ΕΡΓΑΣΙΑ 1

Διαφορικές Εξισώσεις

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i) $y'=y^2$, $y(0)=1$

$$y'=y^2 \Rightarrow$$

$$\frac{dy}{dt}=y^2 \Rightarrow$$

$y(t)=0$ προφανής λύση

$$\frac{dy}{y^2}=dt \Rightarrow$$

όμως $y(0)=1$ άρα δεν μπορεί να ισχύει

$$\int\frac{dy}{y^2}=\int dt + c \Rightarrow$$

$$-\frac{1}{y}=t+c \Rightarrow$$

$$\frac{1}{y}=-t-c \Rightarrow$$

$$y\cdot(-t-c)=1 \Rightarrow$$

$$y=-\frac{1}{t+c} \quad , t\neq -c$$

Γενική λύση: $y(t)=0$, $y(t)=-\frac{1}{t+c}$

Για $y(0)=1$ ισχύει $y(0)=-\frac{1}{0+c}=1 \Rightarrow c=-1$

Άρα $y(t)=-\frac{1}{t-1}$, $t\neq 1$

ii) $y'=y^2$, $y(1)=0$

$$y'=y^2 \Rightarrow$$

$$\frac{dy}{dt}=y^2 \Rightarrow$$

$y(t)=0$ προφανής λύση

$$\frac{dy}{y^2}=dt \Rightarrow$$

ικανοποιεί την αρχική συνθήκη ($y(1)=0$) άρα είναι λύση

$$\int\frac{dy}{y^2}=\int dt + c \Rightarrow$$

$$-\frac{1}{y}=t+c \Rightarrow$$

$$\frac{1}{y} = -t - c$$

$$y \cdot (-t-c) = 1$$

$$y = -\frac{1}{t+c}$$

Γενική λύση $\begin{cases} y(t) = 0 \\ y(t) = -\dfrac{1}{t+c}, \quad t \neq -c \end{cases}$

Για $y(1)=0$ ισχύει $y(1) = -\frac{1}{1+c} = 0 \Rightarrow -1 = 0$ Αδύνατο

Άρα $y(t)=0$

iii) $y' = -y^2$

$y(1) = a$

$y' = -y^2 \Rightarrow \frac{dy}{dt} = -y^2 \Rightarrow -\frac{dy}{y^2} = dt \Rightarrow$

$-\int \frac{dy}{y^2} = \int dt + c \Rightarrow$

$y(t)=0$

προφανής λύση

αν $a=0$

$\frac{1}{y} = t + c \Rightarrow y \cdot (t+c) = 1 \Rightarrow y = \frac{1}{t+c}$

$t \neq -c$

Για $y(1) = a$ ισχύει $y(1) = a = \frac{1}{1+c} \Rightarrow a + ac = 1 \Rightarrow$

$ac = 1 - a \Rightarrow c = \frac{1-a}{a} = \frac{1}{a} - 1$

Άρα $y(t) = \frac{1}{t + \frac{1}{a} - 1}$, $t \neq 1 - \frac{1}{a}$

iv) $y' = t^2 (y-1)$

$y(0) = 2$

$y' = t^2 (y-1)$

Η $y(t) = 1$ δεν είναι λύση της δ.ε αφού δεν ικανοποιείται με αυτή η συνθήκη $y(0) = 2 \ (\neq 1)$

$\frac{y'}{y-1} = t^2 \Rightarrow (\ln|y-1|)' = \left(\frac{t^3}{3}\right)' \Rightarrow \ln|y-1| = \frac{t^3}{3} + c$

$\Rightarrow |y-1| = e^{\frac{t^3}{3} + c} = e^{\frac{t^3}{3}} \cdot e^{c} = c_1 \cdot e^{\frac{t^3}{3}}$

Αν $y>1$ $\quad y-1 = c_1 \cdot e^{t^3/3} \Rightarrow y = 1 + c_1 \cdot e^{t^3/3}$

Για $y(0)=2$ ισχύει $\quad y(0) = 2 = 1 + c_1 \cdot e^0 \Rightarrow c_1 = 1$

Άρα $\quad y = 1 + e^{t^3/3}$

Αν $y<1$ $\quad -y+1 = c_1 \cdot e^{t^3/3} \Rightarrow y = 1 - c_1 \cdot e^{t^3/3}$

