Διάλεξη 9

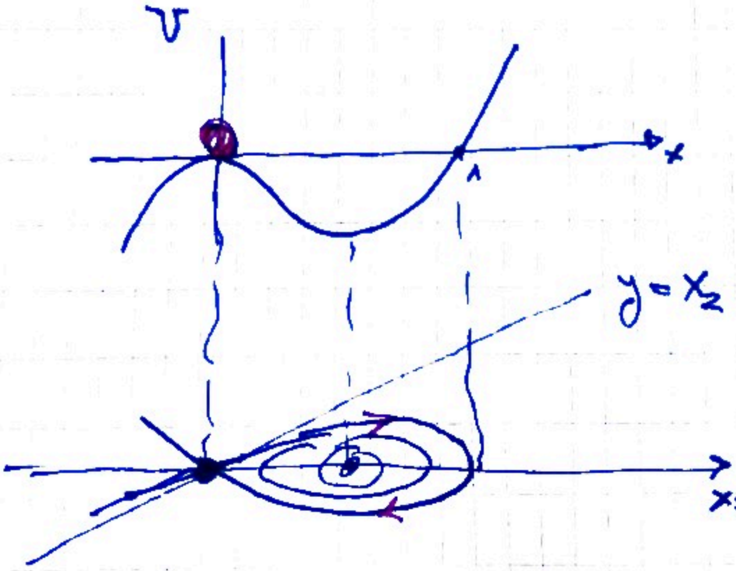
Ομοκλινείς - Ετεροκλινείς

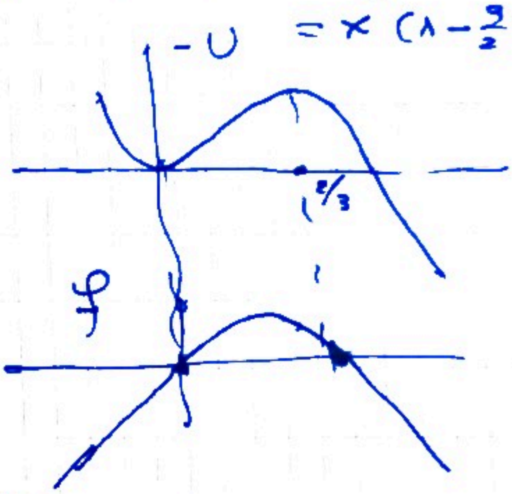
Παραδείγματα

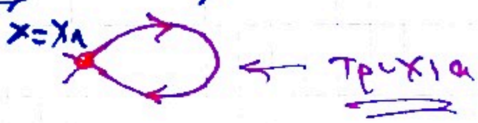
1) Ομοκλινής

$$x'' - x + \frac{3}{2}x^2 = 0 \Leftrightarrow \begin{cases} x_1' = x_2 \\ x_2' = x_1 - \frac{3}{2}x_1^2 \end{cases}$$

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{2}x^3 \quad , \quad f(x) = -U'(x) = x - \frac{3}{2}x^2 = x\left(1 - \frac{3}{2}x\right)$$



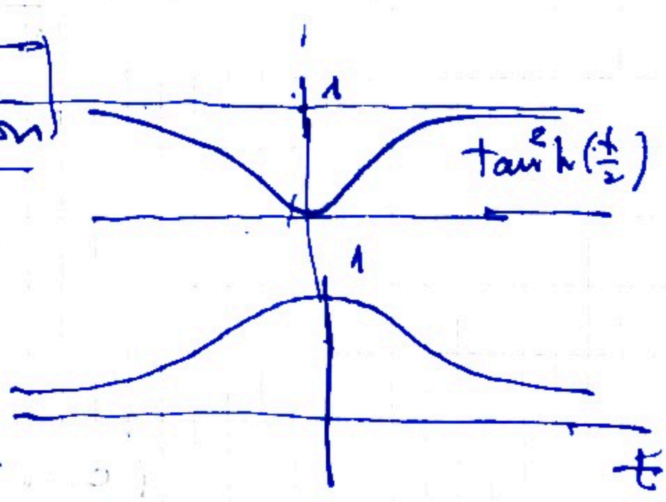




Σημεία Ισορροπίας

$$: \begin{cases} x_2 = 0 \\ x_1 - \frac{3}{2}x_1^2 = 0 \end{cases} \cdot \quad (0,0),\ \left(\tfrac{2}{3},0\right)$$

