Now $f(s) = 1/s$ and $\phi(s) = -s + \ln s$. Laplace's method applies directly to this transformed integral. The maximum of $\phi(s)$ occurs at $s = 1$ so (6.4.19c) gives

$$\Gamma(x) \sim x^x e^{-x}\sqrt{2\pi/x}, \qquad x \to +\infty, \tag{6.4.39}$$

in agreement with (5.4.1). To obtain the next term in the Stirling series we note that $\phi(1) = -1$, $\phi'(1) = 0$, $\phi''(1) = -1$, $\phi'''(1) = 2$, $(d^4\phi/ds^4)(1) = -6$, $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$. Substituting these coefficients into the formula (6.4.35), we obtain

$$\Gamma(x) \sim x^x e^{-x}\sqrt{\frac{2\pi}{x}}\left(1 + \frac{1}{12x}\right), \qquad x \to +\infty, \tag{6.4.40}$$

in agreement with (5.4.1).

The distinction between ordinary and movable maxima is examined in Probs. 6.45 to 6.47.

(I) 6.5 METHOD OF STATIONARY PHASE

There is an immediate generalization of the Laplace integrals studied in Sec. 6.4 which we obtain by allowing the function $\phi(t)$ in (6.4.1) to be complex. Note that, if we wish, we may assume that $f(t)$ is real; if it were complex, $f(t)$ could be decomposed into a sum of its real and imaginary parts. However, allowing $\phi(t)$ to be complex poses new and nontrivial problems. In this section we consider the special case in which $\phi(t)$ is pure imaginary: $\phi(t) = i\psi(t)$, where $\psi(t)$ is real. The resulting integral

$$I(x) = \int_a^b f(t)e^{ix\psi(t)}\, dt \tag{6.5.1}$$

with $f(t)$, $\psi(t)$, a, b, x all real is called a generalized Fourier integral. When $\psi(t) = t$, $I(x)$ is an ordinary Fourier integral. The general case in which $\phi(t)$ is complex is considered in Sec. 6.6.

To study the behavior of $I(x)$ in (6.5.1) as $x \to +\infty$, we can use integration by parts to develop an asymptotic expansion in inverse powers of x so long as the boundary terms are finite and the resulting integrals exist.

Example 1 *Asymptotic expansion of a Fourier integral as $x \to +\infty$.* We use integration by parts to find an asymptotic approximation to the Fourier integral

$$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\, dt.$$

After one integration by parts we obtain

$$I(x) = -\frac{i}{2x}e^{ix} + \frac{i}{x} - \frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt. \tag{6.5.2}$$

The integral on the right side of (6.5.2) is negligible compared with the boundary terms as $x \to +\infty$; in fact, it vanishes like $1/x^2$ as $x \to +\infty$. To see this, we integrate by parts again:

$$-\frac{i}{x}\int_0^1 \frac{e^{ixt}}{(1+t)^2}\, dt = -\frac{1}{4x^2}e^{ix} + \frac{1}{x^2} - \frac{2}{x^2}\int_0^1 \frac{e^{ixt}}{(1+t)^3}\, dt.$$

The integral on the right is bounded because

$$\left|\int_0^1 \frac{e^{ixt}}{(1+t)^3}\, dt\right| \le \int_0^1 (1+t)^{-3}\, dt = \frac{3}{8}.$$

Since the integral on the right in (6.5.2) does vanish like $1/x^2$ as $x \to +\infty$, $I(x)$ is asy
boundary terms: $I(x) \sim -(i/2x)e^{ix} + i/x$ $(x \to +\infty)$.

