ZDM Mathematics Education (2012) 44:641–651
DOI 10.1007/s11858-012-0448-4

ORIGINAL ARTICLE

From the didactical triangle to the socio-didactical tetrahedron: artifacts as fundamental constituents of the didactical situation

Sebastian Rezat · Rudolf Sträßer

Accepted: 23 July 2012 / Published online: 1 August 2012

Abstract Research on the use of artifacts such as textbooks and digital technologies has shown that their use is not a straight forward process but an activity characterized by mutual participation between artifact and user. Taking a socio-cultural perspective, we analyze the role of artifacts in the teaching and learning of mathematics and argue that artifacts influence the didactical situation in a fundamental way. Therefore, we believe that understanding the role of artifacts within the didactical situation is crucial in order to become aware of and work on the relationships between the teacher, their students and the mathematics and, therefore, are worthwhile to be considered as an additional fundamental aspect in the didactical situation. Thus, by expanding the didactical triangle, first to a didactical tetrahedron, and finally to a "socio-didactical tetrahedron", a more comprehensive model is provided in order to understand the teaching and learning of mathematics.

Keywords Artifacts · Cultural–historical activity theory · Didactical tetrahedron · Didactical triangle · Digital technologies · Instrumental approach · Mathematics textbooks · Sociocultural perspective · Teachers' practice · Tools

1 Introduction

In the proposal for this issue of ZDM the questions were raised: "How does/can the introduction of digital technologies to teaching and learning mathematics affect the relationships within the didactic triangle? Does the technology introduce another 'vertex' such that it is necessary to refer to a didactic quadrilateral?" These questions especially attracted our attention. The idea of a new vertex due to the introduction of new technologies has already been expressed in the first ICMI study on the influence of computers and informatics on mathematics and its teaching: "We now have a triangle, student–teacher–computer, where previously only a dual relationship existed" (Churchhouse et al. 1984).

We find it noteworthy for, if not symptomatic of, mathematics education that a broad interest in issues related to tools and tool use in mathematics education was awoken when new technologies were introduced into the mathematics classroom. However, this perspective disregards the fact that tools, understood in a broad meaning as an artificial means incorporated in (mathematical) activity, have always played a major role in teaching and learning mathematics. Thus, the relationship between teacher and students in the mathematics classroom has actually never been dual. Mathematics textbooks, ruler and compasses and log tables are only a few examples of material tools that have been used in mathematics education long before new technologies entered the classrooms. Not to forget about non-physical tools such as language, diagrams and gestures to name but a few. In the end, teaching and learning of mathematics depends heavily on the existence of material representations of the immaterial mathematical structures (if these exist!). As a consequence, the content mathematics is dependent on embodiments (artifacts) to support the teaching/learning process. Dörfler takes "inscriptions" of mathematics as a *sine qua non* for teaching and learning mathematics, if not for the very existence of mathematics (see e.g. Dörfler 2007). Although early protagonists of the use of digital technologies in mathematics education (like Papert or Turkle) have put

S. Rezat (✉) · R. Sträßer
Justus-Liebig-University of Giessen, Giessen, Germany
e-mail: sebastian.rezat@math.uni-giessen.de

forward that digital technologies are different from other artifacts used in teaching and learning, we hold the view that digital technologies are comparable to any other artifact in terms of having affordances and imposing constraints on the user.

From the title of the paper, it is obvious that we answer the question of a new vertex in the didactical situation[1] in the affirmative. Moving from a didactical tetrahedron in the narrow sense, in which artifacts play the role of a fourth vertex, we suggest that social and institutional aspects of teaching and learning should be taken into account by means of an enlarged 'socio-didactical tetrahedron'.

We believe that an awareness of the use of artifacts in mathematics classrooms relates closely to the question "How might teachers be empowered to become aware of and work on relationships between themselves (the teacher), their students and the mathematics?" that was also raised in the proposal for this ZDM issue. Our answer to this question is to suggest that teachers can be empowered to identify and structure the teaching/learning situation through a model, which presents the most important elements influencing the classroom practice of teachers: mathematics, students, artifacts and teachers themselves. By finally suggesting a 'socio-didactical tetrahedron', we hope to contribute to the practicing teacher's (as well as the teacher trainer's, the school administrator's and the researcher's in didactics of mathematics—or mathematics education) awareness of the relationships, potentials and constraints s/he is working in. We start from the assumption that awareness and understanding of a situation has the potential to be effective by assisting the management of the possibilities, both those that promote and constrain learning, as these emerge. All this is triggered by a socio-cultural and activity theoretical perspective on tools in mathematics education.

2 A socio-cultural/activity theoretical perspective on tools in mathematics education

The origins of a psychological theory on tool use can be traced back to the Soviet psychologist Lev Semyonovich Vygotsky who is the founder of the cultural–historical psychology. Vygotsky distinguishes psychological from technical tools: "The most essential feature distinguishing the psychological tool from the technical one is that it is meant to act upon mind and behavior, whereas the technical tool, which is also inserted as a middle term between the activity of man and the external object, is meant to cause changes in the object itself. The psychological tool changes nothing in the object" (Vygotsky 1997). Since the central aim of tool use in didactical situations is to change the students' cognition of mathematics and not the mathematics itself, all tools that are used in the teaching and learning of mathematics can be considered psychological tools. Both kinds of tools are artifacts in the sense of Wartofsky (1979): "in the first place, tools and weapons, but more broadly, in good Aristotelian fashion, anything which human beings create by the transformation of nature and of themselves: thus, also language, forms of social organization and interaction, techniques of production, skills". Since the notion of tools is easily tinged with the idea of something material we prefer the broader notion of artifacts. Vygotsky argues both, psychological and technical tools fundamentally affect the activity that is carried out with the tool: "By being included in the process of behavior, the psychological tool modifies the entire course and structure of the new instrumental act, just as the technical tool modifies the processes of natural adaptation by determining the form of labor operations" (Vygotsky 1997).

