Accepted: 21 September 2024

DOI: 10.1002/cpa.22230

RESEARCH ARTICLE

# Uniqueness of the blow-down limit for a triple junction problem

**Zhiyuan Geng**

Department of Mathematics, Purdue University, West Lafayette, Indiana, USA

**Correspondence**
Zhiyuan Geng, Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA.
Email: geng42@purdue.edu

**Abstract**

We prove the uniqueness of $L^1$ blow-down limit at infinity for an entire minimizing solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ of a planar Allen–Cahn system with a triple-well potential. Consequently, $u$ can be approximated by a triple junction map at infinity. The proof exploits a careful analysis of energy upper and lower bounds, ensuring that the diffuse interface remains within a small neighborhood of the approximated triple junction at all scales.

## 1 | INTRODUCTION

### 1.1 | The problem and main result

This paper is concerned with the uniqueness of the blow-down limit at infinity for an entire, bounded minimizing solution of the system

$$\Delta u - W_u(u) = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2, \tag{1.1}$$

where $W$ is a potential with three global minima. Specifically for $W$ we assume

(H1). $W \in C^2(\mathbb{R}^2; [0, +\infty))$, $\{z : W(z) = 0\} = \{a_1, a_2, a_3\}$, $W_u(u) \cdot u > 0$ if $|u| > M$ and

$$c_2|\xi|^2 \geq \xi^T W_{uu}(a_i)\xi \geq c_1|\xi|^2, \ i = 1, 2, 3.$$

for some positive constants $c_1 < c_2$ depending on $W$.

(H2). For $i \neq j$, $i, j \in \{1, 2, 3\}$, let $U_{ij} \in W^{1,2}(\mathbb{R}, \mathbb{R}^2)$ be an 1D minimizer of the *action*

$$\sigma_{ij} := \min \int_{\mathbb{R}} \left( \frac{1}{2}|U_{ij}'|^2 + W(U_{ij}) \right) d\eta, \quad \lim_{\eta\to-\infty} U_{ij}(\eta) = a_i, \ \lim_{\eta\to+\infty} U_{ij}(\eta) = a_j.$$

$\sigma_{ij}$ satisfies

$$\sigma_{ij} \equiv \sigma > 0 \quad \text{for } i \neq j \in \{1,2,3\} \text{ and some constant } \sigma. \tag{1.2}$$

Note that (1.1) is the Euler–Lagrange equation associated with the energy

$$E(u,\Omega) := \int_\Omega \left( \frac{1}{2}|\nabla u|^2 + W(u) \right) dx. \tag{1.3}$$

An entire *minimizing solution* $u : \mathbb{R}^2 \to \mathbb{R}^2$ is defined in the following local sense.

**Definition 1.1.** A function $u : \mathbb{R}^2 \to \mathbb{R}^2$ is a *minimizing solution* of (1.1) in the sense of De Giorgi if

$$E(u,\Omega) \leq E(u+v,\Omega), \quad \forall \text{ bounded open sets } \Omega \subset \mathbb{R}^2,\ \forall v \in C_0^1(\Omega). \tag{1.4}$$

The solution we seek can be regarded as the diffuse analogue of the minimal 3-partition of the plane. Specifically, we define

$$\mathcal{P} := \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\},$$

which is a partition of the plane into three sectors centered at the origin with opening angles of 120 degrees. Let $\partial\mathcal{P}$ denote the union of three rays that separate $\{\mathcal{D}_i\}_{i=1}^3$, that is,

$$\partial\mathcal{P} := \partial\mathcal{D}_1 \cup \partial\mathcal{D}_2 \cup \partial\mathcal{D}_3.$$

We call $\partial\mathcal{P}$ a *triod*. The *triple junction map* is defined by

$$U_{\mathcal{P}} := a_1\chi_{\mathcal{D}_1} + a_2\chi_{\mathcal{D}_2} + a_3\chi_{\mathcal{D}_3}, \tag{1.5}$$

where $\chi_\Omega$ represents the characteristic function of domain $\Omega$.

