calculating initial vertical velocity were tried. Because the Exploration was driven by this goal, it is reasonably well structured. Rick and David were never in danger of pursuing a single, inappropriate strategy to the exclusion of alternatives, as their local monitoring allowed them to terminate such strategies when they realised that no progress would be possible. At the same time, the constant flow of new ideas increased the likelihood of their finding a way around their impasse.

During the previous Transition, David had started to reread the problem to check that he had noted all the relevant information. After dealing with Rick's interruption, David now returned to reading:

28. D: (Reading) ... ball reaches a maximum height of 4.9 metres –
29. R: All right, fill that in.

Noticing that they had not yet used this information, David and Rick tried to substitute the maximum height for d in the equation $d = v_i t + \frac{1}{2}at^2$. But again they were left with the same problem of two unknowns, v_i and t. Finally, after four minutes of persistence with the same equation of motion, Rick brought David's attention to the other equations they knew, but had not yet considered (Segment 2 in Figure 4):

33. R: You don't just have $v_i t + \frac{1}{2}at^2$, you've got these other ones, $s = u + at$, $v_i = –$
34. D: (Sarcastically) $s = u + at$?
35. R: (Defensively) Something like that.
36. D: You mean $v = u + at$.
37. R: (Casually) Anyhow, it's one of them.

This was a crucial observation which could have started them moving again. Unfortunately, neither student realised the utility of the equation ${v_f}^2 = {v_i}^2 + 2as$: the only one which does not include the troublesome unknown quantity, t, and which would have allowed them to calculate the other unknown of interest, v_i. This lack of assessment was probably caused by the failure to access their knowledge that v_f is zero when the ball is at its maximum height. (They certainly possessed this knowledge, as the final Implementation episode shows.)

Soon afterwards David introduced the angle of projection into the problem for the first time by noting that the initial vertical velocity was $v \sin \theta$, and substituted this information into the distance equation to give $4.9 = v_i t - 4.9t^2$ when the ball is at maximum height. After some skilfully executed and jointly monitored algebraic work, David and Rick rearranged these equations into the standard quadratic form (Segment 3 in Figure 4). This approach was also abandoned when Rick queried whether the angle was known, and David admitted that it was not.

Returning to his list of equations, David briefly considered $s = \frac{v_f + v_i}{2} \cdot t$ but rejected it because, once again, t was not known. Rick started to suggest setting up two equations for s and solving simultaneously for v_i and t, but abandoned this proposal when he realised that he didn't have enough information to make it work.

Throughout this episode Rick was the major source of new ideas and procedures, most of which were inappropriate, and all of which were eventually rejected. Thus the pair's monitoring and assessment helped them to avoid pursuing an irrelevant strategy, but was not thorough enough to reveal the potential usefulness of the equation which would have extricated them from their difficulties. As well as being the idea generator, Rick kept a constant check on the accuracy of David's written calculations and demanded an explanation if he could not follow David's progress.

6.6. *Transition 2*

The previous episode, which was defined by the search for a method of finding initial vertical velocity, ended when David had a sudden flash of insight (Segment 4 in Figure 4):

59. D: [...] Umm ... Oh wait a minute, wait a minute, wait – what happens if we do *that*?
60. R: Ah! So that means – (to MG) can we draw on this? That means (drawing) that we have here ...
61. D: Mm ...
62. R: – and the time starts when he hits it, and the ground will be here at 1.3.
63. D: Yeah.
64. R: He must run as the ball moves.
65. D: Yes, I know –
66. R: – exactly at the same time –
67. D: I *know* –

For the first time, David realised the full significance of the fact that the ball is caught 1.3 metres above the ground. This seemed to trigger Rick's reinterpretation of the problem as one involving the simultaneous motion of two objects, the ball and the fieldsman, both of which start moving at the same time.

6.7. *Episode 4 – Reading*

The discovery of previously overlooked information prompted David to reread the problem statement. His decision to take a new piece of paper and draw a new diagram signalled the opening of the next episode.