Therefore, Vygotsky uses a triangle in order to visualize the structure of the instrumental act. He describes the changes that occur in even more detail: "The inclusion of a tool in the behavioral process, first, sets to work a number of new functions connected with the use and control of the given tool; second, abolishes and makes unnecessary a number of natural processes, whose work is (now) done by the tool; third, modifies the course and the various aspects (intensity, duration, order, etc.) of all mental processes included in the instrumental act, replacing some functions with others, i.e., it recreates, reconstructs the whole structure of behavior just like a technical tool recreates the entire system of labor operations. Mental processes, taken as a whole, form a complex structural and functional unity. They are directed toward the solution of a problem posed by the object, and the tool dictates their coordination and course. They form a new whole—the instrumental act" (Vygotsky 1997). Therefore, it seems likely from the theoretical perspective of cultural–historical-activity-theory that tools affect the students' learning of mathematics in a fundamental way.

Only recently has research in mathematics education come to acknowledge that fundamental transformations of behavior can be caused by the use of tools. At first sight, tools offer a wider range of possibilities for engaging with mathematics and therefore change the student's experience with mathematics as a whole. On a deeper level, the interaction with the tool affects the student's behavior. This is clearly evident in a review of more than 600 publications on the integration of technology into teaching and learning

[1] Here and throughout the text we use 'didactical situation' as a common sense expression and not in the narrow, technical sense of the French "théorie des situations didactiques" (TSD) approach to didactical situations in the sense of Brousseau (1997).

mathematics conducted by Artigue (2002) who concludes "that the complexity of instrumental genesis has been widely under-estimated in research and innovation on TICE [Technologies de l'Information et de la Communication appliquées à l'Enseignement], until quite recently" (Artigue 2002). The notion of instrumental genesis conceptualizes exactly the transformation of behavior that takes place when a new tool is incorporated in use. Artigue traces the introduction of this concept into mathematics education back to the need of frameworks that are more sensitive to the "technical-conceptual-cut" (Artigue 2002), i.e., the relation between technical and conceptual work in mathematics. Rabardel outlines the concept of instrumental genesis as follows: "It would seem that subjects, in line with the objects on which they must act, develop differentiated usage modalities for a given artifact—specific utilization schemes that tend to render the artifact multi-functional—and thus constitute, from the same artifact, several individualized instruments based on the specificity of the objects and the tasks. It would thus be by an adapting differentiation of the scheme component of the instrument that the artifact becomes multi-functional" (Rabardel 2002). An instrument in the sense of Rabardel is defined as "a composite entity made up of an artifact component (an artifact, a fraction of an artifact or a set of artifacts) and a scheme component (one or more utilization schemes, often linked to more general action schemes)" (Rabardel 2002).

Whereas the previous discussion primarily relates to the role of digital technologies in the teaching and learning of mathematics, the introduction of Rabardel's perspective widens the horizon to include a broader definition of artifacts. In research in mathematics education, there is evidence that the same applies to the role of other artifacts. The case of textbooks is especially well analyzed. Remillard (2005), for example, conducted a meta-analysis of studies on teachers' use of curriculum materials (including textbooks). From this review, she concluded that "studies that have focused on how teachers participate with curriculum materials have found that their reading of it is actually a highly interactive and multifaceted activity, rather than a straightforward process as may be assumed" and "that features of the curriculum matter to curriculum use as much as characteristics of the teacher" (Remillard 2005).

Rezat (2009) has recently investigated students' use of mathematics textbooks. One Episode he reports from a German 6th grade mathematics classroom will serve to demonstrate how much the behavior of students is influenced by the presence of textbooks.

Episode from a German 6th grade mathematics classroom:

> The subject of the lesson is the rule for multiplying decimals. The teacher follows a discovery learning approach and hands out problems taken from textbooks, which are not introduced in the school in question. These tasks are supposed to guide students' discovery of the rule for multiplying decimals in different ways: one by estimation in an everyday context, a second one by transforming decimals into ordinary fractions, a third one by using the calculator and the fourth one by calculating as if there was no decimal point and setting the decimal point afterwards on the basis of estimation.
>
> While working on the task in small groups two students use the box with the rule in their textbook. Denise argues that she used her book because she "want[s] to know how to multiply decimals".[2] Mia claims that she uses her book because she "want[s] to know it in advance."
>
> Following the group work the teacher summarizes the findings from the work in small groups in whole class discussion. While writing the rule for multiplying decimals onto the black board, one student complains: "But, in the textbooks it says that you have to determine the algebraic sign first." The teacher answers "I don't care what is written in the book." (Rezat 2011, p. 239).

Two incidents in this episode demonstrate how students' behavior is affected by the presence of the book. Denise's and Mia's comments on using their textbooks show that in both cases the availability of the textbook hinders their involvement in the discovery of the rule. They skip any attempt to discover the rule by themselves and immediately look up the result in their textbooks. Their behavior would have been different if no textbook or other information resource would have been available or if the rule for multiplying decimals would have not been in the book. The same applies to the student who compares what is written on the blackboard with what is written in the textbook. In his case the authority of the textbook allows him to challenge the teacher. This episode shows that the teacher's aims of activating the students, engaging them in the process of finding new rules based on previous knowledge, and using their discoveries and to proceed with their results are thwarted because of students' ways of using their textbooks. To understand how such occurrences arise we must unveil the inner structure of the situation that helps to explain why students and teachers behave like they do in the episode and assists in identifying the forces that influence their behavior. This aim will lead us to the

[2] All quotes from students and teachers were recorded in field notes and are originally in German. They were translated by S. Rezat.

development of a socio-didactical tetrahedron at the end of this paper.