It is well-known that $\mathcal{P}$ is a minimizing partition of $\mathbb{R}^2$ into three phases and $\partial\mathcal{P}$ is a minimal cone. $\mathcal{P}$ is minimizing in the sense that for any $\Omega \subset \mathbb{R}^2$, $\mathcal{P} \llcorner \Omega = \{D_i \cap \Omega\}_{i=1}^3$ is a solution of the following variational problem

$$\min \sum_{i<j} \mathcal{H}^1(\partial(A_i \cap \Omega) \cap \partial(A_j \cap \Omega)),$$
$$\mathcal{A} = \{A_i\}_{i=1}^3 \text{ is a 3-partition of } \mathbb{R}^2 \text{ and } \mathcal{P} \llcorner (\mathbb{R}^2 \setminus \Omega) = \mathcal{A} \llcorner (\mathbb{R}^2 \setminus \Omega).$$

The minimality stated above is related to Steiner's classical result which states that given three points $A, B, C$ on the plane such that the corresponding triangle has no angle greater than or equal to 120 degrees, then if $P$ is a point that minimizes the sum of the distances $|P - A| + |P - B| + |P - C|$, the line segments $PA, PB, PC$ form three 120-degree angles.

In 1996, Bronsard et al. [6] established the existence of an entire solution to (1.1) in the equivariant class of the reflection group $\mathcal{G}$ of the symmetries of the equilateral triangle. The triple-well potential is also assumed invariant under $\mathcal{G}$. The solution is obtained as a minimizer in the equivariant class $u(gx) = gu(x), g \in \mathcal{G}$, hence is not necessarily stable under general perturbations. Their results were extended to the three dimensional case in 2008 by Gui and Schatzman [10]. We refer to the book [2] and to the references therein.

In 2021, Fusco [8] succeeded in establishing essentially the result of [6] in the equivariant class of the rotation subgroup of $\mathcal{G}$ (by $\frac{2}{3}\pi$), thus eliminating the two reflections. The symmetry hypothesis fixes the center of the junction at the origin, which simplifies the analysis.

On bounded domains, there are some constructions of triple junction solutions without imposing symmetry assumptions, see for example, the paper by Sternberg and Ziemer [13] for clover-shaped domains in $\mathbb{R}^2$ via Γ–convergence, and for more general domains by Flores et al. [7] by a mountain pass argument. These Γ-convergence results are not sufficient for establishing the existence of a triple junction solution on $\mathbb{R}^2$.

The existence of an entire minimizing solution in the sense of Definition 1.1, characterized by a triple junction structure at infinity, has been independently established in two recent papers: by Alikakos and the author [3] and by Sandier and Sternberg [11]. Under slightly different hypotheses and employing distinct methods, these two studies have yielded comparable results saying that along a subsequence $r_k \to \infty$, the rescaled function $u_{r_k}(z) := u(r_k z)$ converges in $L^1_{loc}(\mathbb{R}^2)$ to a triple junction map $u_0$ of the form (1.5). However, it remains unclear a priori if there could exist two different sequences of rescalings $\{r_k\}$ and $\{s_k\}$, leading to distinct blow-down limits $u_1$ and $u_2$, corresponding to distinct minimal 3-partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The primary objective of the present paper is to rule out this possibility and demonstrate the uniqueness of the blow-down limit. We now state our main result.

**Theorem 1.2.** *There exists a bounded, minimizing solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ of (1.1) such that for any compact $K \subset \mathbb{R}^2$,*

$$\lim_{r\to\infty} \|u_r(x) - u_0\|_{L^1(K)} = 0,$$

*where $u_0 = \sum_{i=1}^{3} a_i \chi_{D_i}$ for $\mathcal{P} = \{\mathcal{D}_i\}_{i=1}^3$ providing a minimal partition of $\mathbb{R}^2$ into three sectors with angle of 120 degrees and $\partial\mathcal{P}$ is a triod centered at 0.*

*Remark* 1.1. The result above also holds for the case when $\sigma_{ij}$ are not all equal but still satisfy $\sigma_{ij} + \sigma_{jk} > \sigma_{ik}$. In this scenario, the corresponding minimal partition $\mathcal{P}$ consists of three sectors $\mathcal{D}_i$ with opening angles $\theta_i$ satisfying $\sum_{i=1}^3 \theta_i = 2\pi$ and $\frac{\sin\theta_1}{\sigma_{23}} = \frac{\sin\theta_2}{\sigma_{13}} = \frac{\sin\theta_3}{\sigma_{12}}$. The proof for the general $\sigma_{ij}$ case follows the same argument as the proof of Theorem 1.2. In this paper, we will adhere to the equal $\sigma_{ij}$ case for the sake of clarity and simplicity in presentation.