Repeated application of integration by parts gives the complete asymptotic $I(x)$ as $x \to +\infty$: $I(x) = e^{ix}u(x) + v(x)$ where

$$u(x) \sim -\frac{i}{2x} - \frac{1}{4x^2} + \cdots + \frac{(-i)^n(n-1)!}{(2x)^n} + \cdots, \qquad x \to +\infty,$$

$$v(x) \sim \frac{i}{x} + \frac{1}{x^2} + \cdots - \frac{(-i)^n(n-1)!}{x^n} + \cdots, \qquad x \to +\infty.$$

Example 2 *Integration by parts applied to $\int_0^1 \sqrt{t}\, e^{ixt}\, dt$.* Integration by parts can once for the Fourier integral $I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\, dt$. One integration by parts gives

$$I(x) = -\frac{i}{x}e^{ix} + \frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\, dt.$$

The integral on the right side of (6.5.3) vanishes more rapidly than the boun
$x \to +\infty$. We cannot use integration by parts to verify this because the resulting inte
exist. (Why?) However, we can use the following simple scaling argument. We l
obtain

$$\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\, dt = \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\, ds \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{is}}{\sqrt{s}}\, ds, \qquad x \to +\infty.$$

To evaluate the last integral we rotate the contour of integration from the real-
positive imaginary-s axis in the complex-s plane and obtain

$$\int_0^\infty \frac{e^{is}}{\sqrt{s}}\, ds = \sqrt{\pi}\, e^{i\pi/4}.$$

(See Prob. 6.49 for the details of this calculation.) Therefore,

$$I(x) + \frac{i}{x}e^{ix} \sim \frac{i}{2x^{3/2}}\sqrt{\pi}\, e^{i\pi/4}, \qquad x \to +\infty.$$

Clearly, this result cannot be found by direct integration by parts of the integral on
of (6.5.3) because a fractional power of x has appeared. However, it is possible to
asymptotic expansion of $I(x)$ as $x \to +\infty$ by an indirect application of integration
Prob. 6.50).

In Example 1 we used integration by parts to argue that the integ
right side of (6.5.2) vanishes more rapidly than the boundary terms as x -
Example 2 we used a scaling argument to show that the integral on the
of (6.5.3) vanishes more rapidly than the boundary terms as $x \to +\infty$. T
fact, a very general result called the Riemann-Lebesgue lemma that g
that

$$\int_a^b f(t)e^{ixt}\, dt \to 0, \qquad x \to +\infty;$$

provided that $\int_a^b |f(t)|\,dt$ exists. This result is valid even when $f(t)$ is not differentiable and integration by parts or scaling do not work. We will cite the Riemann-Lebesgue lemma repeatedly throughout this section; we could have used it to justify neglecting the integrals on the right sides of (6.5.2) and (6.5.3).

We reserve a proof of the Riemann-Lebesgue lemma for Prob. 6.51. Although the proof of (6.5.6) is messy, it is easy to understand the result heuristically. When x becomes large, the integrand $f(t)e^{ixt}$ oscillates rapidly and contributions from adjacent subintervals nearly cancel.

The Riemann-Lebesgue lemma can be extended to cover generalized Fourier integrals of the form (6.5.1). It states that $I(x) \to 0$ as $x \to +\infty$ so long as $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable for $a \le t \le b$, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$ (see Prob. 6.52). The lemma implies that $\int_0^{10} t^3 e^{ix \sin^2 t}\,dt \to 0$ $(x \to +\infty)$, but it does not apply to $\int_0^{10} t^3 e^{2ix}\,dt$.

Integration by parts gives the leading asymptotic behavior as $x \to +\infty$ of generalized Fourier integrals of the form (6.5.1), provided that $f(t)/\psi'(t)$ is smooth for $a \le t \le b$ and nonvanishing at one of the endpoints a or b. Explicitly,

$$I(x) = \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \Bigg|_{t=a}^{t=b} - \frac{1}{ix} \int_a^b \frac{d}{dt} \frac{f(t)}{\psi'(t)} e^{ix\psi(t)}\,dt.$$

The Riemann-Lebesgue lemma shows that the integral on the right vanishes more rapidly than $1/x$ as $x \to +\infty$. Therefore, $I(x)$ is asymptotic to the boundary terms (assuming that they do not vanish):

$$I(x) \sim \frac{f(t)}{ix\psi'(t)} e^{ix\psi(t)} \Bigg|_{t=a}^{t=b}, \qquad x \to +\infty. \tag{6.5.7}$$

Observe that when integration by parts applies, $I(x)$ vanishes like $1/x$ as $x \to +\infty$.