Για $y(0)=2$ ισχύει $\quad y(0) = 2 = 1 - c_1 \cdot e^0 \Rightarrow c_1 = -1$

Άρα $\quad y = 1 + e^{t^3/3}$

Τελικά $\quad y(t) = 1 + e^{t^3/3}$

v) $y' + y = \int_0^2 y(t)\,dt$ $\qquad \cancel{y' + y = \int_0^2 y(t)\,dt}$

$y(0)=1$

Θέτω $\quad c_1 = \int_0^2 y(t)\,dt$

Άρα $\quad y' + y = c_1 \quad , \mu(t) = e^{\int p(t)\,dt} = e^{\int dt}\, e^t$

$e^t y' + e^t y = e^t c_1$

$(e^t y)' = c_1 \cdot e^t$

$e^t y = c_1 \cdot e^t + c_2$

$\frac{e^t y}{e^t} = \frac{c_1 \cdot e^t}{e^t} + \frac{c_2}{e^t}$

$y = c_1 + \frac{c_2}{e^t}$

Για $y(0)=1$ ισχύει $\quad y(0) = 1 = c_1 + \frac{c_2}{e^0} \Rightarrow c_1 + c_2 = 1 \quad (1)$

$c_1 = \int_0^2 y(t)\,dt = \int_0^2 (c_1 + c_2 \cdot e^{-t})\,dt \Rightarrow c_1 = \int_0^2 c_1 dt + \int_0^2 c_2 \cdot e^{-t} dt \Rightarrow$

$\Rightarrow c_1 = [c_1 t]_0^2 - [c_2 \cdot e^{-t}]_0^2 \Rightarrow c_1 = -c_2 \cdot e^{-2} + c_2 \cdot e^0 \Rightarrow \cancel{c_2 = -c_2 \cdot e^{-2}}$

$\Rightarrow c_2 \cdot e^{-2} = c_2 + c_1 \xRightarrow{(1)} c_2 \cdot e^{-2} = 1 \Rightarrow c_2 = \frac{1}{e^{-2}} = e^2$

Από (1) ~~προκύπτει~~ προκύπτει $c_1 = 1 - c_2 = 1 - e^2$

Άρα $y(t) = 1 - e^2 + e^{2-t}$

vi) $y' + \frac{1}{t} y = 0$, $t > 0$, $y(1) = 2$

$\mu(t) = e^{\int p(t) dt} = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$t \cdot y' + \frac{1}{t} \cdot t \cdot y = 0 \cdot t$

$y' \cdot t + y = 0$

$(yt)' = 0$

$yt = c$

$y = \frac{c}{t}$

Για $y(1) = 2 \Rightarrow 2 = \frac{c}{1} \Rightarrow c = 2$

Άρα $y = \frac{2}{t}$

② $ty' + 6y = 3t y^{4/3} \Rightarrow \frac{ty'}{t} + \frac{6y}{t} = \frac{3ty^{4/3}}{t} \Rightarrow y' + \frac{6}{t} y = 3 y^{4/3}$ (1)

Η εξίσωση είναι Bernoulli με $r = \frac{4}{3}$

$u = y^{1-r} = y^{1-4/3} = y^{-\frac{1}{3}}$

$u' = -\frac{1}{3} y^{-\frac{1}{3}-1} \cdot y' = -\frac{1}{3} y^{-4/3} \cdot y'$

Πολλαπλασιάζω την (1) με το $-\frac{1}{3} y^{-4/3}$:

$-\frac{1}{3} y^{-4/3} \cdot y' - \frac{1}{3} y^{-4/3} \cdot \frac{6}{t} y = -\cancel{3} y^{4/3} \cdot \frac{1}{\cancel{3}} \cdot y^{-\frac{4}{3}} \Rightarrow$

$-\frac{1}{3} y^{-4/3} y' - \frac{2}{t} \cdot y^{-1/3} = -1 \Rightarrow u' - \frac{2}{t} u = -1$

$p(t) = -\frac{2}{t}$ $\quad \mu(t) = e^{-\int \frac{2}{t}} = e^{-2\ln t} = t^{-2}$

$t^{-2}u' - \frac{2}{t}u \cdot t^{-2} = -1 \cdot t^{-2} \Rightarrow t^{-2}u' - 2 \cdot t^{-1} u \cdot t^{-2} = -t^{-2} \Rightarrow$