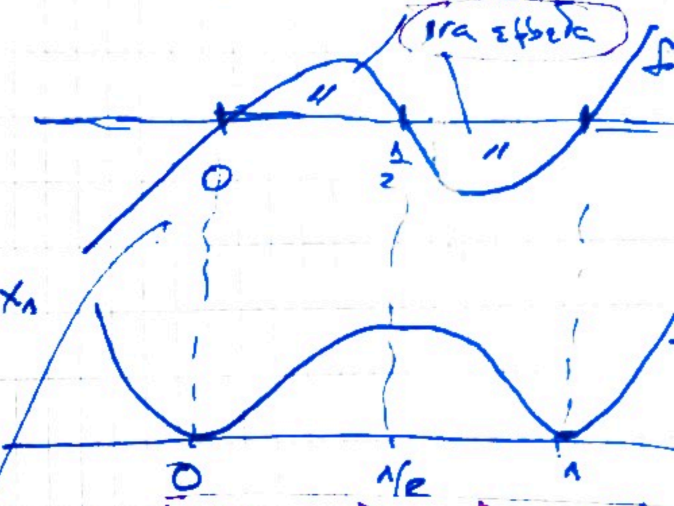
$$\boxed{x(t) = 1 - \tanh^2\frac{t}{2} \quad \text{Λύση}}$$



2) Εφαρμογές

$$x'' + x(1-x)\left(x-\tfrac{1}{2}\right) = 0 \quad , \quad f(x) = -U'(x) = -x(1-x)\left(x-\tfrac{1}{2}\right)$$

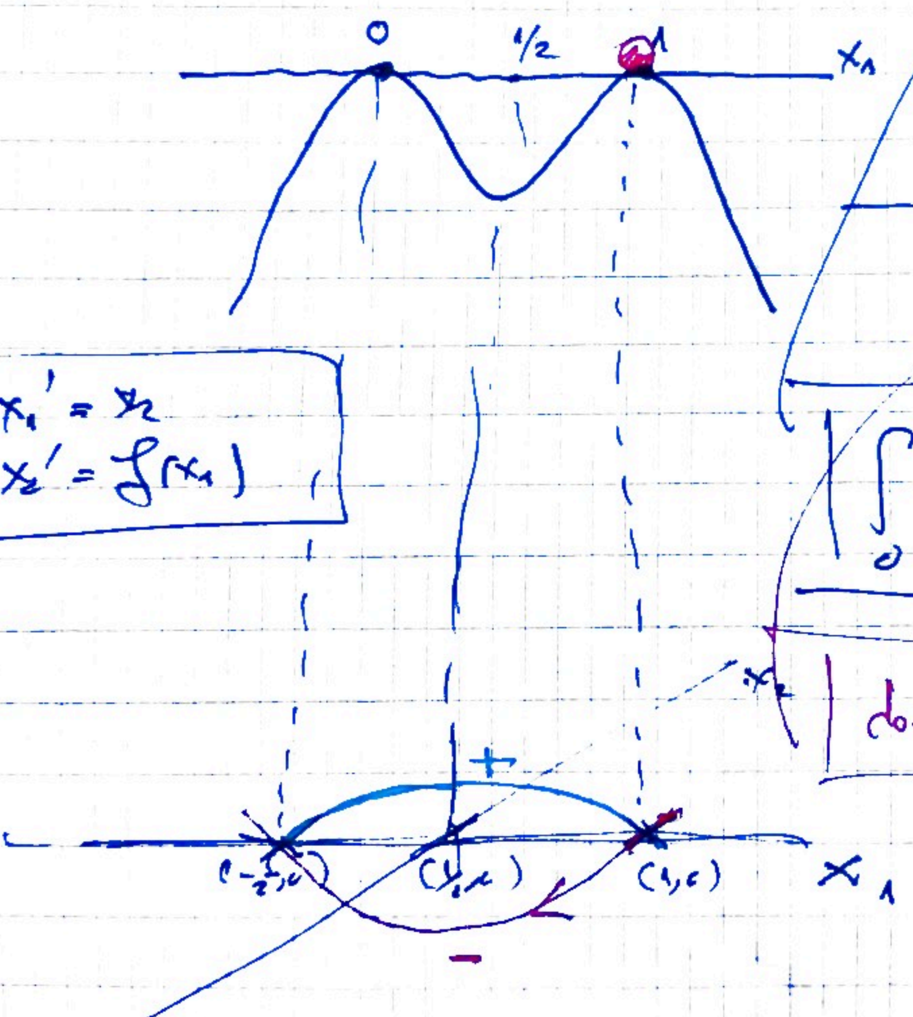
$$V(x) = -\frac{1}{4}x^4 + \frac{1}{2}x^3 - \frac{1}{4}x^2$$