Uniqueness of blow-up or blow-down limits is one of the central questions in the study of singular structures in geometric PDEs. Following the prominent early works by Allard and Almgren [4] and Simon [12], uniqueness questions have been investigated extensively for free boundary problems, harmonic maps, minimal surfaces and geometric flows. Most of these results rely on some type of the Simon–Łojasiewicz inequality or epiperimetric inequalities, showing the decay of certain monotone quantities at a definite rate. In our proof, the uniqueness of the blow-down limit is obtained from a delicate estimate on the localization of the diffuse interface, derived from a purely variational argument, thus avoiding the use of those classical methods.

## 1.2 | An overview of the proof

We now list some key steps and ideas in the proof of our results. As mentioned earlier, we can start with a minimizing solution $u$ of (1.1) as constructed in [3] and [11], which converges to a triple junction map of the form (1.5) along some subsequence $r_k \to \infty$. In particular, for arbitrarily small $\varepsilon$, there exists a sufficiently large $R_0$ such that

$$3\sigma - \varepsilon \le \int_{\partial B_{R_0}} \left( \frac{1}{2}|\nabla_T u|^2 + W(u) \right) d\mathcal{H}^1 \le 3\sigma + \varepsilon,$$

where $\nabla_T$ denotes the tangential derivative.

Starting from this estimate, which basically means there are three phase transitions along $\partial B_{R_0}$, we can derive that $u(z)$ should "behave nicely" on $\partial B_{R_0}$. To be more specific, $u$ is close to phases $a_1, a_2, a_3$ on three arcs $I_1, I_2, I_3$, respectively. Between these phases, there will be three small arcs $I_{ij}$ $(i \neq j \in \{1,2,3\})$ separating them, which can be regarded as the place where phase transition happens. We pick points $A \in I_{12}$, $B \in I_{13}$, $C \in I_{23}$ and determine the point $D$ such that $|DA| + |DB| + |DC|$ is minimized. Then we let $T_{ABC} := DA \cup DB \cup DC$ be the approximate triod on $B_{R_0}$.

With such a nicely behaved boundary data on $\partial B_{R_0}$, we obtain the following energy upper bound and lower bound in $B_{R_0}$.

$$\sigma(|DA| + |DB| + |DC|) - CR_0^{\alpha} \le \int_{B_{R_0}} \left( \frac{1}{2}|\nabla u|^2 + W(u) \right) dz \le \sigma(|DA| + |DB| + |DC|) + CR_0^{\alpha}, \tag{1.6}$$

for some constants $C = C(W)$ and $\alpha \in (0,1)$. For the upper bound, we utilize the construction of an energy competitor outlined in [11, Proposition 3.3], while for the lower bound we mimic the proof of [3, Proposition 3.1].

We define the diffuse interface as

$$I_\gamma := \{z : |u(z) - a_i| \ge \gamma,\ \forall i = 1,2,3\}.$$

The energy bound (1.6) implies that $I_\gamma \cap B_{R_0}$ is located in an $O(R_0^{\beta})$ (for some $\beta < 1$) neighborhood of the approximate triod $T_{ABC}$. Moreover, away from $T_{ABC}$, the distance of $u(z)$ to $a_i$ will decay exponentially with respect to $\mathrm{dist}(z, T_{ABC})$ thanks to Hypothesis (H1) and standard elliptic theories.