6.8. *Episode 5 – Analysis*

This improved diagram (Segment 5 in Figure 4) contained almost all the relevant information and was constructed with much more care than either student's earlier efforts. Rick and David now carefully checked each other's understanding of their new perspective on the problem:

70. R: 1.3 is there (pointing to diagram).
71. D: Yeah, but we're only interested in here (pointing to the horizontal distance covered by the ball).
72. R: That's right – see, that doesn't go because he catches it (meaning that the hypothetical range is of no interest) – does he catch it?
73. D: Yeah, he's supposed to.

With Rick's help, David then began to systematically organise the given information by listing known and unknown quantities in two columns representing the ball's vertical and horizontal motion (Segment 6 in Figure 4). At this point Rick tried to direct David to write down the 'initial horizontal distance' as 65 metres, which suggests that he still had only an imperfect understanding of the need to consider the motion of the ball and the fieldsman separately. Thus the next exchange serves as a Transition during which the new analytical perspective was evaluated, and the need for an explicit statement of the plan of attack became apparent.

6.9. *Transition 3*

With Rick again threatening to lead the solution attempt astray, David took responsibility for keeping matters on track:

76. R: I'm telling you, write it down! Initial distance – he starts off at 65 metres away from where the ball was hit –
77. D: (emphatically) No, ignore that for now –

78. R: But you can't ignore it!
79. D: Yes we can!

6.10. *Episode 6 – Planning*

Rick's intransigence ultimately forced David to articulate the plan which had probably been evolving in his mind since the start of the previous Analysis episode.

79. D: [...] Because that's later, that's the second part of it. We only want to find out now how far it's going and how far (he meant to say, how *long*) it'll take until it's 1.3 metres in the air. We can find out how far it's gone (gestures horizontally on diagram) and then we take that away from that (meaning: subtract the distance the ball travelled from the initial distance between batsman and fieldsman) and work out how long the ball will take to get there, and you work out the velocity from there.
80. R: (pause) True. It's the long way round, but it'll work.

6.11. *Episode 7 – Implementation*

The episode began with David once again listing all the equations of motion (Segment 7 in Figure 4). Without hesitation he selected the previously overlooked formula which would yield the ball's initial vertical velocity: $v_f^2 = v_i^2 + 2\,as$. At first, Rick thought David was still trying to use the distance equation which had dominated their early attempts to solve the problem:

81. D: Zero squared equals v_i squared plus –
82. R: What zero squared? It has a distance –

This challenge provoked David into revealing the reason for his choice of formula, bringing to light the fact that vertical velocity is zero when the ball is at its maximum height.

David's confidence in his plan and its implementation is nicely illustrated by his reaction to a minor calculation error. In substituting into the maximum-height equation he inadvertently used a vertical acceleration of $+9.8$ m/s^2, instead of -9.8 m/s^2. This gave an impossible result: $v_i^2 = -96.04$ (Segment 8 in Figure 4). He responded with 'Uh oh ... mm ... ignore it – v_i is 9.8', which suggests that he had decided not to investigate a possibly unimportant error, unless further difficulties arose.

Although both David and Rick monitored Implementation successfully at the local level, there was a period when lack of global assessment caused them to waste their efforts in calculating the time the ball would have taken to hit the ground if it had not been caught (Segment 9 in Figure

4). However, this turned out to be a minor detour which did not adversely affect the solution.

6.12. *Episode 8 – Verification*

After David had successfully calculated the time the ball was in the air (Segment 10 in Figure 4), Rick tried to assess the reasonableness of the result by relating it to something more familiar: the speed of a car at the recent Touring Car race at Bathurst.

104. R: One second? It's going to travel 65 metres in – ?
105. D: Listen, it makes two seconds till it hits the ground! (continues calculating)
106. R: In 1.8 seconds the ball's going to go 80 metres? Boy, that's moving – do you know how many clicks an hour that's doing? (uses calculator) Eighty ... it's doing 288 km/hr, mate, it'll overtake those Nissans – hit it down at Bathurst!