Διευστάσεις Δυναμικού

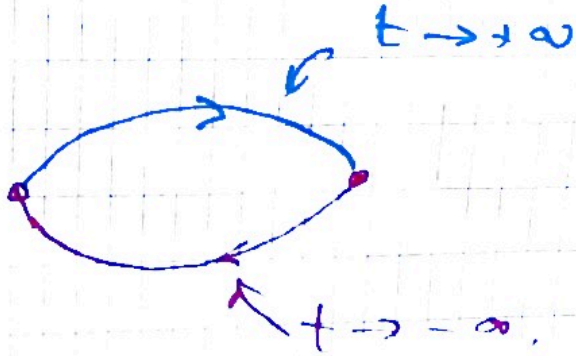
$$\boxed{\int_0^1 f(x)\,dx = 0}$$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= f(x_1) \end{aligned}}$$



double-well

$$x(t) = \frac{\pm 1}{1 + e^{-\frac{t}{\sqrt{2}}}}$$



3) (α) Να μελετηθεί επίπεδο φάσης

$$x'' - x + g'(x) = 0 \quad , \quad g(x) = Ax^{2k}, \ A > 0, \ k \geq 1$$

$$f(x) = x - g(x)$$

$$= x - Ax^{2k} = x(1 - Ax^{2k-1})$$