We proceed by rescaling $B_{R_0}$ to the unit disk $B_1$ through the function $u_{R_0} := u(R_0 z)$. The exponential decay above implies that $u_{R_0}$ is closely approximated in $L^1$ norm by a minimal partition $\{\mathcal{D}^i_{R_0}\}_{i=1}^3$ of $B_1$ determined by the rescaled version of $T_{ABC}$. Specifically, we have

$$\|u_{R_0} - U_{R_0}\|_{L^1} \le CR_0^{\beta-1}, \tag{1.7}$$

where $U_{R_0} = \sum a_i \chi_{\mathcal{D}^i_{R_0}}$. We point out that (1.7) holds for any larger scaling $R_i \sim 2^i R_0$, where $U_{R_i}$ is the corresponding approximate triple junction map at the scale $R_i$.

A key observation is that approximate triple junction maps at two consecutive scales are close to each other, that is,

$$\|U_{R_i} - U_{R_{i+1}}\| \le CR_i^{\beta-1}. \tag{1.8}$$

This is established by (1.7) and the fact that $u_{R_i}$ and $u_{R_{i+1}}$ are obtained by rescaling the same function $u$. Finally, we can iterate (1.8) and deduce that $U_{R_i}$ will converge to a unique triple junction map, and thereby concluding the proof.

The article is organized as follows. In Section 2, we present some preliminary results from [1] and [2]. In Sections 3 and 4, we establish the existence of a minimizing solution $u$ and fix a well behaved boundary data on $\partial B_{R_0}$. Next, we establish the energy bound (1.6) in Section 5. The localization of the diffuse interface within an $O(R_0^\beta)$ neighborhood of the triod $T_{ABC}$ is proved in Section 6. Then in Section 7 we rescale $B_{R_0}$ to $B_1$ and prove (1.7). Lastly, we conclude the proof of the main theorem utilizing the estimate (1.8) in Section 8.

## 2 | PRELIMINARIES

Throughout the paper we denote by $z = (x, y)$ a 2D point and by $B_r(z)$ the 2D ball centered at the point $z$ with radius $r$. In addition, we let $B_r$ denote the 2D ball centered at the origin. We recall the following basic results which play a crucial part in our analysis.

**Lemma 2.1** (Lemma 2.1 in [1]). *The hypotheses on $W$ imply the existence of $\delta_W > 0$, and constants $c_W, C_W > 0$ such that*

$$|u - a_i| = \delta$$
$$\Rightarrow \frac{1}{2}c_W\delta^2 \leq W(u) \leq \frac{1}{2}C_W\delta^2, \quad \forall \delta < \delta_W,\ i = 1, 2, 3.$$

*Moreover if $\min_{i=1,2,3} |u - a_i| \geq \delta$ for some $\delta < \delta_W$, then $W(u) \geq \frac{1}{2}c_W\delta^2$.*

**Lemma 2.2** (Lemma 2.3 in [1]). *Take $i \neq j \in \{1, 2, 3\}$, $\delta < \delta_W$ and let $s_+ > s_-$ be two real numbers. Let $v : (s_-, s_+) \to \mathbb{R}^2$ be a smooth map that minimizes the energy functional*

$$J_{(s_-,s_+)}(v) := \int_{s_-}^{s_+} \left(\frac{1}{2}|\nabla v|^2 + W(v)\right) dx$$

*subject to the boundary condition*

$$|v(s_-) - a_i| = |v(s_+) - a_j| = \delta.$$

*Then*

$$J_{(s_-,s_+)}(v) \geq \sigma - C_W\delta^2,$$

*where $C_W$ is the constant in Lemma 2.1.*

**Lemma 2.3** (Variational maximum principle, Theorem 4.1 in [2]). *There exists a positive constant $r_0 = r_0(W)$ such that for any $u \in W^{1,2}(\Omega, \mathbb{R}^2) \cap L^\infty(\Omega, \mathbb{R}^2)$ being a minimizer of $E(\cdot, \Omega)$, if $u$ satisfies*

$$|u(x) - a_i| \leq r \text{ on } \partial\Omega, \text{ for some } r < r_0,\ i \in \{1, 2, 3\},$$

*then*

$$|u(x) - a_i| \le r \ \ \forall x \in \Omega.$$

## 3 | EXISTENCE OF AN ENTIRE MINIMIZING SOLUTION

By the constructions in [3] and [11], there exists a minimizing solution $u : \mathbb{R}^2 \to \mathbb{R}^2$ of (1.1) such that the following hold:

(1) There exists $M > 0$ such that

$$|u(z)| + |\nabla u(z)| \le M, \quad \forall z \in \mathbb{R}^2. \tag{3.9}$$

(2) [11, Theorem 1.1] For any sequence $r_k \to \infty$, there is a subsequence, still denoted by $\{r_k\}$, such that

$$u(r_k z) \to u_0(z) \ \text{ in } L^1_{loc}(\mathbb{R}^2), \tag{3.10}$$

where $u_0(z) = \sum_{i=1}^{3} a_i \chi_{\mathcal{D}_i}$. for $\mathcal{P} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ providing a minimal partition of $\mathbb{R}^2$ into three sectors of the angle $\frac{2}{3}\pi$ and $\partial\mathcal{P}$ is a triod centered at 0. Moreover, along the same subsequence we have the following energy estimate

$$\lim_{r_k \to \infty} \frac{1}{r_k} E(u, B_{r_k}(0)) = 3\sigma.$$

This energy estimate follows from the $\Gamma$–convergence result in Baldo [5] that holds also without the mass constraint (see Gazoulis [9]).

(3) By [11, Lemma 3.4 & Lemma 3.5], $u$ is asymptotically homogeneous and satisfies an asymptotic energy equipartition property at large scale,

$$\lim_{R \to \infty} \frac{1}{R} \int_{B_R} W(u)\, dz = \lim_{R \to \infty} \frac{1}{R} \int_{B_R} \frac{1}{2} |\nabla u|^2\, dz = \lim_{R \to \infty} \frac{1}{R} \int_{B_R} \frac{1}{2} |\nabla_T u|^2\, dz = \frac{3\sigma}{2}. \tag{3.11}$$

Here $\nabla_T$ denotes the tangential derivative.

Utilizing (3.11), for any $\varepsilon > 0$, there exists a large $R(\varepsilon)$ such that for any $R > R(\varepsilon)$,

$$\begin{aligned}
3\sigma - \varepsilon \le & \frac{1}{R} \int_{B_R} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dz \le 3\sigma + \varepsilon, \\
\frac{3\sigma - \varepsilon}{2} \le & \frac{1}{R} \int_{A_{2R,R}} W(u)\, dz \le \frac{3\sigma + \varepsilon}{2}, \\
\frac{3\sigma - \varepsilon}{2} \le & \frac{1}{R} \int_{A_{2R,R}} \frac{1}{2} |\nabla_T u|^2\, dz \le \frac{3\sigma + \varepsilon}{2},
\end{aligned}$$

where $A_{2R,R}$ represents the annulus $\{z \in \mathbb{R}^2 : R < |z| < 2R\}$. Fixing $\varepsilon$ as a small parameter to be determined later, by Fubini's theorem we can find $R_0(\varepsilon) \in (R(\varepsilon), 2R(\varepsilon))$ such that

$$3\sigma - \varepsilon \leq \int_{\partial B_{R_0}} \left(\frac{1}{2}|\nabla_T u|^2 + W(u)\right) d\mathcal{H}^1 \leq 3\sigma + \varepsilon. \tag{3.12}$$

$$3\sigma - \varepsilon \leq \frac{1}{R_0}\int_{B_{R_0}} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dz \leq 3\sigma + \varepsilon. \tag{3.13}$$

As our analysis progresses, we will gradually determine the conditions on the smallness of $\varepsilon$.

## 4 | "WELL-BEHAVED" PROFILE OF $u(z)$ ON $\partial B_{R_0}$

Firstly, we take a fixed small constant $\delta = \delta(W)$ which is independent of $\varepsilon$, to be determined later. We keep in mind that in certain places of our analysis, $\varepsilon$ is considered to be significantly smaller than $\delta$. From the energy bound (3.12) and Lemma 2.1 we conclude that

$$\mathcal{H}^1(\{z : \min_{i=1,2,3} |u(z) - a_i| > \delta\} \cap \partial B_{R_0}) \leq \frac{C}{\delta^2}, \tag{4.14}$$

for some $C = C(W)$. In other words, most of points on $\partial B_{R_0}$ are close to one of the phases $a_i$. We set

$$Y_i := \{z : |u(z) - a_i| \leq \delta\} \cap \partial B_{R_0}, \quad i = 1, 2, 3.$$