Because David was occupied with the final step of the solution, calculating the fieldsman's speed (Segment 11 in Figure 4), he did not notice Rick's error (the ball travelled 65 metres, not 80, in the 1.8 seconds).

6.13. *Summary of Episode Structure*

The protocol could be divided into three main parts:

1. The initial careless Reading and Analysis, followed by an ill-considered jump into Implementation.
2. The Exploration episode, whose purpose was to find a way out of the difficulties caused by Rick and David's early impulsiveness.
3. An orderly progression of activity (Reading – Analysis – Planning – Implementation – Verification) which led to a successful solution.

6.14. *Analysis of Metacognitive Strategies*

Metacognitive decision points are listed and classified in Table III. The protocol contains only three Transitions, each of which resulted in a major reallocation of resources and a change in problem solving direction. New Information/New Procedure points were concentrated in the Exploration episode and subsequent Transition. The majority of Local Assessment points occurred in two separate parts of the protocol: the first Transition, during which David pointed out the inadequacies of Rick's initial implementation strategy, and the final Implementation/Verification episode, where both students shared the responsibility for evaluating the accuracy and reasonableness of their results.

TABLE III

Metacognitive decisions – CRICKET protocol

Episode[a]	Move[b]	Initiator[c]	Type[d]	Context	NI/NP Useful?	Response to NI/NP	Response Appropriate?
E2 Implementation	4	D	LA: knowledge	Unknown variables			
	5	R	NP	Use $t = \frac{v_f - v_i}{a}$		D rejects – v_i unknown	✓
	6	D	LA: usefulness of procedure	D's response to R's NP			
T1 Transition	10	D	LA: usefulness of procedure	R's procedure for finding t			
	14	D	LA: accuracy of result	R's calculation of t			
	24	D	LA: usefulness of procedure	R's procedure for finding t			
E3 Exploration	29	R	NI	Maximum height	✓	D substitutes into distance equation: rejects – too many unknowns	✓
	33	R	NP	List other equations of motion	✓	D lists them but does not assess their usefulness	
	40	R	NP	Use calculus		D ignores	✓
	41	D	NP	Introduce θ and substitute into distance equation		Tried and rejected – too many unknowns	✓
	44	R	NP	Use quadratic formula		Tried and rejected – too many unknowns	✓
	55	D	NP	$s = \frac{v_f + v_i}{2} t$		D rejects without trial – too many unknowns	✓
	58	R	NP	Solve via simultaneous equations		D rejects without trial – too many unknowns	✓

TABLE III

(continued)

Episode[a]	Move[b]	Initiator[c]	Type[d]	Context	NI/NP Useful?	Response to NI/NP	Response Appropriate?
T2 Transition	59	D	NI	Ball caught at height of 1.3 m	✓	R accepts and marks on new diagram	✓
	62	R	NI	Ball and fieldsman move at same time	✓	D accepts	✓
	64	R	NI		✓		
T3 Transition	77	D	LA: usefulness of procedure	R's intention to use "initial distance = 65"			
E6 Planning	80	R	LA: usefulness of procedure	D's plan			
E7 Implementation	82	R	LA: accuracy of procedure	D's use of formula $v_f^2 = v_i^2 + 2as$			
	83	D	NI	Velocity = 0 at maximum height	✓	R challenges, but then accepts	✓
	89	D	LA: accuracy of result	$v_i^2 = -96.04$			
	90/92	R	LA: accuracy of result	D's calculation of vertical velocity			
	95	D	LA: task difficulty	Easy to calculate t			
	98/100	R	LA: accuracy of result	D's calculation of t			
E8 Verification	104	R	LA: reasonableness of result	Value of t			