Integration by parts may not work if $\psi'(t) = 0$ for some t in the interval $a \le t \le b$. Such a point is called a *stationary* point of ψ. When there are stationary points in the interval $a \le t \le b$, $I(x)$ must still vanish as $x \to +\infty$ by the Riemann-Lebesgue lemma, but $I(x)$ usually vanishes less rapidly than $1/x$ because the integrand $f(t)e^{ix\psi(t)}$ oscillates less rapidly near a stationary point than it does near a point where $\psi'(t) \ne 0$. Consequently, there is less cancellation between adjacent subintervals near the stationary point.

The method of stationary phase gives the *leading* asymptotic behavior of generalized Fourier integrals having stationary points. This method is very similar to Laplace's method in that the leading contribution to $I(x)$ comes from a small interval of width ε surrounding the stationary points of $\psi(t)$. We will show that if c is a stationary point and if $f(c) \ne 0$, then $I(x)$ goes to zero like $x^{-1/2}$ as $x \to +\infty$ if $\psi''(c) \ne 0$, like $x^{-1/3}$ if $\psi''(c) = 0$ but $\psi'''(c) \ne 0$, and so on; as $\psi(t)$ becomes flatter at $t = c$, $I(x)$ vanishes less rapidly as $x \to +\infty$.

Since any generalized Fourier integral can be written as a sum of integrals in which $\psi'(t)$ vanishes only at an endpoint, we can explain the method of stationary phase for the special integral (6.5.1) in which $\psi'(a) = 0$ and $\psi'(t) \ne 0$ for $a < t \le b$.

We decompose $I(x)$ into two terms:

$$I(x) = \int_a^{a+\varepsilon} f(t)e^{ix\psi(t)}\,dt + \int_{a+\varepsilon}^b f(t)e^{ix\psi(t)}\,dt,$$

where ε is a small positive number to be chosen later. The second integra right side of (6.5.8) vanishes like $1/x$ as $x \to +\infty$ because there are no st points in the interval $a + \varepsilon \le t \le b$.

To obtain the leading behavior of the first integral on the right side o we replace $f(t)$ by $f(a)$ and $\psi(t)$ by $\psi(a) + \psi^{(p)}(a)(t-a)^p/p!$ where $\psi^{(p)}(a)$ $\psi'(a) = \cdots = \psi^{(p-1)}(a) = 0$:

$$I(x) \sim \int_a^{a+\varepsilon} f(a) \exp\left\{ix\left[\psi(a) + \frac{1}{p!}\psi^{(p)}(a)(t-a)^p\right]\right\} dt, \qquad x \to +\infty$$

Next, we replace ε by ∞, which introduces error terms that vanish lik $x \to +\infty$ and thus may be disregarded, and let $s = (t - a)$:

$$I(x) \sim f(a)e^{ix\psi(a)} \int_0^\infty \exp\left[\frac{ix}{p!}\psi^{(p)}(a)s^p\right] ds, \qquad x \to +\infty.$$

To evaluate the integral on the right, we rotate the contour of integration f real-s axis by an angle $\pi/2p$ if $\psi^{(p)}(a) > 0$ and make the substitution

$$s = e^{i\pi/2p}\left[\frac{p!\,u}{x\psi^{(p)}(a)}\right]^{1/p}$$

with u real or rotate the contour by an angle $-\pi/2p$ if $\psi^{(p)}(a) < 0$ and m substitution

$$s = e^{-i\pi/2p}\left[\frac{p!\,u}{x|\psi^{(p)}(a)|}\right]^{1/p}$$

Thus,

$$I(x) \sim f(a)e^{ix\psi(a) \pm i\pi/2p}\left[\frac{p!}{x|\psi^{(p)}(a)|}\right]^{1/p} \frac{\Gamma(1/p)}{p}, \qquad x \to +\infty,$$

where we use the factor $e^{i\pi/2p}$ if $\psi^{(p)}(a) > 0$ and the factor $e^{-i\pi/2p}$ if $\psi^{(p)}($

The formula in (6.5.12) gives the leading behavior of $I(x)$ if $f(a)$ $\psi'(a) = 0$. If $f(a)$ vanishes, it is necessary to decide whether the contributi the stationary point still dominates the leading behavior. When it does, th ior is slightly more complicated than (6.5.12) (see Prob. 6.53).