Then (4.14) implies

$$\sum_{i=1}^{3} \mathcal{H}^1(Y_i) \geq 2\pi R_0 - \frac{C}{\delta^2}. \tag{4.15}$$

**Lemma 4.1.** *For any $i \in \{1, 2, 3\}$, $Y_i \neq \emptyset$.*

*Proof.* We need to rule out the following two cases.

*Case* 1. Two of $Y_i$'s are empty sets. Without loss of generality, assume

$$Y_2 = Y_3 = \emptyset.$$

We construct the following energy competitor in $B_{R_0}$ by writing in polar system $z = (x, y) = re^{i\theta}$:

$$\tilde{u}(r,\theta) = \begin{cases} u(R_0, \theta), & r = R_0 \\ (1 + r - R_0)u(R_0, \theta) + (R_0 - r)a_1, & r \in (R_0 - 1, R_0), \\ a_1, & r \in [0, R_0 - 1]. \end{cases}$$

In view of Lemma 2.1, we have that

$$\begin{aligned}
&\int_{A_{R_0,R_0-1}} |\partial_r \tilde{u}|^2\, dz \\
&= \int_{R_0-1}^{R_0} \int_0^{2\pi} |u(R_0,\theta) - a_1|^2 r\, dr\, d\theta \\
&\le \int_{Y_1} |u-a_1|^2\, d\mathcal{H}^1 + \int_{\partial B_{R_0}\setminus Y_1} |u-a_1|^2\, d\mathcal{H}^1 \\
&\le \int_{Y_1} CW(u)\, d\mathcal{H}^1 + \frac{C}{\delta^2} M^2 \\
&\le C(\delta, W).
\end{aligned} \tag{4.16}$$

$$\int_{A_{R_0,R_0-1}} W(u)\, dz \le \int_{A_{R_0,R_0-1}} C|u-a_1|^2\, dz \le C(\delta, W). \tag{4.17}$$

$$\int_{A_{R_0,R_0-1}} |\partial_T \tilde{u}|^2\, dz \le \int_{\partial B_{R_0}} |\partial_T \tilde{u}|^2\, d\mathcal{H}^1 \le 3\sigma. \tag{4.18}$$

Adding (4.16), (4.17) and (4.18) together gives

$$\int_{B_{R_0}} \left( \frac{1}{2} |\nabla u|^2 + W(u) \right) dz \le C(\delta, W), \tag{4.19}$$

which yields a contradiction with (3.13) given that $R_0$ can be arranged to be much larger than $C(\delta, W)$. As a result, we eliminate Case 1.

*Case* 2. One of $Y_i$'s is empty. Assume without loss of generality that $Y_3 = \emptyset$. Then on $\partial B_{R_0}$, essentially there are only two phases appearing, namely $a_1$ and $a_2$. There exist $z_1,\ z_2 \in \partial B_{R_0}$ such that

$$|u(z_1) - a_1| \le \delta, \quad |u(z_2) - a_2| \le \delta, \quad \text{for } z_i := R_0 e^{i\theta_i},\ i = 1,2. \tag{4.20}$$

where $\theta_i$ denotes the polar angle of $z_i$ ($i = 1,2$). Then $z_1$ and $z_2$ split the circle $\partial B_{R_0}$ into two arcs, denoted by $A$ and $B$, respectively. Then we define

$$\begin{aligned}
A_1 &:= \{z \in A : |u(z) - a_1| \le \delta\}, \\
A_2 &:= \{z \in A : |u(z) - a_2| \le \delta\}.
\end{aligned}$$

By (4.20) and the continuity of $u$ we know that $A_1$ and $A_2$ are two disjoint non-empty closed sets in $A$. Set

$$\begin{aligned}
z_A^1 &:= \text{ the point from } A_1 \text{ that is closest to } A_2, \\
z_A^2 &:= \text{ the point from } A_2 \text{ that is closest to } A_1.
\end{aligned}$$

