are nontrivial solutions for both $\lambda > \lambda_0$ and $\lambda < \lambda_0$. On the other hand, if $a_2 = 0$, the bifurcating branch is locally quadratic and the location of solutions depends on the sign of λ_2. If $\lambda_2 < 0$ there are two nontrivial solutions for $\lambda < \lambda_0$ and none for $\lambda > \lambda_0$ (locally, of course). If $\lambda_2 > 0$ the opposite occurs, namely there are two nontrivial solutions for $\lambda > \lambda_0$ and none for $\lambda < \lambda_0$. This situation ($a_2 = 0$) is called a **pitchfork** bifurcation. Higher order degeneracies can also occur and their representative solution structure can be determined (see Problem 11.3.5).

11.3.1 Lyapunov-Schmidt Method

It is apparent that small solutions of the nonlinear eigenvalue problem (11.14) exist. However, the implicit function theorem cannot be invoked for the problem as formulated. The **Lyapunov-Schmidt method** provides a way to reformulate nonlinear problems to overcome this difficulty.

Suppose that we let $\lambda = \lambda_0 + \mu$ and $u = \epsilon\phi + \epsilon v$, where $\langle v, \phi\rangle = 0$. In terms of these new variables, the equation (11.14) becomes

$$Lv + \lambda_0 v = -\mu\epsilon(\phi + v) - f(\epsilon\phi + \epsilon v).$$

This equation is solvable if and only if its right hand side is orthogonal to ϕ, so we project its right hand side onto the range of the operator $L + \lambda_0$ by writing

$$\begin{aligned} Lv + \lambda_0 v &= -\mu\epsilon(\phi + v) - f(\epsilon(\phi + v)) + \langle \mu\epsilon(\phi + v) + f(\epsilon(\phi + v)), \phi\rangle\phi \\ &= -\mu\epsilon v - f(\epsilon(\phi + v)) + \langle f(\epsilon(\phi + v)), \phi\rangle\phi. \end{aligned} \quad (11.18)$$

However, for this to be the same problem, we must also require

$$\langle \mu\epsilon(\phi + v) + f(\epsilon(\phi + v)), \phi\rangle = \mu\epsilon + \langle f(\epsilon(\phi + v)), \phi\rangle = 0. \quad (11.19)$$

We collect (11.18) and (11.19) into the system of equations for v and μ

$$\begin{pmatrix} Lv + \lambda_0 v \\ \mu \\ \langle v, \phi\rangle \end{pmatrix} = \begin{pmatrix} -\mu\epsilon v - f(\epsilon(\phi + v)) + \langle f(\epsilon(\phi + v)), \phi\rangle\phi \\ \frac{1}{\epsilon}\langle f(\epsilon(\phi + v)), \phi\rangle \\ 0 \end{pmatrix}. \quad (11.20)$$

We observe that the right hand side of (11.20) is fully nonlinear, i.e., has no linear terms when $\epsilon = 0$, and a solution of this problem is $v = 0$, $\mu = 0$ when $\epsilon = 0$. Furthermore, the linear operator is invertible, so that the implicit function theorem can be invoked: There is a unique solution $v = v(\epsilon)$, $\mu = \mu(\epsilon)$ for all ϵ sufficiently small. The perturbation calculation given above is one way to explicitly determine this solution.

11.4 Oscillations and Periodic Solutions

Another problem related to the nonlinear eigenvalue problem is the calculation of periodic solutions of weakly nonlinear differential equations. In this section we consider four examples.

11.4.1 Advance of the Perihelion of Mercury

In this first example, we determine the relativistic effects on the orbit of planets. Using Newtonian physics and the inverse square law of gravitational attraction, one can show (Problem 11.4.1b) that the motion of a satellite about a massive planet is governed by the equation

$$\frac{d^2u}{d\theta^2} + u = a, \quad (11.21)$$

where $a = GM/h^2$, M is the mass of the central planet, G is the gravitational constant $r = 1/u$ is the radius of the orbit, θ is the angular variable of the orbit and $r^2\frac{d\theta}{dt} = h$ is the angular momentum per unit mass of the satellite. The solutions of this equation are given by

$$r = \frac{1}{a + A\sin\theta},$$

which is the equation in polar coordinates of an ellipse with eccentricity $\epsilon = A/a$. The period of rotation is

$$T = \frac{2\pi a}{h(a^2 - A^2)^{3/2}}.$$

Following Einstein's theory of general relativity, Schwarzschild determined that with the inclusion of relativistic effects the equation (11.21) is modified to

$$\frac{d^2u}{d\theta^2} + u = a(1 + \epsilon u^2), \quad (11.22)$$

where $a\epsilon = 3GM/c^2$, c^2 is the speed of light. (A derivation of equation (11.22) is given in Chapter 5; see (5.8).) We want to find periodic solutions of this equation for small ϵ. As a first attempt we try a solution of the form

$$u = a + A\sin\theta + \epsilon u_1 + \epsilon^2 u_2 + \cdots.$$

This guess incorporates knowledge of the solution for $\epsilon = 0$.