[a] E = Episode, T = Transition

[b] Moves represent completed turns at speaking

[c] D = David, R = Rick

[d] LA = Local Assessment, NI = New Information, NP = New Procedure

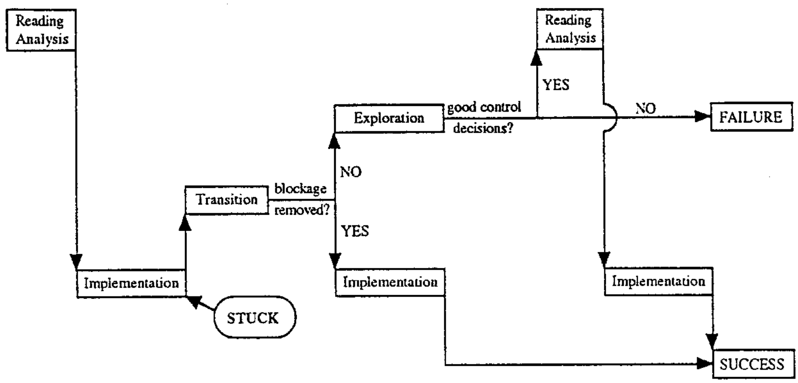


Fig. 5. Characteristic structure of subject's problem solving protocols.

6.15. *Getting Stuck – Reasons and Responses*

Both subjects were stuck at the end of the first Implementation episode, during which they had pursued separate strategies. Their difficulties were caused by insufficient and careless reading and analysis of the problem statement. Because Rick was unaware that he was stuck, he would not have been able to extricate himself without David's help. David, on the other hand, recognised the flaws in both his partner's and his own strategies. In the former case he responded by making a series of retrospective Local Assessments of his friend's work, and eventually persuaded Rick he was in error. David responded to his own lack of progress by considering a slightly modified problem in an Exploration episode driven by an appropriately selected subgoal. Although this subgoal was not achieved (because of some ineffective local monitoring), new information which came to light triggered a return to Reading and Analysis episodes and, ultimately, a successful solution.

7. SUMMARY

In the following discussion the arguments are based on the full set of data obtained from all four problem solving protocols, only one of which (CRICKET) has been provided in this paper.

7.1. *Structure of Problem Solving Attempts*

With respect to the first research question the most striking feature that the solution attempts have in common is the immediate jump into implementation after an initial hasty reading and analysis of the problem. Because these Reading and Analysis episodes included, at best, minimal assessment of the state of the students' knowledge and of the problems' goals and conditions, Rick and David usually became stuck soon after implementation began. Recognition of the difficulty prompted a short Transition which, if the blockage was removed, allowed implementation to resume and the problem to be solved. If, however, the difficulty was not resolved quickly, an Exploration episode was necessary and its success depended on the quality of the subject's control decisions.

The characteristic structure of the subjects' four problem solving protocols is shown in Figure 5.

7.2. *Metacognitive Strategy Use*

The second research question was illuminated by determining the extent to which Rick and David exploited their knowledge, and the manner in which they monitored their progress, through collating the numbers and types of New Information/New Procedure points, and Local/Global Assessments. From these data assembled from the four protocols it can be inferred that

1. Rick consistently generated more new ideas than David.
2. David usually produced more Local Assessments than Rick (an exception was the MASCOT problem).
3. Only Rick checked the accuracy of procedures as they were executed.
4. Only David evaluated task difficulty, and assessed what was known or not known.
5. Rick and David shared the responsibility for assessing the accuracy and reasonableness of results.

It appears that Rick and David have differing, but complementary, metacognitive strengths. Rick played two roles during problem solving: he was both the *idea generator* and the *checker* of David's calculations. However many of Rick's ideas were irrelevant or unworkable. Because he failed to assess the usefulness of his ideas, Rick was in constant danger of setting off on wild goose chases. The task of rescuing him from this fate fell to David, who effectively filled the role of *procedural assessor* in all but the MASCOT problem. (In the latter protocol it was Rick who made the majority of procedural assessments as he tried, unsuccessfully, to convince David that his strategy was wrong.)