Here "closest" refers to the distance along $\partial B_{R_0}$. Now we have the key observation that on $A$ the arcs $\widehat{z_1 z_A^1}$ and $\widehat{z_2 z_A^2}$ do not overlap. Suppose by contradiction this is not the case, we have

$$z_A^2 \in \widehat{z_1 z_A^1} \cap A, \quad z_A^1 \in \widehat{z_2 z_A^2} \cap A.$$

If we start from $z_1$ and traverse along the clockwise direction of $\partial B_{R_0}$ for a complete circle, we will encounter at least four phase transitions occurring between the two phases $a_1$ and $a_2$, namely

$$z_1 \to z_2 \to z_A^1 \to z_A^2 \to z_1.$$

Then we have

$$\int_{\partial B_{R_0}} \left( \frac{1}{2} |\nabla_T u|^2 + W(u) \right) d\mathcal{H}^1 \geq 4\sigma - C\delta^2,$$

which contradicts (3.12). Therefore $\widehat{z_1 z_A^1}$ and $\widehat{z_2 z_A^2}$ cannot overlap. Moreover, appealing to (3.9) we have that

$$\mathcal{H}^1(\widehat{z_A^1 z_A^2}) \geq C,$$

for some positive constant $C$ depending on $|a_1 - a_2|$ and the uniform bound of $|\nabla u|$.

Now for any $z \in \widehat{z_A^1 z_A^2} = A \setminus \left( \widehat{z_1 z_A^1} \cup \widehat{z_2 z_A^2} \right)$, it holds

$$|u(z) - a_i| > \delta, \quad \forall i \in \{1, 2, 3\}.$$

Utilizing the energy bound (3.12) and the hypothesis (H1), we obtain

$$\mathcal{H}^1(\widehat{z_A^1 z_A^2}) \leq \frac{C}{\delta^2},$$

for some constant $C$ depending only on $W$.

In the same manner, on $B$ we can define

$$B_1 := \{z \in B : |u(z) - a_1| \leq \delta\},$$

$$B_2 := \{z \in B : |u(z) - a_2| \leq \delta\}.$$

and

$$z_B^1 := \text{ the point from } B_1 \text{ that is closest to } B_2,$$

$$z_B^2 := \text{ the point from } B_2 \text{ that is closest to } B_1.$$

By the same argument we have $\widehat{z_1 z_B^1}$ and $\widehat{z_2 z_B^2}$ do not overlap and

$$C_1 \leq \mathcal{H}^1(\widehat{z_B^1 z_B^2}) \leq \frac{C_2}{\delta^2},$$

for some positive constants $C_1$, $C_2$. Denote the polar angles for $z_A^1$, $z_A^2$, $z_B^1$, $z_B^2$ by $\theta_A^1$, $\theta_A^2$, $\theta_B^1$, $\theta_B^2$, respectively. Set

$$\Theta_1 := \widehat{\theta_B^1\theta_1} \cup \widehat{\theta_1\theta_A^1}, \quad \Theta_2 := \widehat{\theta_A^2\theta_2} \cup \widehat{\theta_2\theta_B^2}, \quad \Theta_0 := \widehat{\theta_A^1\theta_A^2} \cup \widehat{\theta_B^1\theta_B^2}.$$

Here $\Theta_i$ $(i = 1, 2)$ approximately represents the set of polar angles for $a_i$ phase points on $\partial B_{R_0}$, where $\Theta_0$ represents the set of polar angles for the transition layer. The size of $\Theta_0$ can be controlled by

$$\frac{C_1}{R_0} \le |\Theta_0| \le \frac{C_2}{R_0\delta^2}. \tag{4.21}$$

Now we first define the following function on $\partial B_{R_0-1}$:

$$\tilde{u}(R_0-1,\theta) = \begin{cases} a_1, & \theta \in \Theta_1, \\ a_2, & \theta \in \Theta_2, \\ \text{smooth connection of } a_1,\ a_2 & \theta \in \Theta_0. \end{cases} \tag{4.22}$$