Example 3 *Leading behavior of* $\int_0^{\pi/2} e^{ix \cos t}\,dt$ *as* $x \to +\infty$. The function $\psi(t) = \cos t$ tionary point at $t = 0$. Since $\psi''(0) = -1$, (6.5.12) with $p = 2$ gives $I(x) \sim \sqrt{\pi/2}$ $(x \to +\infty)$.

Example 4 *Leading behavior of* $\int_0^\infty \cos(xt^2 - t)\,dt$ *as* $x \to +\infty$. To use the method of stationary phase, we write this integral as $\int_0^\infty \cos(xt^2 - t)\,dt = \text{Re}\int_0^\infty e^{i(xt^2-t)}\,dt$. The function $\psi(t) = t^2$ has a stationary point at $t = 0$. Since $\psi''(0) = 2$, (6.5.12) with $p = 2$ gives $\int_0^\infty \cos(xt^2 - t)\,dt \sim \text{Re}\,\frac{1}{2}\sqrt{\pi/x}\,e^{i\pi/4} = \frac{1}{2}\sqrt{\pi/2x}$ $(x \to +\infty)$.

Example 5 *Leading behavior of* $J_n(n)$ *as* $n \to \infty$. When n is an integer, the Bessel function $J_n(x)$ has the integral representation

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt \tag{6.5.13}$$

(see Prob. 6.54). Therefore, $J_n(n) = \text{Re}\int_0^\pi e^{in(\sin t - t)}\,dt/\pi$. The function $\psi(t) = \sin t - t$ has a stationary point at $t = 0$. Since $\psi''(0) = 0$, $\psi'''(0) = -1$, (6.5.12) with $p = 3$ gives

$$\begin{aligned} J_n(n) &\sim \frac{1}{\pi}\text{Re}\left[\frac{1}{3}e^{-i\pi/6}\left(\frac{6}{n}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)\right], \qquad x \to +\infty, \\ &= \frac{1}{\pi}2^{-2/3}3^{-1/6}\Gamma\left(\frac{1}{3}\right)n^{-1/3}, \qquad n \to \infty. \end{aligned} \tag{6.5.14}$$

Observe that because $\psi''(0) = 0$, $J_n(n)$ vanishes less rapidly than $n^{-1/2}$ as $n \to \infty$.

If n is not an integer, (6.5.14) still holds (see Prob. 6.55).

In this section we have obtained only the leading behavior of generalized Fourier integrals. Higher-order approximations can be complicated because nonstationary points may also contribute to the large-x behavior of the integral. Specifically, the second integral on the right in (6.5.8) must be taken into account when computing higher-order terms because the error incurred in neglecting this integral is usually algebraically small. By contrast, recall that the approximation in (6.4.2) for Laplace's method is valid to all orders because the errors are exponentially, rather than algebraically, small. To obtain the higher-order corrections to (6.5.12), one can either use the method of asymptotic matching (see Sec. 7.4) or the method of steepest descents (see Sec. 6.6).

(I) 6.6 METHOD OF STEEPEST DESCENTS

The method of steepest descents is a technique for finding the asymptotic behavior of integrals of the form

$$I(x) = \int_C h(t)e^{x\rho(t)}\,dt \tag{6.6.1}$$

as $x \to +\infty$, where C is an integration contour in the complex-t plane and $h(t)$ and $\rho(t)$ are analytic functions of t. The idea of the method is to use the analyticity of the integrand to justify deforming the contour C to a new contour C' on which $\rho(t)$ has a constant imaginary part. Once this has been done, $I(x)$ may be evaluated asymptotically as $x \to +\infty$ using Laplace's method. To see why, observe that on the contour C' we may write $\rho(t) = \phi(t) + i\psi$, where ψ is a real constant and $\phi(t)$ is a real function. Thus, $I(x)$ in (6.6.1) takes the form

$$I(x) = e^{ix\psi}\int_{C'} h(t)e^{x\phi(t)}\,dt. \tag{6.6.2}$$

Although t is complex, (6.6.2) can be treated by Laplace's method because $\phi(t)$ is real.