3 From triangles to tetrahedron

In the previous section it was argued that artifacts, in general, and digital technologies and textbooks, in particular, have structuring effects on teaching and learning activities. The problematic of instrumental genesis indicates that artifacts are not passive resources that teachers and students draw on but 'actively' shape activities. This potential is also acknowledged within different theoretical perspectives that are not necessarily related to the instrumental approach. For example, Adler (2000) argues for a conceptualization of resources in terms of the verb *re-source*, meaning to source again or differently, in order to draw attention to resources and their use.

The theoretical approaches discussed in the previous section only relate to a single user of an artifact. In order to relate to the characteristic of the didactical situation with two different users or user groups, namely students and teachers sharing the same artifacts, it seems necessary to relate artifacts to the didactical situation as a whole. Because of their ability to restructure the didactical situation as a whole we argue that they must be considered as a fourth constituent, i.e., as a fourth vertex, of the didactical situation. We suggest that the inclusion of artifacts does not simply expand the didactical triangle to a quadrilateral, but adds a new dimension and thus leads to a tetrahedron model of the didactical situation.

Our move from the didactical triangle to a tetrahedron is not absolutely new, but has its forerunners in mathematics education. David Tall, for example, suggested the expansion of the triangle to a tetrahedron in 1986. In a paper published online, Tall used the important role of computers equipped with mathematical software to expand the classical didactical triangle to a didactical tetrahedron (Tall 1986). The same type of reconstruction is seen in a paper by Olive et al. (2010). While these tetrahedra only relate to technological artifacts in mathematics education, Rezat (2006) has also argued that mathematics textbooks can be considered as a fourth vertex related to the didactical triangle and developed an associated tetrahedron model. Starting from Rezat's tetrahedron, Sträßer (2009) suggests the range of circumstances for which this tetrahedron is valid is broader than first theorized. He argues that the tetrahedron model is applicable not only to mathematics textbooks or digital technologies but to artifacts in general.

Most of these contributions introduce the didactical tetrahedron as a ready made model without a genesis. Here, we want to summarize arguments that reveal the step by step development of the didactical tetrahedron based on the didactical triangle.

Starting from a Vygotskian perspective we will analyze the relationship of textbooks and digital technologies to the vertices of the didactical triangle. There are two reasons why we start from a Vygotskian perspective: first of all, Vygotsky explicitly refers to psychological tools which are a particular focus of this ZDM special issue besides digital technologies. Secondly, the instrumental act, as it was introduced by Vygotsky, is the core of many socio-cultural and semiotic theories which have proven fruitful in mathematics education research, such as activity theory (Engeström 1987), socio-cultural theory of action (Wertsch 1998), the ergonomic theory of the instrument (Rabardel 2002), and the theory of semiotic mediation (Bartolini Bussi and Mariotti 2008). By referring to this core we find it likely that our arguments will also be valid for these other theoretical approaches following Vygotsky.

As pointed out in the introduction, any encounter with mathematics is mediated through artifacts. Therefore, we can regard the student's activity, which we assume to be directed towards the knowing and understanding of mathematics, as an instrumental act in Vygotskian terms as depicted in Fig. 1.

Mediating artifacts might be mathematics textbooks, digital technologies, as well as tasks and problems, language.

A particularity of institutionalized learning of mathematics is that students do not always choose the mediating artifacts themselves, but teachers "orchestrate" the artifacts. The notion of orchestration refers in the first place to "the intentional and systematic organization of the various artifacts available in a computerized learning environment by the teacher for a given mathematical situation, in order to guide students' instrumental genesis" (Drijvers and Trouche 2008) but may as well be generalized to any artifact in a learning environment. A similar and widely accepted notion in research on the use of mathematics textbooks is that teachers mediate textbook use (Griesel and Postel 1983; Pepin and Haggarty 2001; Love and Pimm 1996). Evidence for the teacher's mediatory role is given by Rezat in his study on students' use of mathematics textbooks (Rezat 2011, 2009). These examples of teacher mediation show that the impact of the teacher on students' use of artifacts is twofold: On the one hand the teacher

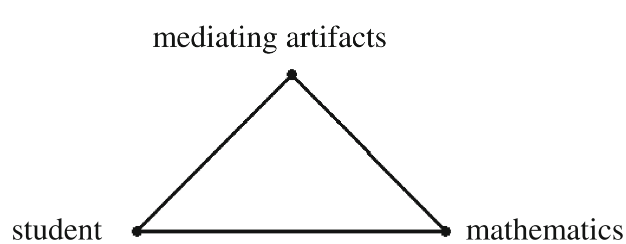

Fig. 1 Instrument mediated activity of learning mathematics

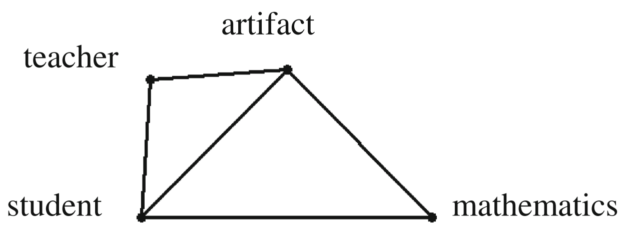

Fig. 2 Teacher mediation of students' artifact use

influences the instrumental act depicted in Fig. 1 in the way that s/he decides which artifacts to use and when they are to be used and on the other hand s/he even mediates the actual use of artifacts by students in terms of influencing the instrumental genesis. Figure 2 includes the second aspect of the teacher's mediatory role in the instrumental act of students.