(β) Δείξτε ότι η ομοκλινική εντός του κύκλου

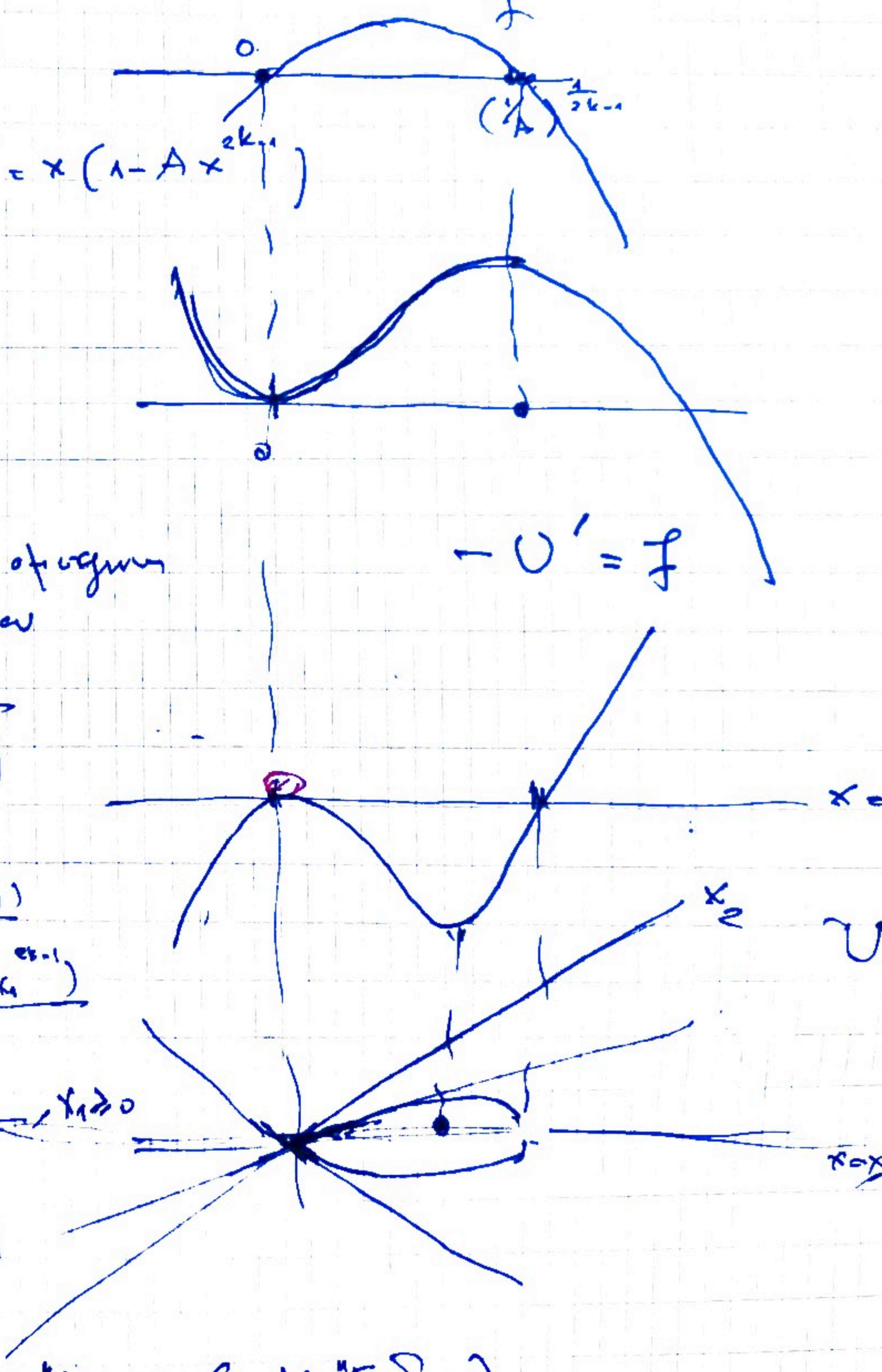
$$\boxed{x_2 = \pm x_1}$$

$$\frac{dx_2}{dx_1} = \frac{x_1 - g(x_1)}{x_2} = \frac{x_1(1 - Ax_1^{2k-1})}{x_2}$$

$$\leq \frac{x_1}{x_2}, \ x_1 \geq 0$$

$$\Rightarrow x_2 dx_2 \leq x_1 dx_1$$

$$\boxed{x_2^2 \leq x_1^2}$$



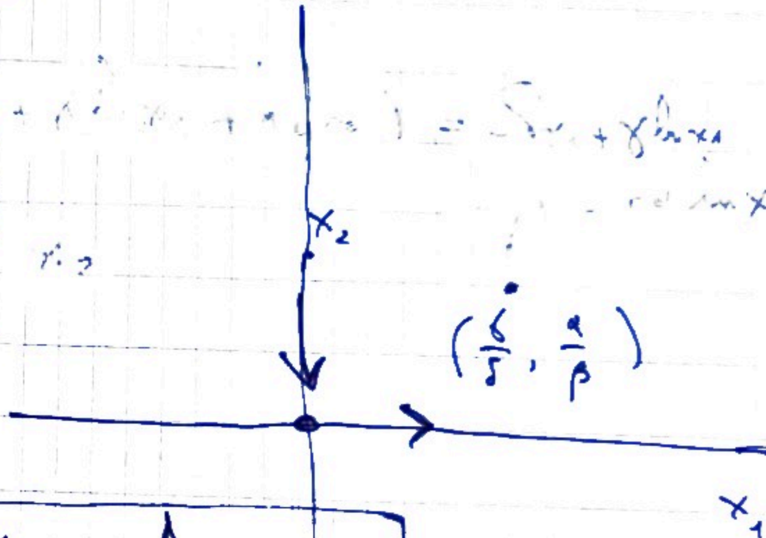
Ασκήσεις (Α-Κ, 2η Έκδοση)
10.16 10.20, 10.27, 10.28.