Our motivation for deforming C into a path C' on which Im $\rho(t)$ is to eliminate rapid oscillations of the integrand when x is large. Ol could also deform C into a path on which Re $\rho(t)$ is a constant and th method of stationary phase. However, we have seen that Laplace's much better approximation scheme than the method of stationary ph the full asymptotic expansion of a generalized Laplace integral is de the integrand in an arbitrarily small neighborhood of the point wher maximum on the contour. By contrast, the full asymptotic expansio alized Fourier integral typically depends on the behavior of the inte the entire contour. As a consequence, it is usually easier to obtain the totic expansion of a generalized Laplace integral than of a generali integral.

Before giving a formal exposition of the method of steepest consider three preliminary examples which illustrate how shifting c tours can greatly simplify asymptotic analysis. In the first example w Fourier integral whose asymptotic expansion is difficult to find by used in Sec. 6.5. However, deforming the contour reduces the integra integrals that are easy to evaluate by Laplace's method.

Example 1 *Conversion of a Fourier integral into a Laplace integral by deforming th* behavior of the integral

$$I(x) = \int_0^1 \ln t\, e^{ixt}\,dt$$

as $x \to +\infty$ cannot be found directly by the methods of Sec. 6.5 because there i point. Also, integration by parts is useless because $\ln 0 = -\infty$. Integration by par fail because, as we will see, the leading asymptotic behavior of $I(x)$ contains the fa is not a power of $1/x$.

To approximate $I(x)$ we deform the integration contour C, which runs from real-t axis, to one which consists of three line segments: C_1, which runs up the i from 0 to iT; C_2, which runs parallel to the real-t axis from iT to $1 + iT$; and down from $1 + iT$ to 1 along a straight line parallel to the imaginary-t axis (s Cauchy's theorem, $I(x) = \int_{C_1+C_2+C_3} \ln t\, e^{ixt}\,dt$. Next we let $T \to +\infty$. In this lim tion from C_2 approaches 0. (Why?) In the integral along C_1 we set $t = is$, and in th C_3 we set $t = 1 + is$, where s is real in both integrals. This gives

$$I(x) = i\int_0^\infty \ln(is)\,e^{-xs}\,ds - i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds.$$

The sign of the second integral on the right is negative because C_3 is traversed

Observe that both integrals in (6.6.4) are Laplace integrals. The first integr exactly. We substitute $u = xs$ and use $\ln(is) = \ln s + i\pi/2$ and the identity $\int_0^\infty e^{-}$ where $\gamma = 0.5772\ldots$ is Euler's constant, and obtain

$$i\int_0^\infty \ln(is)\,e^{-xs}\,ds = -i(\ln x)/x - (i\gamma + \pi/2)/x.$$

We apply Watson's lemma to the second integral on the right in (6.6.4) using the Ta $\ln(1 + is) = -\sum_{n=1}^\infty (-is)^n/n$, and obtain

$$-i\int_0^\infty \ln(1 + is)\,e^{ix(1+is)}\,ds \sim ie^{ix}\sum_{n=1}^\infty \frac{(-i)^n(n-1)!}{x^{n+1}}, \qquad x \to +\infty.$$

(I) **6.33** Show that the last integral on the right side of (6.3.23) vanishes like $x^{-3/2}$ as $x \to +\infty$.

(D) **6.34** Calculate two terms in the asymptotic expansion of $\int_0^\infty e^{-t-x/t^2}\,dt$ and $\int_0^\infty e^{-xt}(1+t)^{-1/2}\,dt$ as $x \to 0+$ and as $x \to +\infty$.

(I) **6.35** Find the leading behavior of $\int_0^\infty e^{-t-x/t^\alpha}\,dt$ for $\alpha > 0$ as $x \to 0+$ and as $x \to +\infty$.