This model, which leads to a quadrilateral representation of the didactical situation, neglects the fact that the teacher himself is a user of most of the artifacts implemented in mathematics classrooms. Research on teachers' use of textbooks and digital technologies has shown that the didactical situation is fundamentally influenced by the teachers' use of these artifacts. Within the Third International Mathematics and Science Study (TIMSS), Schmidt et al. (2001) found out that there is a positive correlation between textbook coverage in terms of space allocated for a topic to teacher implementation in terms of both, the average percent of instructional time allocated to that topic and to the percentage of a country's teachers who teach that topic. Another indication of the interaction between artifact use and teacher characteristics can be taken from the use of digital technologies: "When an analysis is conducted between countries it emerges that there is a positive association between teacher confidence in using ICT and actual use of ICT" (Ainley et al. 2010).

Gueudet and Trouche (2012) draw attention to teachers' use of artifacts and resources: "The teacher interacts with resources, selects them and works on them (adapting, revising, reorganizing, etc.) within processes where design and enacting are intertwined". In order to conceptualize teachers' professional work with artifacts and resources the authors suggest the documentational approach as "a general perspective for the study of teachers' professional evolution, where the researcher's attention is focused on the resources, their appropriation and transformation by the teacher or by a group of teachers working together" (Gueudet and Trouche 2009).

With respect to these considerations, it seems important to regard the teachers' own use of artifacts as an important aspect influencing the didactical situation and therefore include the teacher's instrumental act in the model of the didactical situation. However, Fig. 2 neither includes the teacher's own use of artifacts nor does it represent the relation of the teacher to the subject matter taught—to mathematics—at all. Including these relations in the model will render it almost impossible to depict the situation in terms of a quadrilateral. Moreover, incorporating teachers' and students' relations to mathematics and artifacts into a single two-dimensional representation is likely to lead to a confusing picture. Therefore, we argue that the complex relationships among the four constituents of the didactical situation—teacher, students, mathematics and artifacts—can be best described in a tetrahedron model as depicted in Fig. 3, which incorporates different instrumental acts into a new whole.

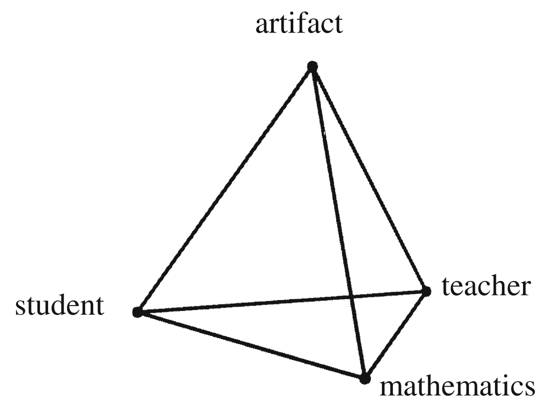

Fig. 3 Tetrahedron model of the didactical situation

Each of the triangular faces of the tetrahedron stands for a particular perspective on the role of artifacts within mathematics education: the didactical triangle is the basis of the model; the triangle *student–artifact–mathematics* depicts the instrument mediated activity of learning mathematics; the role of the teacher in relation to this triangle is best described as an orchestrator and a mediator as depicted in the triangle *teacher–artifact–student*; finally, the triangle *teacher–artifact–mathematics* depicts the teacher's instrument mediated activity of doing mathematics and planning mathematics instruction. The students are affected by this activity in terms of the teacher's beliefs about the nature of mathematics and about learning mathematics as well as the teacher's own 'instrumentalization' of artifacts (in the sense of Rabardel 2002). Thus, the whole tetrahedron is a comprehensive model of the didactical situation in the sense that it combines these different perspectives and also encompasses their interplay. All four constituents are interlaced beyond the possibility of separation.

Due to the dependency of mathematics on artifacts it is not by chance that the expansion of the didactical triangle to a tetrahedron, by adding the artifact as fourth vertex, was done in mathematics education and not with other school subjects so far.[3]

Additionally, we believe that considering artifacts as a fourth vertex of the didactical situation is also important for reasons that relate to the question of how teachers might be empowered to become aware of and work on relationships between themselves (the teacher), their students and the mathematics posed in the proposal for this ZDM special issue. The short episode from a 6th grade German mathematics classroom, presented above, shows how students' use of artifacts might interfere with the teacher's intentions and plans and thus is a constitutive element in classroom interaction. Furthermore, an insight into students' use of artifacts provides a better understanding of students' learning activities. In the episode it becomes clear that some students use their textbooks as a source of information in order to gain new knowledge or to constantly compare what is going on in mathematics class with what is written in the textbook. The behavior of the students elicits a focus on adopting (and not deducing or discovering) given rules in mathematics and thus a particular view on mathematics. It also points to the authority of textbooks in the learning process (cf. Love and Pimm 1996). Because of the preceding two arguments, the use of artifacts can be seen as one focal point of classroom interaction. This is consistent with Wartofsky's position: "artifacts not only have a use, but also are understood as representing the mode of activity in which they are used" (Wartofsky 1979). More awareness of students' use of artifacts means to become more aware of the relationships within the didactical situation in general.

4 The social and the cultural dimension

Sträßer (2009) argues that the didactical tetrahedron in the narrow sense "models the basic relations important for a scientific analysis of instrumented teaching and learning mathematics". However, it only relates to the 'surface' of students' and teachers' involvement with mathematics. Bauersfeld (1980) first drew attention to the "hidden dimensions of the so-called reality of a mathematics classroom". As one important aspect of mathematics education research Bauersfeld has pointed to the role of institutions: "Teaching and learning mathematics is realized in *institutions* which the society has set up explicitly to produce shared meanings among their members. Institutions are represented and reproduced through their members and that is why they have characteristic impacts on human interactions inside of the institutional. They constitute norms and roles; they develop rituals in actions and in meanings; they tend to seclusion and self-sufficiency; and they even produce their own content—in this case, school mathematics" (Bauersfeld 1980).

