Łojasiewicz Θεώρημα

$(*)\quad x' = -\nabla E(x)$, $E:\mathbb{R}^n \to \mathbb{R}$ πραγμ. αναλυτική

$x(0) = x_0$

Θεώρημα

Έστω $x(t)$ φραγμένη λύση, $\sup_{t \geq 0} |x(t)| < +\infty$

Τότε ισχύει

$$(1)\qquad \lim_{t \to +\infty} x(t) = \xi, \quad \nabla E(\xi) = 0.$$

Σχόλιο: Το βασικό εργαλείο είναι η Ανισότητα Łojasiewicz:

Έστω $\nabla E(x^*) = 0$. $\Longrightarrow \exists\ \gamma > 0$, $\theta \in [\frac{1}{2}, 1)$, και $C > 0$

τ.ω.

$$(2)\qquad |E(x) - E(x^*)|^{1-\theta} \leq C\,|\nabla E(x)|, \quad x \text{ σε περιοχή } \omega \text{ του } x^*$$

Παρατηρήσεις - Σχόλια

1) Η (1) ισχύει για $n=1$, χωρίς υπόθεση αναλυτικότητας

$\longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot$ (μονοτονικότητα)

2) Η (1) δεν ισχύει για C^∞ (αλλά όχι C^ω) συναρτήσεις

(Αντιπαράδειγμα Palis - De Melo σ.13)

3) Προφανές πόρισμα της (2) ότι αν $\xi \in \boxed{Eq := \{\nabla E = 0\}}$

$\xi \in \omega$, τότε $E(\xi) = E(x^*)$, δηλ. ξ, x^* στο ίδιο σύνολο στάθμης (που δεν ισχύει για C^∞ συναρτήσεις)

4) Έστω f αναλυτική μιας μεταβλητής, $f(z)$

$f(z) = a_m z^m + a_{m+1} z^{m+1} + \dots \qquad a_m \neq 0$

$f'(z) = m a_m z^{m-1} + \dots$

$x^* = 0, \qquad |f(z)|^{\frac{m-1}{m}} \leq C\,|f'(z)|$

Απόδειξη της (2): Πολλές μεταβλητές.

$z(t) := E(x(t))$

(6), (5) $\Rightarrow$

$$(7) \quad z'(t) \leq -C\,(z(t))^{2\alpha}$$

$$\Rightarrow \quad z(t) \leq K_1 t^{-\frac{1}{2\alpha-1}} \qquad (7\tfrac{1}{2})$$

$$(8) \quad z'(t) = \frac{d}{dt} E(x(t)) = -|\nabla E(x(t))|^2 = -|x'(t)|^2$$

$\Rightarrow$

$$(9) \quad \int_t^{2t} |x'(s)|^2 ds = z(t) - z(2t) \overset{(7\frac{1}{2})}{\leq} K_1 t^{-\beta}, \quad \beta = \frac{1}{2\alpha-1}$$

CS $\Rightarrow$

$$\int_t^{2t} |x'(s)|\, ds \leq \left(\int_t^{2t} |x'(s)|^2 ds\right)^{1/2} \left(\int_t^{2t} ds\right)^{1/2}$$

$$\leq C\, t^{-\beta/2}\, t^{1/2} = C\, t^{\frac{1-\beta}{2}},$$

$$\frac{1-\beta}{2} = -\gamma = \frac{\alpha-1}{2\alpha-1}.$$

$$\therefore \quad (10) \quad \int_t^{2t} |x'(s)|\, ds \leq C t^{-\gamma}$$

5) Αρχη Αναλλοιωτω LaSalle $\Rightarrow$ $\omega(x_0) \subset Eq$,

E σταθερα στο $\omega(x_0)$.

6) Θα δειξουμε περαιτερω τις εκτιμησεις

$$\alpha := 1-\theta$$

(3) $$|x(t)-\xi| \leq M t^{-\gamma} \quad , \gamma = \frac{1-\alpha}{2\alpha-1}$$

Αν $\theta = \frac{1}{2}$ τοτε ισχυει

(4) $$|x(t)-\xi| \leq M e^{-\delta t}, \text{ καποιο } \delta > 0$$

Αποδειξη

Χωρις βλαβη γενικοτητος

$E = 0$ στο $\omega(x_0)$

Επισης $\operatorname{dist}(x(t), \omega(x_0)) \longrightarrow 0, \quad t \to +\infty$.