(I) **6.36** (*a*) Verify that the integral representation in (6.4.20) satisfies the differential equation for $K_\nu(x)$.
(*b*) Use the integral representation (6.4.20) to show that $K_\nu(x) \sim \sqrt{\pi/2\nu}\,(2\nu/ex)^\nu$ $(\nu \to +\infty)$.

(I) **6.37** (*a*) Show that

$$\int_0^\infty \frac{t^{x-1}e^{-t}}{t+x}\,dt \sim \frac{1}{2x}\Gamma(x), \qquad x \to +\infty.$$

(*b*) Find the leading behavior as $x \to +\infty$ of

$$\frac{\int_0^\infty [t^{x-1}/(x+t)]\,e^{-t^x}\,dt}{\int_0^\infty t^{x-1}e^{-t^x}\,dt}.$$

(I) **6.38** Solve Prob. 3.77 using Watson's lemma.

(I) **6.39** The logarithmic integral function li (x) is defined as li $(x) = P\int_0^x dt/\ln t$, where P indicates that the Cauchy principal part of the integral is taken when $x > 1$. Show that li $(e^a) \sim e^a \sum_{n=0}^\infty n!/a^{n+1}$ $(a \to +\infty)$.

(I) **6.40** Prove that

$$\sum_{n=0}^{2N+1} (-1)^n \frac{t^{2n+1}}{(2n+1)!} < \sin t < \sum_{n=0}^{2N} (-1)^n \frac{t^{2n+1}}{(2n+1)!}$$

for all $t > 0$ and all integers N.

Clue: Prove that $\sin t < t$ by integrating $\cos t < 1$. In the same way, use repeated integration to establish the general result.

(I) **6.41** Show that (6.4.27) is an integral representation of the modified Bessel function $I_n(x)$. In other words, show that the integral satisfies the differential equation $x^2y'' + xy' - (x^2+\nu^2)y = 0$ and the relation $I_n(x) \sim (x/2)^n/n!$ $(x \to 0+)$.

(I) **6.42** Use Laplace's method for a movable maximum to find the next correction to (6.4.40). In particular, show that

$$\Gamma(x) \sim x^{x-1/2}e^{-x}\sqrt{2\pi}\left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right), \qquad x \to +\infty.$$

(I) **6.43** (*a*) Show that Laplace's method for expanding integrals consists of approximating the integrand by a δ function. In particular, show how the representation $\delta(t) = \lim_{x\to\infty}\sqrt{x/\pi}\,e^{-xt^2}$ reproduces the leading behavior of a Laplace integral for which $\phi'(c) = 0$ but $\phi''(c) < 0$. [See (6.4.19*c*) and (1.5.10*c*).]

(*b*) What is the appropriate δ-function representation for the case in which $\phi(t) < \phi(a)$ for $a < t < b$ and $\phi'(a) < 0$? [See (6.4.19*a*).]

(*c*) What is the appropriate δ-function representation for the case in which $\phi'(c) = \phi''(c) = \cdots = \phi^{(p-1)}(c) = 0$, $\phi^{(p)}(c) < 0$ with p even? [See (6.4.19*d*).]

(D) (*d*) Extend the δ-function analysis of parts (*a*) to (*c*) to give the higher-order corrections to the leading behavior.

Clue: The answer is given in (6.4.35).

(D) **6.44** Find the leading behavior of the double integral $\int_0^\infty ds \int_0^\pi dt\, e^{-x[\nu s + (\cos t)(\sin s)]}$ as $x \to +\infty$ for $0 < \nu < 1$, $\nu = 1$, and $\nu > 1$. Sketch the function for large x.

Clue: Show that when $0 < \nu < 1$, the exponent has four stationary points. As $\nu \to 1-$, these stationary points merge into two. When $\nu > 1$, there are no stationary points.