This is exactly what Sträßer points out as one shortcoming of the tetrahedron model: "As a consequence of this being a model, it is obvious that it may be worthwhile to think of something surrounding this tetrahedron, e.g., all those persons and institutions interested in the teaching and learning of mathematics, the "noosphere" (Chevallard 1985). For a full account of institutional and societal influences on teaching and learning mathematics with the help of instruments, it may be appropriate to have even more spheres surrounding this tetrahedron (Sträßer 2009).

Geiger has taken up this idea of surrounding spheres. He shows that "social interactions, in concert with available secondary artifacts, influence the transformation of students' understanding of mathematical knowledge, as well as their modes of reasoning and meaning making, in different ways according to the particular social setting in which learning is situated" (Geiger, in print) and thus expands the tetrahedron model by surrounding it with spheres of social context (SSC). These SSCs are inspired by Chevallard's (1985) noosphere but they differ as they are specific to the types of interaction that take place in individual, small group and whole group settings. Furthermore, his analysis suggests "that SSCs are not concentric and independent entities but rather that SSCs interact" (Geiger, in print).

Geiger's extension of the tetrahedron model draws particular attention to social settings in the classroom and their effect on instrumented learning, but it does not yet include societal and institutional influences. In order to overcome this shortcoming of the model we will further develop the tetrahedron model proposed in the previous section by drawing on Engeström's model of the activity system from the perspective of cultural–historical-activity theory as depicted in Fig. 4.

Engeström sees 'activity' as "a collective, systemic formation that has a complex mediational structure" (Engeström 1998). The major advantage of Engeström's

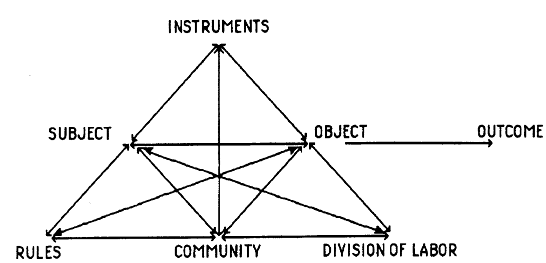

Fig. 4 Engeström's (1998) model of the activity system

[3] As far as we can see, a didactical tetrahedron was only also introduced in didactics of chemistry. But there it played a different role.

model is that "the less visible social mediators of activity—rules, community, and division of labor—are depicted at the bottom of the model" (Engeström 1998). Although a variety of interrelations between different aspects of human activity are depicted in this triangle Engeström points out that the primary task is to conceive the whole system. Nevertheless, it is important to stress that artifacts play a crucial role because they serve to focalize the other aspects of the whole system. As Wertsch (1998) points out from the perspective of socio-cultural analysis "this is so because the meditational means, or cultural tools, are inherently situated culturally, institutionally, and historically. […] the fact that cultural tools are involved means that the socio-cultural embeddedness of the action is always built into one's analysis." From this perspective the explication of socio-cultural aspects such as rules, community and division of labor is redundant, because these aspects are mirrored in the use of artifacts, but still it is a good reminder that they are explicated and thus will not be forgotten in the analysis.

Engeström applies his model to the two interrelated activity systems of teachers and students which he calls "traditional teaching" and "school-going" (Engeström 1998) as depicted in Fig. 5.

According to Engeström both activity systems are connected by sharing the same object of the activity: the school text. This promotes a pessimistic view on schooling by seeing the reproduction of the texts as the predominant object of the activity. However, we prefer a more optimistic view on teaching and learning mathematics by regarding mathematics as the object of the activity systems. Engeström's approach offers two perspectives—namely the teacher's and the student's perspective—on activities related to schooling. But, his description in terms of two interrelated activity systems does not provide a full account of the communities that teachers and students are a part of. On the one hand teachers and students belong to different and more or less discrete communities: students belong to the community of their peers, their families and maybe tutors; the community of teachers is made up of other teachers and mathematics educators—or in a broader sense the noosphere. But teachers and students do not only collaborate in a joint activity in the classroom. They are also members of one community with its norms and roles within the same institution. Therefore, we argue that an extension of the didactical tetrahedron based on Engeström's model of the activity system as depicted in Fig. 6 would provide a more comprehensive model.

By taking rules, communities, and division of labor of teachers and students into account it relates to the societal and institutional context of teaching and learning. The tetrahedron incorporates both, the distinct communities of teachers and students as well as the institution (in most cases the school) as a third and shared community. As Engeström pointed out, conventions and norms about being a student, about learning, about being a teacher, and about teaching relate to institutional settings such as the standardized time schedule, the organization of school subjects and grading. This is also true for the particular roles of teachers and students with regard to different approaches to teaching and learning, e.g. the teacher as the mediator of knowledge or the teacher as the moderator of classroom interaction and organizer of learning opportunities. In addition, the socio-didactical tetrahedron makes room for social and socio-mathematical norms as analyzed for example by Yackel and Cobb (1996). In their analysis Yackel and Cobb (1996) unveil underlying layers of conventions and norms that are negotiated in conversations between students and teachers about mathematics and argue that these are opportunities for teachers' and students' learning. Reinterpreting their analysis using the socio-didactical tetrahedron we can see that they focused on the role of 'discussion' as an artifact related to the negotiation of conventions and norms about mathematics between teachers and students. The fact that socio-mathematical norms are negotiated supports the separation of teachers' and students' system of conventions and norms within the socio-didactical tetrahedron. Furthermore,