We substitute this proposed solution into the equation (11.22), collect like powers of ϵ, and find the hierarchy of equations

$$u_1'' + u_1 = au_0^2,$$

$$u_2'' + u_2 = au_0u_1,$$

and so on, where $u_0 = a + A\sin\theta$. The first equation already exposes the difficulty. The operator $Lu = u'' + u$ with periodic boundary conditions, has a two dimensional null space spanned by the two periodic functions $\sin\theta$ and $\cos\theta$. To find u_1, we must have au_0^2 orthogonal to $\sin\theta$ and $\cos\theta$, which it is not. Therefore u_1 does not exist.

The difficulty is that, unlike the nonlinear eigenvalue problems of the previous section, there is no free parameter to adjust so that the orthogonality condition can be satisfied. If there were a free parameter somewhere there might be a hope, so we need to look around for a free parameter.

After some reflection, the unknown parameter becomes apparent. Notice that we are looking for 2π-periodic solutions of this problem, but why, one might ask, should we expect the period to remain fixed at 2π as ϵ changes? Indeed, the logical choice for unknown parameter is the period of oscillation.

To incorporate the unknown period into the problem, we make the change of variables $x = \omega\theta$ and then seek fixed period 2π periodic solutions of the equation $\omega^2 u'' + u = a(1 + \epsilon u^2)$ where ω is an unknown eigenvalue parameter.

Now the problem looks much like a nonlinear eigenvalue problem from the last section so we try

$$u = a + A\sin x + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \qquad \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots.$$

We substitute this guess into the governing equation (11.22), collect like powers of ϵ, and find the hierarchy of equations

$$u_1'' + u_1 = au_0^2 - 2\omega_1 u_0'', \tag{11.23}$$

$$u_2'' + u_2 = 2au_0u_1 - 2\omega_2 u_0'' - \omega_1^2 u_0'' - 2\omega_1 u_1'', \tag{11.24}$$

and so on.

To solve for u_1, we require that the right hand side of (11.23) be orthogonal to both $\sin x$ and $\cos x$. This right hand side is

$$au_0^2 - 2\omega_1 u_0'' = a(a^2 + 2aA\sin x + A^2\sin^2 x) + 2\omega_1 A\sin x,$$

which is orthogonal to $\sin x$ and $\cos x$ if and only if $\omega_1 = -a^2$. We can now determine that

$$u_1 = a^3 + \frac{aA^2}{2} + \frac{aA^2}{6}\cos 2x,$$

which is uniquely specified by the requirement $\langle u_1, \sin x\rangle = \langle u_1, \cos x\rangle = 0$.

We could continue in this fashion generating more terms of the expansion. One can show that this procedure will work and in fact, converge to the periodic solution of the problem. For the physicist, however, one term already contains important information. We observe that the solution is

$$u = a + A\sin((1 - a^2\epsilon + O(\epsilon)^2)\theta) + O(\epsilon).$$

The perihelion (the point most distant from the center of the orbit) occurs when the sine is maximized, approximately when

$$(1 - a^2\epsilon)\theta = 2\pi n + \pi/2,$$

so that two successive maxima are separated by

$$2\pi + \Delta\theta = \frac{2\pi}{1 - a^2\epsilon},$$

which is greater than 2π. That is, the perihelion advances by $\Delta\theta = \frac{2\pi a^2\epsilon}{1-a^2\epsilon}$ radians. In terms of measurable physical quantities this is

$$\Delta\theta = 2\pi a^2\epsilon = \frac{6\pi}{1 - e^2}\frac{1}{c^2}\left(\frac{2\pi GM}{T}\right)^{2/3}.$$

The perihelion of Mercury is known to advance 570 seconds of arc per century. Of this amount all but 43 seconds of arc can be accounted for as due to the effects of other planets. To see how well the above formula accounts for the remaining 43 seconds of arc, we must do a calculation. Using that the eccentricity of the orbit of Mercury is 0.205, the period of Mercury is 87.97 days, the mass of the sun is 1.990×10^{30} kg, the gravitational constant G is 6.67×10^{-11} Nm^2/kg and the speed of light c is 3.0×10^8 m/sec, we find that $\Delta\theta = 50.11 \times 10^{-8}$ radians/period. This converts to 42.9 seconds of arc per century, which is considered by physicists to be in excellent agreement with observation.