συμπαγες

$\therefore$ για t μεγαλο $x(t)$ σε περιοχη του $\omega(x_0)$ οπου ισχυει η ανισοτητα (2)

(5) $$|E(x(t))|^{\alpha} \leq C\,|\nabla E(x(t))|$$

$(*) \Rightarrow$

(6) $$\frac{d}{dt} E(x(t)) = -|\nabla E(x(t))|^2$$

(αι) $\int_t^{\infty} |x'(s)|ds \leq \sum_{k\geq 1}^{\infty} \int_{2^{k-1}t}^{2^k t} |x'(s)|ds \leq C \sum (2^k t)^{-\gamma}$
$\leq C_*^{*} t^{-\gamma}$.

$\{x(t) \,;\, t \geq 0\}$ Cauchy

$\lim x(t)$ $\exists$.

$\left(\int^{\infty} |x'(s)|ds < +\infty \Rightarrow \lim_{t\to+\infty} x(t) \text{ υπάρχει} \right)$.

$\square$

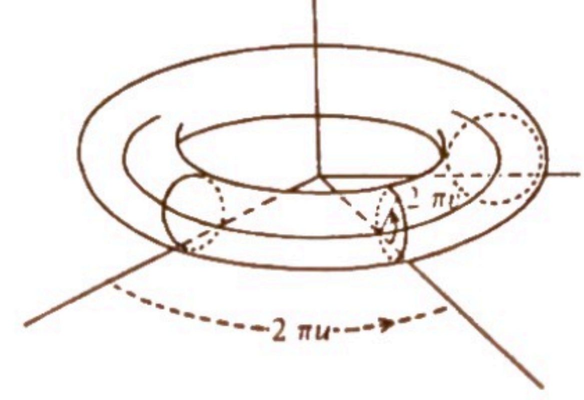

Figure 6

$\varphi(\Delta_{\bar{c}}) = \varphi(\bigcup_{c \in C} \Delta_c)$ is dense in T^2. To show that C is dense in $\mathbb{R}$ it is enough to prove that $G = \{m\alpha + n; m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$, because $c \in C$ if and only if $c - \bar{c} \in G$. As G is a subgroup of the additive group $\mathbb{R}$ we know that G is either dense or discrete. It remains, therefore, to show that G is not discrete. But for each $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $u_m = m\alpha + n$ belongs to the interval $[0, 1]$. The sequence u_m has a cluster point and, as α is irrational, its terms are distinct. Thus G is dense.

The vector field Y^α above is called the *rational* or *irrational field* on T^2 according as to whether α is rational or not. If α is rational the ω-limit of any orbit is itself. If α is irrational, the ω-limit of any orbit is the whole torus T^2.

EXAMPLE 3 (Gradient Vector Fields). Consider a manifold $M^m \subset \mathbb{R}^k$. At each point $p \in M$ we take in TM_p the inner product $\langle\ ,\ \rangle_p$ induced by $\mathbb{R}^k$. We denote the norm induced by this inner product by $\|\ \|_p$ or, simply, by $\|\ \|$. If X and Y are C^∞ vector fields on M then the function $g: M \to \mathbb{R}$, $g(p) = \langle X(p), Y(p)\rangle_p$ is of class C^∞. Let $f: M \to \mathbb{R}$ be a C^{r+1} map. For each $p \in M$ there exists a unique vector $X(p) \in TM_p$ such that $df_p v = \langle X(p), v\rangle_p$ for all $v \in TM_p$. This defines a vector field X which is of class C^r. It is called the *gradient* of f and written as $X = \operatorname{grad} f$. We shall now indicate some basic properties of gradient fields. Firstly, $\operatorname{grad} f(p) = 0$ if and only if $df_p = 0$. Along nonsingular orbits of $X = \operatorname{grad} f$ we have f strictly increasing because $df_p X(p) = \|X(p)\|^2$. In particular $\operatorname{grad} f$ does not have closed orbits. Moreover, the ω-limit of any orbit consists of singularities. For let us suppose that $X(q) \neq 0$ and $q \in \omega(p)$ for some $p \in M$. Let S be the intersection of $f^{-1}(f(q))$ with a small neighbourhood of q. We see that S is a submanifold of dimension $m - 1$ orthogonal to $X = \operatorname{grad} f$ and, by the continuity of the flow, the orbit through any point near q intersects S. As $q \in \omega(p)$ there exists a sequence p_n in the orbit of p converging to q. Thus the orbit of p intersects S in more than one point (in fact, in infinitely many points) which is absurd since f is increasing along orbits. On the other hand, it is clear that, if the ω-limit of an orbit of a gradient vector field contains more than one singularity, it must contain infinitely many. We are going to show that this can in fact occur.