(4)

Άλλα Παραδείγματα Συστημάτων – Γενικά – ΌΧΙ της μορφής $x'' = f(x)$

4) Παρ. Lotka-Volterra

$$(*)\begin{cases} x_1' = (\alpha - \beta x_2)x_1 \\ x_2' = (\delta x_1 - \gamma)x_2 \end{cases}, \quad \alpha, \beta, \gamma, \delta$$



Σ.Ι.

$$\boxed{\begin{aligned} (\alpha - \beta x_2)x_1 &= 0 \\ (\delta x_1 - \gamma)x_2 &= 0 \end{aligned}}$$

$$x_1 = 0,\ x_2 = 0$$

$$\boxed{x_2 = \frac{\alpha}{\beta}}, \quad \boxed{x_1 = \frac{\gamma}{\delta}}$$

$\boxed{\text{Άξονες Αναλλοίωτοι}}$

Διανυσματικό πεδίο

$$\left\langle (\alpha - \beta x_2)x_1,\ (\delta x_1 - \gamma)x_2 \right\rangle$$

$\boxed{x_2 = 0}$

$$\langle \alpha x_1,\ 0 \rangle$$

$\boxed{x_1 = 0}$

$$\langle 0,\ -\gamma x_2 \rangle$$

$$\frac{dx_2}{dx_1} = \frac{(\delta x_1 - \gamma)x_2}{(\alpha - \beta x_2)x_1} = \frac{x_1 x_2\left[\delta - \frac{\gamma}{x_1}\right]}{x_1 x_2\left[\frac{\alpha}{x_2} - \beta\right]}$$

$$= \frac{\delta - \frac{\gamma}{x_1}}{\frac{\alpha}{x_2} - \beta}$$

εκτός αξόνων

$\Updownarrow$ (I)$\Leftrightarrow$(II)

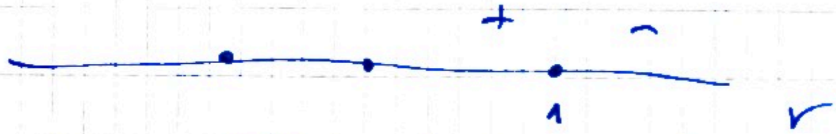
$$(\mathrm{II})\quad \boxed{\begin{aligned} y_1' &= \frac{\alpha}{y_2} - \beta \\ y_2' &= \delta - \frac{\gamma}{y_1} \end{aligned}}$$