7.3. *Responses to Being Stuck, and Influence of Metacognitive Behaviour on Outcomes*

The third and fourth (related) research questions cannot be effectively addressed when students work on exercises (as distinct from problems) except when a mistake requires corrective action to achieve a solution. Within this study the PULLEY and GOLF questions were expected to be exercises. In fact, identification of a mistake in the GOLF question resulted in the correction of mechanical errors and a re-reading of the question leading to successful solution. However the students were only alerted to their mistake by an error message returned by a calculator. Without this prompt Rick and David would not have realised they were 'stuck' with an inappropriate trigonometric formulation. The outcome could have been either success or failure in terms of their utilisation of their own metacognitive resources and the GOLF protocol was therefore classified as *control neutral.*

In the CRICKET problem the students effectively co-ordinated their complementary roles (described above) and the resulting good control decisions promoted success. The CRICKET protocol was classified as *control positive.*

The MASCOT problem was the only problem the students failed eventually to solve. Their difficulties arose from their initial faulty analysis of the problem, which led them to pursue a trigonometric strategy for calculating the mascot's angle of deflection rather than formulate the problem in terms of the forces acting on the mascot. There were at least two opportunities for the students to rescue themselves through an executive review of progress leading to a fruitful control decision (Schoenfeld, 1985a). At either of these points the students could have averted failure if they had been willing to re-evaluate their perspective on the problem. In this problem it was Rick who made the most valuable monitoring contributions – contributions which could have saved the situation if only David had treated them seriously. Because David insisted on being the final judge of both students' New Information/New Procedure statements and Local Assessments, he was ultimately responsible for the many poor decisions and missed opportunities which finally guaranteed their failure. Thus, the students persisted with an unproductive strategy, useful knowledge about the relationship between force and acceleration remained unexploited, and a dead end was not avoided. For these reasons the MASCOT protocol was classified as *control negative.*

8. CONCLUSION

Apart from noting some similarities between the general structure of the students' problem solving protocols and the typical behaviour demonstrated by Schoenfeld's (1985a) college students we believe that the findings of the present study are significant for two reasons – one substantive, the other methodological.

First, the results add to the limited knowledge that currently exists concerning metacognitive strategy used by secondary school students. Most published research has used college students or younger students as subjects (see Kroll, 1988; Lester et al., 1989; Schoenfeld, 1985a).

Second, the analytical scheme devised for the study contains some novel features which not only classify the individual students' contributions, but allow the interactions between the students to be described. This extension of Schoenfeld's (1985a) protocol analysis technique differs from that devised by Kroll (1988) for her investigation of co-operative problem solving among female college students.

The interactive exchanges of the students further underline the influence of factors other than metacognitive processes such as *impulsiveness* (displayed in initial hasty analysis of goals and conditions); and *social interaction* in which the perceived order of merit (David regarded himself as more competent than Rick) prevented some of the latter's fruitful ideas from being followed. Given that peer collaboration has been put forward as an effective means of developing self-regulation (Campione et al., 1989; Schoenfeld, 1987b) these influences have important implications for practice. The data emerging from this study point to some negative effects of peer interaction and these need to be studied further. Using Vygotskian terminology Forman (1989) names three conditions needed for a bi-directional Zone of Proximal Development, created by collaborating students, to be effective:

1. Students must have mutual respect for each other's perspective on the task.
2. There must be an equal distribution of knowledge.
3. There must be an equal distribution of power.

Breakdowns in the collaborative problem solving process that occurred in this study can be linked to breakdowns in one or more of these conditions.

In the learning of applications of mathematics there are two arenas within which collaborating students need to function – the mathematics world and the real world in which the mathematics is to be applied.

Challenges to conditions for effective collaboration are thus intensified in applied contexts – for example if one student is more able mathematically, and the other has a greater practical sense. Clearly much remains to be learned about collaboration between students applying mathematics to real situations in order to make this process more effective. This study has attempted to extract and highlight some of the issues that need further attention.

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