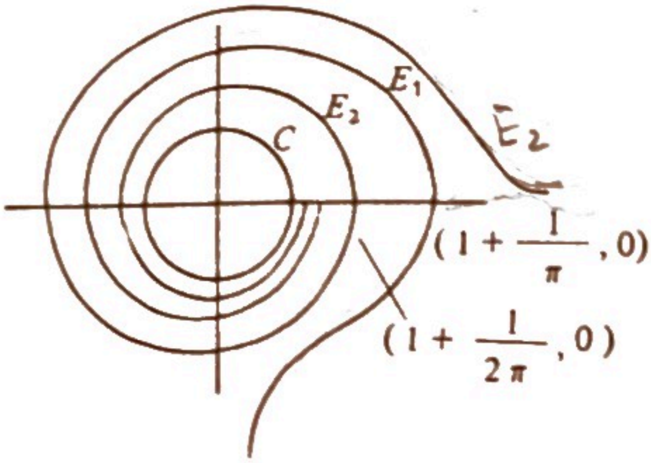

Figure 7

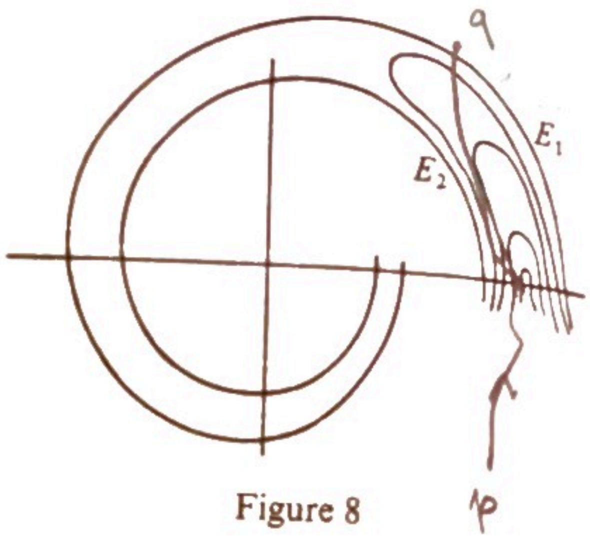

Figure 8

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(r\cos\theta, r\sin\theta) = \begin{cases} e^{1/(r^2-1)}, & \text{if } r < 1; \\ 0, & \text{if } r = 1; \\ e^{-1/(r^2-1)} \sin(1/(r-1) - \theta), & \text{if } r > 1. \end{cases}$$

Let $X = \operatorname{grad} f$. We have $X(r\cos\theta, r\sin\theta) = 0$ if and only if $r = 0$ or $r = 1$. We are going to show that there exists an orbit of X whose ω-limit is the circle C with centre at the origin and radius 1. Note that $f^{-1}(0) = C \cup E_1 \cup E_2$, where E_1 and E_2 are the spirals (Figure 7) defined by

$$E_1 = \{(r\cos\theta, r\sin\theta);\, r = 1 + 1/(\pi + \theta),\, -\pi < \theta < \infty\},$$

$$E_2 = \{(r\cos\theta, r\sin\theta);\, r = 1 + 1/(2\pi + \theta),\, -2\pi < \theta < \infty\}.$$

Let us consider the region $U = \{(r\cos\theta,\ r\sin\theta);\ 1 + 1/(2\pi + \theta) < r < 1 + 1/(\pi + \theta), \theta \geq 0\}$ and let I be the interval $\{(x, 0); 1 + 1/2\pi \leq x \leq 1 + 1/\pi\}$. We shall show the existence of a point $p_0 \in I$ whose positive orbit remains in the region U. Hence the ω-limit of p_0 will be the circle C. In Figure 8 we draw some level curves of the function f on U.

The intersection of the level curve through a point $p \in I$ with U is a compact segment whose ends are in I. The length of this segment tends to infinity as p approaches the ends of I.