11.4.2 Van der Pol Oscillator

Our second example is to determine periodic orbits of the van der Pol equation. In 1928, the Dutch engineers van der Pol and van der Mark published an electrical circuit exhibiting periodic oscillations that they proposed as a model for the pacemaker of the heart. Today we know that this model is far too crude to model the heartbeat in a reliably quantitative way, but in certain parameter regions it has some qualitative similarities with the cardiac pacemaker (the sinoatrial node).

The van der Pol oscillator is often described by the electrical circuit shown in Fig. 11.3. It is a device with three parallel circuits, one a capacitor, the second a resistor, inductor and voltage source in series, and the last some nonlinear device, such as a tunnel diode. Kirchhoff's laws for this circuit give that

$$I_0 = i_1 + i_2 + i_3,$$

where I_0 is the current input into the circuit, i_1, i_2, and i_3 are the currents in the three branches of the circuit. The voltage-current relationships in the three branches are

$$C\frac{dv}{dt} = i_1,$$

$$Ri_2 + L\frac{di_2}{dt} = v + v_0,$$

$$i_3 = f(v),$$

where v is the voltage drop across the circuit, and $f(v)$ is the current-voltage response function of the nonlinear device.

If we eliminate i_1 and i_3, these equations reduce to the two equations

$$C\frac{dv}{dt} = I_0 - i_2 - f(v), \qquad L\frac{di_2}{dt} = v + v_0 - Ri_2$$

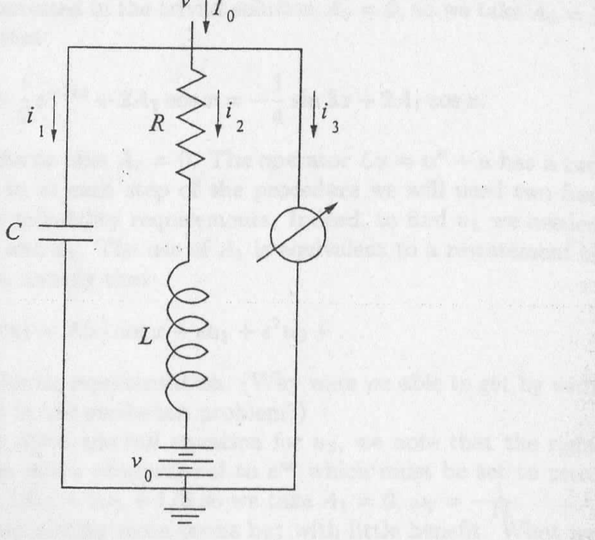

Figure 11.3: Circuit diagram for van der Pol equation.

and, if $R = 0$, we have

$$C\frac{d^2v}{dt^2} + f'(v)\frac{dv}{dt} + \frac{v+v_0}{L} = \frac{dI_0}{dt}.$$

The important property of the tunnel diode is that it has a nonlinear current-voltage response function with a negative resistance region (although tunnel diodes are rarely used in modern electronic circuits). A reasonable model is to take $f(v)$ to be the cubic polynomial

$$f(v) = Av\left(\frac{v^2}{3} - \frac{1}{2}(v_1+v_2)v + v_1v_2\right).$$

We tune the voltage source so that $v_0 = -(v_1+v_2)/2$, set

$$v = \left(\frac{v_2-v_1}{2}\right)u + \frac{v_1+v_2}{2}$$

and scale time so that $t = \sqrt{LC}\tau$. The resulting equation (with $\frac{dI_0}{dt} = 0$) is the **van der Pol equation**

$$u'' + \epsilon u'(u^2-1) + u = 0, \tag{11.25}$$

where ϵ is the dimensionless parameter

$$\epsilon = A\left(\frac{v_2-v_1}{2}\right)^2 \cdot \sqrt{\frac{L}{C}}.$$

In this chapter we seek solutions of the van der Pol equation when the parameter $\epsilon > 0$ is small. The region ϵ large is of greater relevance to the study of the heartbeat, but requires significantly different techniques, which we defer to Chapter 12.