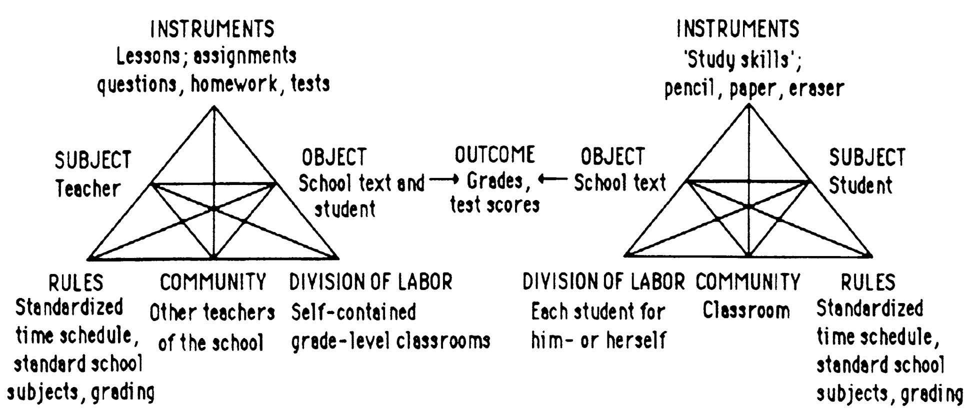

Fig. 5 Traditional teaching and school-going as interrelated activity systems according to Engeström (1998)

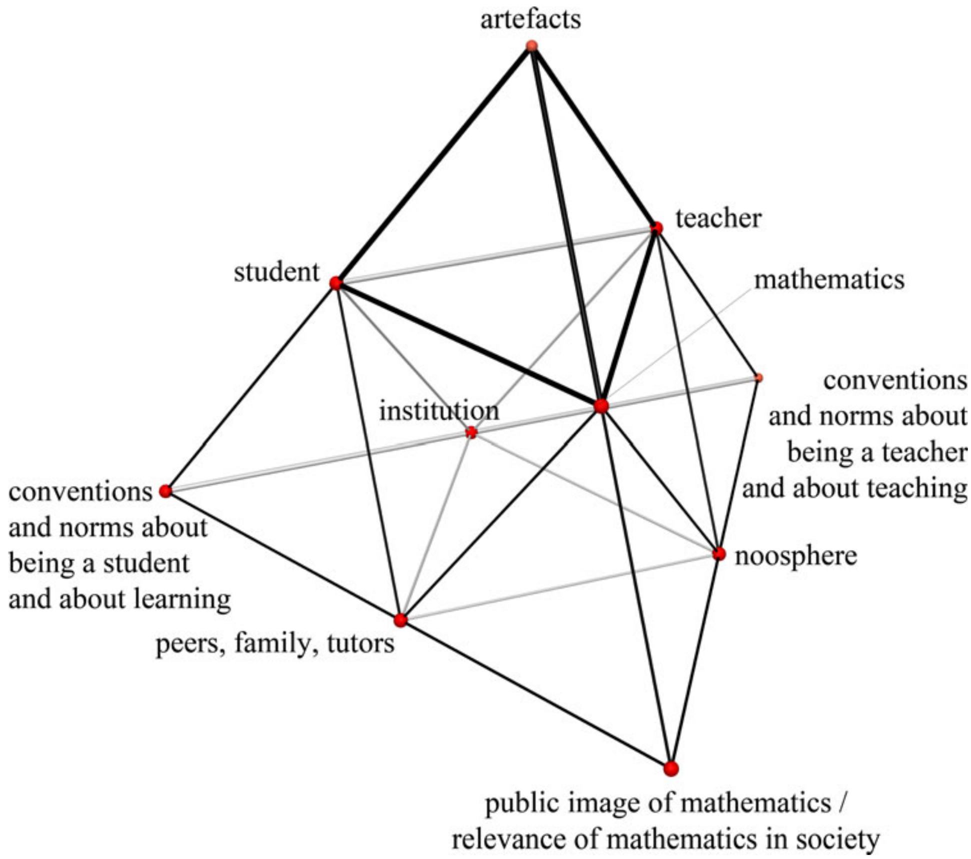

Fig. 6 Socio-didactical tetrahedron

separating conventions and norms related to teachers and teaching from conventions and norms related to students and learning also takes into account that both teacher and students act according to their actual subjective realities, which are constructed based on these norms and conventions. Bauersfeld (1980) has shown how teacher and students do not react to one another, but in fact react to their self-constructed interpretation of the situation. The third vertex of the base of the socio-didactical tetrahedron (public image of mathematics/relevance of mathematics in society) not only acknowledges the broad influence of the public view and relevance of the subject matter taught onto the two major agents of the didactical triangle. It also relates to more general questions on the role of mathematics in society in terms of division of labor. These have been raised recently by the ICMI-ICIAM study on Educational Interfaces between Mathematics and Industry (EIMI). This study seeks to better understand the "intimate connections between innovation, science, mathematics and the production and distribution of goods and services in society" (Damlamian and Sträßer 2009). In other words, the fundamental role of mathematics in relation to the division of labor in society is addressed explicitly as a starting point of the study. In particular, Sects. 1 and 2 of the discussion document show specific features of the public image of mathematics and the role, which mathematics visibly or invisibly plays in society and which are interlaced with issues of mathematics education (see Damlamian and Sträßer 2009, pp. 525–527).

For reasons of clarity and comprehensibility we neglected some connections within each side of the tetrahedron compared to Engeström's triangles in our representation in Fig. 6, e.g. the connection of the student with the public image of mathematics/the role of mathematics in society and the connection of norms and conventions about learning and being a student with mathematics. Nevertheless, we think that these also are important to keep in mind. But, as Engeström points out for his model of the activity system, it is most important to capture the systemic whole of the relations within the socio-didactical tetrahedron.