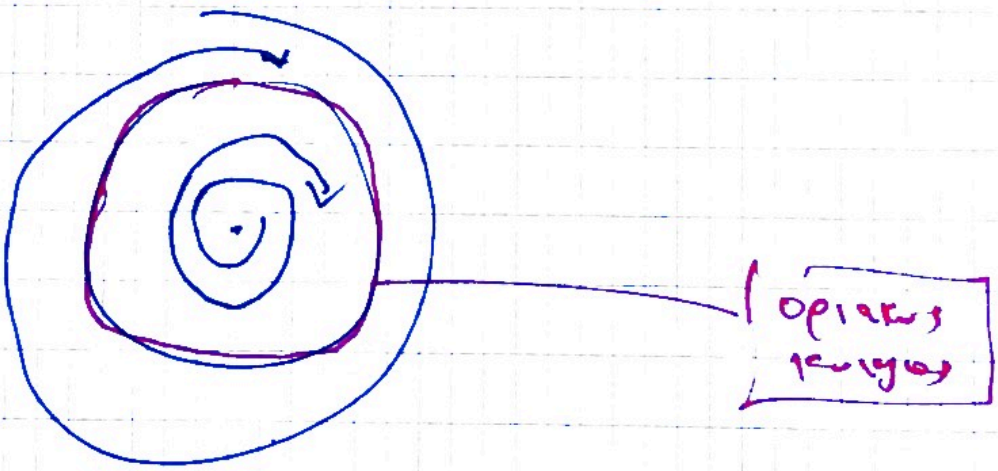
Σύγκριση με

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}}$$

Πολικές : $x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$$r' = r(1-r^2), \quad \theta' = -1$$

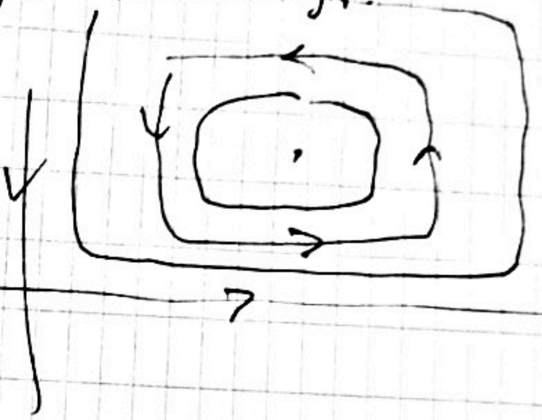




5

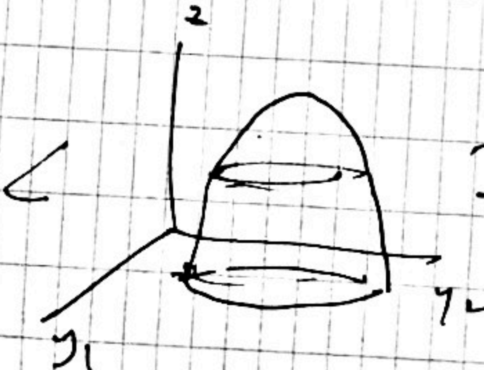
$$H(y_1,y_2) = -\delta y_1 + \gamma \ln y_1 - \beta y_2 + \alpha \ln y_2$$

$$\frac{\partial H}{\partial y_2} = -\beta + \frac{\alpha}{y_2}, \quad \frac{\partial H}{\partial y_1} = -\delta + \frac{\gamma}{y_1}$$



Ασ 1.52 ΑΙΚ

$$z = H(y_1,y_2)$$

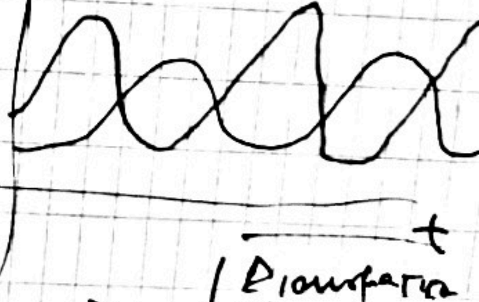


$\to 0$ $\boxed{H_c}$

Σχέση (I), (II)

$$x_1' = f_1(x_1,x_2)$$
$$x_2' = f_2(x_1,x_2)$$

$$x_1' = a(x_1,x_2) f_1(x_1,x_2) =: g_1(x_1,x_2)$$
$$x_2' = a(x_1,x_2) f_2(x_1,x_2) =: g_2(x_1,x_2)$$



5) Οριακός Κύκλος

$$x_1' = x_2 + x_1(1 - x_1^2 - x_2^2)$$

$$x_2' = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

$$\langle g_1(x_1,x_2), g_2(x_1,x_2)\rangle = a(x_1,x_2)\langle f_1(x_1,x_2), f_2(x_1,x_2)\rangle$$

Διανυσματικά πεδία γεωμετρικά ισοδύναμα