With our experience from the previous problem, we know to let the period of oscillation be a free parameter, and make the change of variables $x = \omega\tau$, after which the equation (11.25) becomes

$$\omega^2\frac{d^2u}{dx^2} - \epsilon\omega\frac{du}{dx}(1-u^2) + u = 0.$$

We expect solutions to depend on ϵ, and furthermore, we know how to solve the reduced problem with $\epsilon = 0$. We try a power series solution of the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots, \qquad \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \ldots$$

and as usual, we find a hierarchy of equations

$$\begin{aligned} u_0'' + u_0 &= 0, & (11.26)\\ u_1'' + u_1 &= -u_0'(u_0^2-1) - 2\omega_1 u_0'', & (11.27)\\ u_2'' + u_2 &= -\omega_1^2 u_0'' - \omega_1 u_0'(u_0^2-1) - 2\omega_1 u_1'' \\ &\quad -(u_0^2-1)u_1' - 2u_0u_0'u_1 - 2\omega_2 u_0'', & (11.28) \end{aligned}$$

and so on.

The solution of the first equation (11.26) is

$$u_0 = A\cos(x+\phi).$$

Since the van der Pol equation is autonomous, the phase shift ϕ is arbitrary and it is convenient to take it to be zero. Actually, the calculations that follow below are much easier if we use complex exponentials rather than trigonometric functions, but regardless, they should be done using symbolic computation. Thus, we take

$$u_0 = A_0(e^{ix} + e^{-ix}) = 2A_0\cos x,$$

and then, according to (11.27), u_1 must satisfy

$$u_1'' + u_1 = -A_0^2 i e^{3ix} - (A_0^3 i - A_0 i + 2A_0\omega_1)e^{ix} + cc.$$

By cc we mean the complex conjugate of the preceding complex functions.

It is an easy matter to see what to do. The operator $Lu = u'' + u$ has a two dimensional null space spanned by $\sin x$ and $\cos x$ and in order to find u_1, the right hand side of the equation for u_1 must be orthogonal to these basis functions. One need not evaluate any integrals but simply observe that this occurs if and only if the coefficients of e^{ix} and e^{-ix} are identically zero. Thus, we require

$$A_0^3 - A_0 = 0, \qquad A_0\omega_1 = 0.$$

We are certainly not interested in the trivial solution $A_0 = 0$, so we take $A_0 = 1$ and $\omega_1 = 0$. It follows that

$$u_1 = \frac{i}{8}e^{3ix} - \frac{i}{8}e^{-3ix} + 2A_1 \cos x = -\frac{1}{4}\sin 3x + 2A_1 \cos x.$$

In general, it is a mistake to take $A_1 = 0$. The operator $Lu = u'' + u$ has a two dimensional null space so at each step of the procedure we will need two free parameters to meet the solvability requirements. Indeed, to find u_1 we needed the two parameters A_0 and ω_1. The use of A_1 is equivalent to a restatement of our original assumption, namely that

$$u(x) = A(\epsilon)\cos x + \epsilon u_1 + \epsilon^2 u_2 + \ldots$$

where $A(\epsilon)$ has a power series representation. (Why were we able to get by with only one free parameter in the perihelion problem?)

Rather than writing down the full equation for u_2, we note that the right hand side of (11.28) has terms proportional to e^{ix} which must be set to zero. The coefficient of e^{ix} is $2A_1 i + 2\omega_2 + 1/8$ so we take $A_1 = 0$, $\omega_2 = -\frac{1}{16}$.

We could continue calculating more terms but with little benefit. What we now know is that there is a periodic solution of the van der Pol equation of the form

$$u = 2\cos\omega\tau - \frac{\epsilon}{4}\sin 3\omega\tau + O(\epsilon^2), \qquad \omega = 1 - \frac{\epsilon^2}{16} + O(\epsilon^3).$$

In other words, as ϵ changes, the circular orbit becomes slightly noncircular with a slight decrease of frequency. The radius of the unique limit cycle is about 2 for ϵ small.

11.4.3 Knotted Vortex Filaments

The goal of this example is to determine the motion of a knotted vortex filament. A more prosaic way of describing this problem is to say we are going to find the motion of thin smoke rings which are trefoil knots. Exactly how to blow a knotted smoke ring is not known. To be sure, this example is a bit more complicated than the previous examples, but in spirit it is identical to the calculation of planetary orbits.