In order to demonstrate the benefit of the socio-didactical tetrahedron we would like to return to the episode from Sect. 2. Our previous analysis served to highlight the important role of artifacts in teaching and learning mathematics. At that point we could only analyze the surface level of the situation: the teacher used tasks from different textbooks in order to introduce the rule for multiplying decimals. Some students looked up the rule in their textbooks and used it to solve the tasks. Using the socio-didactical tetrahedron we can now analyze the structure of the situation and its driving forces. However, we have to highlight that at this point these driving forces are only hypothetical and cannot be verified through the data, because data on these aspects were not gathered in the study. Nevertheless, we believe that it would be possible to collect data on these aspects in future research. Let us look at the teacher first: the teacher chooses tasks from different textbooks. The analysis of these tasks reveals that these tasks can be characterized as guided discovery of the rule of multiplying decimals. This is in line with conventions and norms about teaching and being a teacher where the teacher has to stimulate students' learning and mathematical exploration rather than teaching them the right mathematics. This view is supported by the noosphere in terms of didactical principles like "discovery learning" (Bruner 1960) or "guided reinvention" of mathematics (Freudenthal 1991) and is in line with a view of mathematics as an activity. This way of teaching seems to be in conflict with the students' view of mathematics which can be characterized as looking upon mathematics as a fixed body of knowledge that has to be acquired. Presumably, they do not seem to see mathematics as an activity but mathematics as defined by what is written in the book. The students' behavior seems to be rooted in conventions and norms about being a student and about learning that might be characterized as "success in school means to get the right answer", and "learning mathematics is learning the rules". Furthermore it seems like that the two students who looked up the rule in their book in order to know it in advance are cooperative in the sense that they play the game of guided discovery with the teacher, because there was no moment during the lesson where it was apparent that they looked up the rule in their book. This seems to be different for the other student who compares the rule in the book with the rule on the blackboard. One way his behavior might be interpreted is that he seems to rely on the authority of the book in order to question the teacher.

This short analysis reveals that the driving forces of the actors only become apparent by taking into account the aspects at the bottom level of the socio-didactical tetrahedron. Only a comprehensive analysis of actions within the didactical tetrahedron and their social and institutional grounding reveal a picture that contributes to a situated understanding of the complex network of actions and their backings in the situation at hand.

5 Conclusion

Starting from the assumption that mathematics is heavily dependent on tools in conjunction with a socio-cultural/activity-theoretical view on tools we argued that tools or artifacts in general must be considered as a fourth fundamental constituent of the didactical situation in mathematics education. Accordingly, we summarized arguments that call for an expansion of the well known didactical triangle to a didactical tetrahedron. This is our response to the question about how technology affects the relationships within the didactic triangle in a way that it seems necessary to introduce another 'vertex'. We suggest that not only the introduction of technology, but any use of artifacts in the teaching and learning of mathematics calls for an expansion of the didactical triangle to a "didactical tetrahedron in the narrow sense".

The absence of social and institutional aspects of teaching and learning mathematics that affect classroom interaction was pointed out as a major shortcoming of the didactical tetrahedron in the narrow sense. This led us to an expansion of the didactical tetrahedron building on Engeström's triangles of the activity system including the societal and institutional aspects 'rules', 'communities' and 'division of labor'. Nevertheless, artifacts remain the focal point of the whole system in the sense that artifact use is culturally, institutionally, and historically situated and therefore the societal and cultural domain is always included.

We are aware that this so called 'socio-didactical tetrahedron' also has its limitations. The tetrahedron should be taken as a model with the usual constraints and affordances of such models. To be more specific and in accordance to the usual mathematical definition, we do not intend to see the tetrahedron as a regular one, we do not even intend to have metrics on it. The segments between the points merely indicate relations, no distances or other measures. Moreover, we do not claim that the socio-didactical tetrahedron is presenting all possible and researchable features of the didactical situation. In this respect, a mere hint to the fact that the (individual as well as the societal) didactical situation has a development in time will suffice. There is no indication of this in the tetrahedron as we have presented it above.

Nevertheless, compared to Geiger's surrounding spheres, our model in Fig. 6 offers a structure of societal and institutional influences and also comprises their interrelations. Consequently, the model also structures the scientific field of mathematics education by relating different perspectives on the didactical situation represented in the didactical triangle to one another. It can also be used to locate different kinds of research within the system, e.g. student-centered research may focus on the edge "student-mathematics", research on artifacts may focus on the edge "teachers-mathematics" or "peers-mathematics". In that respect it is also valuable to examine different edges, triangles or sub-tetrahedra within the socio-didactical tetrahedron. We do not claim that every sub-tetrahedron of our model makes sense of its own or relates to a particular aspect of the whole activity system, but most of the sub-tetrahedra provide insight or a better understanding of the whole system. This also applies to other 'sub-polyhedra' of the socio-didactical tetrahedron. E.g., if the four limiting vertices of the larger tetrahedron are neglected, an interesting octahedron is left, which is constituted by the didactical triangle *teacher–student–mathematics*, but is complemented by a 'lower', societal level of the institution, people associated with the student and those who are interested in mathematics education at large—the "noosphere" of Chevallard. The relations between these six vertices represent important forces influencing and shaping the teaching and learning of mathematics, but have the disadvantage of not representing the importance of the artifacts used in the teaching–learning process—the starting point for our elaborations.

Acknowledgments We sincerely thank the three unknown reviewers, who reviewed the first version of this paper, for their constructive and pertinent comments and suggestions. We particularly appreciate the work of Vincent Geiger who turned our text into readable Australian English.