$t^{-2}u' - 2t^{-3}u = -t^{-2} \Rightarrow$

$(t^{-2}u)' = -t^{-2} \Rightarrow$

$t^{-2}u = \int -t^{-2} + c \Rightarrow t^{-2}u = t^{-1} + c \Rightarrow \frac{u}{t^2} = \frac{1}{t} + c \Rightarrow$

$\Rightarrow u = \frac{t^2}{t} + ct^2 \Rightarrow u = t + ct^2 \; (1)$

$u = y^{-\frac{1}{3}} \Rightarrow u = \frac{1}{y^{1/3}} \quad \overset{(1)}{\Rightarrow} : ct^2 + t = \frac{1}{y^{1/3}} \Rightarrow$

$y^{1/3} = \frac{1}{ct^2 + t} \Rightarrow y = \sqrt[3]{\frac{1}{ct^2+t}}$

(3) $y' = 1 + t^2 - 2ty + y^2 \qquad y = y_1 + \frac{1}{u} \qquad y^2 = y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}$

$y' + 2ty = y^2 + t^2 + 1 \qquad y' = y_1' - \frac{1}{u^2} \cdot u'$

$y' + p(t)y = q(t)y^2 + f(t)$

$y_1' - \frac{1}{u^2} + p(t) \cdot \left(y_1 + \frac{1}{u}\right) = q(t) \cdot \left(y_1^2 + \frac{2y_1}{u} + \frac{1}{u^2}\right) + f(t)$

$y_1' - \frac{u'}{u^2} + p(t)y_1 + \frac{p(t)}{u} = q(t) \cdot y_1^2 + \frac{2q(t)y_1}{u} + \frac{q(t)}{u^2} + f(t)$

$-\frac{u'}{u^2} + \frac{p(t)}{u} = 2q(t)\frac{y_1}{u} + \frac{q(t)}{u^2}$

Πολλαπλασιάζω με u^2 $\Big\{ -u' + up(t) = 2q(t)y_1 \cdot u + q(t) \Rightarrow$

$-u' + up(t) - 2q(t)y_1 u = q(t) \Rightarrow$

$-u' + (p(t) - 2q(t)y_1)\, u = q(t) \Rightarrow$

$u' + (2q(t)y_1 - p(t)) \cdot u = -q(t) \Rightarrow$

$u' + (2y_1 - 2t) \cdot u = -1$

$$\mu(t) = e^{\int(2y_1-2t)dt}$$

~~Άρα~~ $$u' \cdot e^{\int(2y_1-2t)dt} + (2y_1-2t) \cdot u \cdot e^{\int(2y_1-2t)dt} = -1e^{\int(2y_1-2t)dt}$$

$$\left(u \cdot e^{\int(2y_1-2t)dt}\right)' = -e^{\int(2y_1-2t)dt}$$

$$u \cdot e^{\int(2y_1-2t)dt} = \int -e^{\int(2y_1-2t)dt} + c \Rightarrow$$

$$u = \frac{\int -e^{\int(2y_1-2t)dt} + c}{e^{\int(2y_1-2t)dt}}$$

$$y(t) = y_1 + \frac{1}{u} = y_1 + \frac{1}{\frac{\int -e^{\int(2y_1-2t)dt} + c}{e^{\int(2y_1-2t)dt}}} = y_1 + \frac{e^{\int(2y_1-2t)dt}}{\int -e^{\int(2y_1-2t)dt} + c}$$

(4) $t \cdot y' - y = t^2$, $t > 0$

Διαιρώ με t: $\frac{ty'}{t} - \frac{y}{t} = \frac{t^2}{t} \Rightarrow$

$$y' - \frac{y}{t} = t$$

$$\mu(t) = e^{\int p(t)dt} = e^{-\int \frac{1}{t}dt} = e^{-\ln t} = t^{-1} = \frac{1}{t}$$

$$y' \cdot \frac{1}{t} - \frac{1}{t} \cdot \frac{y}{t} = t \cdot \frac{1}{t} \Rightarrow$$

$$y' \cdot \frac{1}{t} - \frac{y}{t^2} = 1 \Rightarrow$$

$$\left(y \cdot \frac{1}{t}\right)' = 1 \Rightarrow$$

$$y \cdot \frac{1}{t} = t + c \Rightarrow$$

$$y = t^2 + ct$$