(I) **6.45** What happens if we try to treat an ordinary Laplace integral $\int_a^b f(t)e^{x\phi(t)}\,dt$ using the methods appropriate for a moving maximum? Suppose we rewrite the integral as $\int_a^b e^{x\phi(t) + \ln f(t)}\,dt$ and expand about the maximum of the integrand. Show that now an interior maximum is shifted slightly fr $t = c$ where $\phi'(c) = 0$, but that this does not affect the result given by Laplace's method in (6.4.19*c* tc

6.46 Show that naive application of Laplace's method for a moving maximum to the inte $I(x,\alpha) = \int_0^\infty t^\alpha e^{-xt}\,dt = x^{-\alpha-1}\Gamma(\alpha+1)$ gives the wrong answer! Show that the maximum of the i grand occurs at $t = \alpha/x$ and retaining only quadratic terms gives

$$I(x,\alpha) \sim e^{-\alpha}\alpha^{\alpha+1/2}x^{-\alpha-1}\sqrt{2}\int_{-\sqrt{\alpha}/2}^\infty e^{-u^2}\,du, \qquad x \to +\infty.$$

Explain why we have obtained the wrong answer.

6.47 (*a*) Show that

$$\int_0^{1/e} \frac{e^{-xt}}{\ln t}\,dt \sim -\frac{1}{x\ln x}, \qquad x \to +\infty.$$

Clue: See Example 3 of Sec. 6.6.

(*b*) Show that

$$\int_0^{1/e} \frac{e^{-xt}}{\ln t}\,dt \sim -\frac{1}{x\ln x}\sum_{n=0}^\infty (\ln x)^{-n}\int_0^\infty (\ln s)^n e^{-s}\,ds, \qquad x \to +\infty.$$

(*c*) Explain why naive use of Laplace's method for a moving maximum fails to give the results and (*b*) above.

6.48 Find the leading behaviors of

(*a*) $\int_0^\infty e^{-xt}e^{-a(\ln t)^2}t^{-1}\,dt$ as $x \to \infty$;

(*b*) $(d^n/dx^n)\Gamma(x)|_{x=1}$ as $n \to \infty$.

Section 6.5

6.49 Show that $\int_0^\infty e^{is}s^{\alpha-1}\,ds = e^{i\pi\alpha/2}\Gamma(\alpha)$ for $0 < \text{Re}\,\alpha < 1$.

Clue: Substitute $s = it$ and rotate the contour of integration from the negative imaginary-t axi the positive real-t axis.

6.50 Use integration by parts to show that the full asymptotic expansion $I(x)$ in (6.5.3) is

$$I(x) \sim \frac{i\sqrt{\pi}}{2x^{3/2}}e^{i\pi/4} - \frac{i}{x\sqrt{\pi}}e^{ix}\left[1 + \frac{1}{2}\sum_{n=0}^\infty \left(\frac{-i}{x}\right)^{n+1}\Gamma\left(n+\frac{1}{2}\right)\right], \qquad x \to +\infty.$$

Clue: Write

$$\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \int_0^\infty \frac{e^{ixt}}{\sqrt{t}}\,dt - \int_1^\infty \frac{e^{ixt}}{\sqrt{t}}\,dt$$

and use integration by parts on the second integral on the right.

6.51 Prove the Riemann-Lebesgue lemma by showing that $\int_a^b f(t)e^{ixt}\,dt \to 0$ $(x \to +\infty)$ provided t $|f(t)|$ is integrable.

Clue: Break up the region of integration into small subintervals and bound the integral on e subinterval.

6.52 Show that $\int_a^b f(t)e^{ix\psi(t)}\,dt \to 0$ $(x \to +\infty)$ provided that $|f(t)|$ is integrable, $\psi(t)$ is continuo differentiable, and $\psi(t)$ is not constant on any subinterval of $a \le t \le b$.

Clue: Use the Riemann-Lebesgue lemma.

6.53 Find the leading behavior of $\int_a^b f(t)e^{ix\psi(t)}\,dt$ under the following assumptions: $\psi'(a) = \cdots$ $\psi^{(p-1)}(a) = 0$; $\psi^{(p)}(a) \ne 0$; $f(t) \sim A(t-a)^\alpha$ $(t \to a+)$ with $\alpha > -1$.

(*a*) What is the leading contribution to the behavior of $I(x)$ from the neighborhood of stationary point at $t = a$? This result is a generalization of the formula in (6.5.12).