References

Adler, J. (2000). Conceptualizing resources as a theme for teacher education. *Jorunal of Mathematics Teacher Education, 3*(3), 205–224.

Ainley, J., Eveleigh, F., Freeman, C., & O'Malley, K. (2010). ICT in the teaching of science and mathematics in year 8 in Australia: report from the IEA Second International Technology in Education Study (SITES) survey. ACER Research Monographs. http://research.acer.edu.au/acer_monographs/6. Accessed 17 July 2012.

Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*, 245–274.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 750–787). Mahwah: L. Erlbaum Associates.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11*(1), 23–41. doi:10.1007/bf00369158.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dordrecht: Kluwer Academics Publishers.

Bruner, J. S. (1960). *The process of education*. Harvard: Harvard University Press.

Chevallard, Y. (1985). *La Transposition Didactique. Du savoir savant au savoir enseigné.* Grenoble: Pensées sauvages.

Churchhouse, R. F., Cornu, B., Ershov, A. P., Howson, A. G., Kahane, J. P., van Lint, J. H., et al. (1984). The influence of computers and informatics on mathematics and its teaching. An ICMI discussion document. *L'Enseignement Mathématique, 30*, 161–172.

Damlamian, A., & Sträßer, R. (2009). ICMI Study 20: educational interfaces between mathematics and industry. *ZDM—The International Journal on Mathematics Education, 41*(4), 525–533.

Dörfler, W. (2007). Matrices as Peircean diagrams: A hypothetical learning trajectory. In D. Pitta-Pantazi & G. Philippou. Larnaca (Eds.), *European research in mathematics education. Proceedings of the fifth congress of the European Society for Research in Mathematics Education* (pp. 852–861). Cyprus: European Society for Research in Mathematics Education (ERME), Department of Education, University of Cyprus.

Drijvers, P., & Trouche, L. (2008). From artifacts to instruments. A theoretical framework behind the orchestra metaphor. In G. W. Blume & M. K. Heid (Eds.), *Research on technology and the teaching and learning of mathematics: Volume 2. Cases and perspectives* (pp. 363–391). Charlotte: Information Age.

Engeström, Y. (1987). *Learning by expanding. An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit Oy.

Engeström, Y. (1998). Reorganizing the motivational sphere of classroom culture: an activity-theroretical analysis of planning in teacher team. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp. 76–103). Cambridge: Cambridge University Press.

Freudenthal, H. (1991). *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer Academic Publishers.

Geiger, V. (in print). The role of social aspects of teaching and learning in transforming mathematical activity: Tools, tasks, individuals and learning communities. In S. Rezat, M. Hattermann, & A. Peter-Koop (Eds.), *Transformation—A Big Idea in Mathematics Education.* New York: Springer.

Griesel, H., & Postel, H. (1983). Zur Theorie des Lehrbuchs – Aspekte der Lehrbuchkonzeption. *Zentralblatt für Didaktik der Mathematik, 83*(6), 287–293.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199–218.

Gueudet, G., & Trouche, L. (2012). Teachers' work with resources: Documentational geneses and professional geneses. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources* (Vol. 7, pp. 23–41, Mathematics Teacher Education). Dordrecht: Springer.

Love, E., & Pimm, D. (1996). 'This is so': a text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (Vol. 1, pp. 371–409). Dordrecht: Kluwer.

Olive, J., Makar, K., Hoyos, V., Kor, L., Kosheleva, O., & Sträßer, R. (2010). Mathematical knowledge and practices resulting from access to digital technologies. In C. Hoyles & J. B. Lagrange (Eds.), *Mathematics education and technology—rethinking the terrain* (pp. 133–177). New York: Springer.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: a way to understand teaching and learning cultures. *ZDM—The International Journal on Mathematics Education, 33*(5), 158–175.

Rabardel, P. (2002). People and Technology: a cognitive approach to contemporary instruments. http://ergoserv.psy.univ-paris8.fr. Accessed 17 July 2012.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211–246.

Rezat, S. (2006). A model of textbook use. In J. Novotná, H. Moraová, M. Krátká, & N. A. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 409–416). Prague: Charles University, Faculty of Education.

Rezat, S. (2009). *Das Mathematikbuch als Instrument des Schülers. Eine Studie zur Schulbuchnutzung in den Sekundarstufen*. Wiesbaden: Vieweg+Teubner.

Rezat, S. (2011). Interactions of teachers' and students' use of mathematics textbooks. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources. Mathematics curriculum* *materials and teacher development* (pp. 231–246). New York: Springer.

Schmidt, W. H., McKnight, C. C., Houang, R. T., Wang, H., Wiley, D. E., Cogan, L. S., et al. (2001). *Why schools matter: A cross-national comparison of curriculum and learning*. San Francisco: Jossey-Bass.

Sträßer, R. (2009). Instruments for learning and teaching mathematics. An attempt to theorise about the role of textbooks, computers and other artefacts to teach and learn mathematics. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the 33rd conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 67–81). Thessaloniki: PME.

Tall, D. (1986). Using the computer as an environment for building and testing mathematical concepts. A tribute to Richard Skemp. Warwick. http://www.warwick.ac.uk/staff/David.Tall/themes/computers.html. Accessed 17 July 2012.

Vygotsky, L. (1997). The instrumental method in psychology. In R. W. Rieber & J. Wollock (Eds.), *The collected works of L. S. Vygotsky. Volume 3. Problems of the theory and history of psychology* (pp. 85–89). New York: Plenum Press.

Wartofsky, M. W. (1979). *Models—representation and the scientific understandig* (Vol. 48, Boston Studies in the Philosophy of Science). Dordrecht: Reidel.

Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477.