To begin this discussion we must know how a vortex filament moves in space. Vorticity is defined as the curl of the velocity vector field of a fluid, and we suppose that the vorticity is confined to be nonzero only along a thin filament in space. Then the position vector $\mathbf{R}$ of the filament has a velocity given by

$$\mathbf{R}_t = \kappa\mathbf{B}, \tag{11.29}$$

where κ is the curvature, and $\mathbf{B}$ is the binormal vector of the filament. This equation is derived in the book by Batchelor [8]. The derivation begins with the Biot-Savart law (8.20) derived in Chapter 8, applied to an infinitely thin vortex line, but actually shows that an infinitely thin vortex filament with nonzero curvature moves infinitely fast. However, a change of time scale is made to

make velocities finite and the equation $\mathbf{R}_t = \kappa\mathbf{B}$ results. How this equation describing the self induced motion of a vortex filament relates to real fluids is not completely resolved, although there is some evidence that it is a reasonable approximation in supercooled helium.

We want to find invariant solutions of the equation of motion (11.29). By invariant, we mean solutions of the equation that move in space without change of shape. Thus, they can rotate rigidly or translate. To allow these we take the space curve $\mathbf{R}$ to be represented by

$$\mathbf{R}(s,t) = \rho(s,t)\mathbf{r}(\theta(s,t) + \Omega t) + (z(s,t) + Vt)\mathbf{k}, \tag{11.30}$$

where $\mathbf{k}$ is a unit vector in the z direction, $\mathbf{r}$ is a unit vector in the x, y plane, orthogonal to the z direction, and the argument of $\mathbf{r}$ denotes the polar angle direction in the x, y plane in which $\mathbf{r}$ points. Thus, $\mathbf{r}' = d\mathbf{r}/d\theta$ is a unit vector orthogonal to $\mathbf{r}$, and the three vectors $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{k}$ form a right handed coordinate system. The unkown number V is the velocity of translation along the z axis, and Ω is the rate of rotation about the z axis. Here s is the arclength coordinate, so we must have

$$\left|\frac{d\mathbf{R}}{ds}\right|^2 = \rho_s^2 + \rho^2\theta_s^2 + z_s^2 = 1. \tag{11.31}$$

A brief review of space curves is in order. For any space curve $\mathbf{R}$, its unit tangent vector $\mathbf{T}$ is defined by $\mathbf{T} = \mathbf{R}_s$, where s is the arclength coordinate of $\mathbf{R}$. The variation of $\mathbf{T}$ in space gives definition to the unit normal vector $\mathbf{N}$, defined by $\mathbf{T}_s = \kappa\mathbf{N}$. An orthogonal triad, the **Frenet frame** is completely defined with the unit binormal vector $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. The torsion of a curve, τ, is uniquely defined by $\mathbf{N}_s = -\kappa\mathbf{T} + \tau\mathbf{B}$, and for consistency it must be that $\mathbf{B}_s = -\tau\mathbf{N}$. These three relationships between $\mathbf{T}, \mathbf{N}$, and $\mathbf{B}$ are called the **Frenet-Serret equations.**

Now we observe that, if $\mathbf{R}_t = \kappa\mathbf{B}$, then $\mathbf{R}_t \cdot \mathbf{R}_s = \kappa\mathbf{B}\cdot\mathbf{T} = 0$, and $\mathbf{R}_t \times \mathbf{R}_s = \kappa\mathbf{B}\times\mathbf{T} = \kappa\mathbf{N} = \mathbf{R}_{ss}$. In addition, we make the assumption that the functions ρ, θ and z are functions of the single variable $\zeta = s - ct$. This change of variables allows us to look for steadily progressing but invariant objects. We now use $\mathbf{R}_t \cdot \mathbf{R}_s = 0$ with (11.30) and (11.31) to calculate that

$$-c + \rho^2\theta'\Omega + Vz' = 0, \tag{11.32}$$

and from the $\mathbf{k}$ component of the equation $\mathbf{R}_t \times \mathbf{R}_s = \mathbf{R}_{ss}$, we learn that

$$z'' = -\Omega\rho\rho'.$$

This can be integrated once to

$$z' = \Omega(A - \frac{1}{2}\rho^2), \tag{11.33}$$

where A is an unknown constant. Finally, the arclength restriction (11.31) reduces to

$$\rho'^2 + \rho^2\theta'^2 + z'^2 = 1. \tag{11.34}$$