Let $q \in E_1$. As $X(q)$ is orthogonal to E_1 and points out of U (becau[se f is] negative in U), we see that the negative orbit of q intersects one of t[he level] curves through a point in the interior of I. So the negative orbit of q in[tersects] I. Therefore, the set $J = \{p \in I;\, X_t(p) \in U$ for $0 \leq t < s$ and $X_s(p) \in$ [E_1] nonempty. Moreover, given $q \in E_1$, there exists $p \in J$ such that the po[sitive] orbit of p contains q and the segment of the orbit between p and q is in U. [On] the other hand, given $q \in E_2$, the negative orbit of q also intersects I so th[at] $J \neq I$.

Let p_0 be the infimum of J. We claim that the positive orbit of p_0 remains in U. For if this is not the case there exists a point q in the positive orbit of p_0 such that the segment of the orbit between p_0 and q is contained in U and $X_t(q) \notin U$ for sufficiently small $t > 0$. Thus $q \in E_1$ or $q \in E_2$ or $q \in I$. If $q \in E_1$ then each positive orbit through a point of J intersects E_1 in a point of the segment between $(1 + 1/\pi, 0)$ and q. This is absurd because the negative

An orbit can not intersect the same level set twice.

orbit through any point of E_1 intersects I and therefore J. If $q \in E_2$ or $q \in I$ then the positive orbit of each point near p_0 leaves U without meeting E_1 which is absurd since p_0 is the infimum of J. Thus the positive orbit of p_0 is contained in U, which proves our claim.

We remark that the vector field on S^2 in Example 1 is the gradient of the height function that measures height above the plane tangent to the sphere S^2 at p_S. Other simple examples can be obtained by considering the function on a surface in $\mathbb{R}^3$ which measures the distance from its points to a plane. Some of these examples will be considered later.

Next we shall discuss some general properties of ω-limit sets.

1.4 Proposition. *Let $X \in \mathfrak{X}^r(M)$ where M is a compact manifold and let $p \in M$. Then*

(a) $\omega(p) \neq \varnothing$,
(b) $\omega(p)$ *is closed,*
(c) $\omega(p)$ *is invariant by the flow of X, that is $\omega(p)$ is a union of orbits of X, and*
(d) $\omega(p)$ *is connected.*

Proof. Let $t_n \to \infty$ and $p_n = X_{t_n}(p)$. As M is compact p_n has a convergent subsequence whose limit belongs to $\omega(p)$. Thus $\omega(p) \neq \varnothing$. Suppose now that $q \notin \omega(p)$. Then it has a neighbourhood $V(q)$ disjoint from $\{X_t(p); t \geq T\}$ for some $T > 0$. This implies that the points of $V(q)$ do not belong to $\omega(p)$ and so $\omega(p)$ is closed. Next suppose that $q \in \omega(p)$ and $\tilde{q} = X_s(q)$. Take $t_n \to \infty$ with $X_{t_n}(p) \to q$. Then $X_{t_n+s}(p) = X_s X_{t_n}(p)$ converges to $X_s(q) = \tilde{q}$ and so $\tilde{q} \in \omega(p)$. This shows that $\omega(p)$ is invariant by the flow. Suppose that $\omega(p)$ is not connected. Then we can choose open sets V_1 and V_2 such that $\omega(p) \subset V_1 \cup V_2$, $\omega(p) \cap V_1 \neq \varnothing$, $\omega(p) \cap V_2 \neq \varnothing$ and $\overline{V}_1 \cap \overline{V}_2 = \varnothing$. The orbit of p accumulates on points of both V_1 and V_2 so, given $T > 0$, there exists $t > T$ such that $X_t(p) \in M - (V_1 \cup V_2) = K$, say. Thus there exists a sequence $t_n \to \infty$ with $X_{t_n}(p) \in K$. Passing to a subsequence, if necessary, we have $X_{t_n}(p) \to q$ for some $q \in K$. But this implies that $q \in \omega(p) \subset V_1 \cup V_2$ which is absurd. □

Remark. Clearly the properties above are also true for the α-limit set. On the other hand, if the manifold were not compact we should have to restrict attention to an orbit contained in a compact set for positive time (or for negative time). Figure 9 shows an orbit of a vector field on $\mathbb{R}^2$ whose ω-limit is not connected.