Then we extend the energy competitor $\tilde{u}$ to $B_{R_0}$,

$$\tilde{u}(r,\theta) := \begin{cases} u(R_0,\theta), & r = R_0 \\ (r-R_0+1)u(R_0,\theta) + (R_0-r)\tilde{u}(R_0-1,\theta), & r \in (R_0-1,R_0) \\ \tilde{u}(R_0-1,\theta), & r = R_0-1 \\ \text{energy minimizer}, & r \in [0,R_0-1) \end{cases} \tag{4.23}$$

where we require that $\tilde{u}$ minimizes the energy $E(\cdot, B_{R_0-1})$ with respect to the Dirichlet boundary constraint $\tilde{u}|_{B_{R_0-1}}$.

We first estimate the energy in the annulus $A_{R_0,R_0-1}$.

$$\begin{aligned} &\int_{A_{R_0,R_0-1}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz \\ &= \left(\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_1\}} + \int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_2\}} + \int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_0\}}\right)\left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz. \end{aligned}$$

From (4.21) and (3.9) it follows that

$$\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_0\}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz \le C(\delta, W).$$

For any $\theta_0 \in \Theta_1$, if $|u(R_0,\theta_0) - a_1| < \delta$, then

$$|\tilde{u}(r,\theta_0) - a_1| \le |u(R_0,\theta_0) - a_1| < \delta, \quad \forall r \in (R_0-1,R_0),$$

which implies

$$c_1|\tilde{u}(r,\theta_0) - a_1|^2 \le W(\tilde{u}(r,\theta_0)) \le CW(u(R_0,\theta_0)), \quad \text{for some } C = C(W),\ \forall r \in (R_0-1,R_0).$$

On the other hand if $|u(R_0,\theta_0)-a_1| \geq \delta$, together with the definition of $\Theta_1$ it holds that

$$|u(R_0,\theta_0)-a_i| \geq \delta, \forall i \in \{1,2,3\},$$

and

$$\max\{|\tilde{u}(r,\theta_0)-a_1|^2,\ W(\tilde{u}(r,\theta_0))\} \leq \frac{C}{\delta^2} W(u(R_0,\theta_0)), \quad \forall r \in (R_0-1,R_0).$$

Following the similar computation as in (4.16), (4.18), and (4.17), we obtain

$$\begin{aligned}
&\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_1\}} |\partial_r \tilde{u}|^2\, dz \\
=&\int_{R_0-1}^{R_0}\int_{\Theta_1} |u(R_0,\theta)-a_1|^2 r\, dr\, d\theta \\
\leq& C(\delta,W) R_0 \int_{\Theta_1} W(u(R_0,\theta))\, d\theta \\
\leq& C(\delta,W).
\end{aligned}$$

$$\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_1\}} |\partial_T \tilde{u}|^2\, dz \leq \int_{\partial B_{R_0}} |\partial_T \tilde{u}|^2\, d\mathcal{H}^1 \leq 3\sigma.$$

$$\begin{aligned}
&\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_1\}} W(\tilde{u})\, dz \\
\leq& C(\delta,W)\int_{R_0-1}^{R_0} dr \int_{\Theta_1} W(u(R_0,\theta)) r\, d\theta \leq C(\delta,W).
\end{aligned}$$

Summing up the inequalities above implies

$$\int_{A_{R_0,R_0-1}\cap\{\theta\in\Theta_1\}} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dz \leq C(\delta,W).$$

This estimate also applies to the energy on $A_{R_0,R_0-1}\cap\{\theta\in\Theta_2\}$. Consequently we have

$$\int_{A_{R_0,R_0-1}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \leq C(\delta,W), \tag{4.24}$$

We are left with the the estimation of $E(\tilde{u},B_{R_0-1})$. Set

$$P_1 := (R_0-1)e^{i\frac{\theta_A^1+\theta_A^2}{2}}, \quad P_2 := (R_0-1)e^{i\frac{\theta_B^1+\theta_B^2}{2}}.$$

Then $P_1P_2$ intersects with $\partial B_{R_0-2}$ at two points $P_1'$ and $P_2'$, respectively. Up to a possible rotation, we can assume $P_1P_2$ is parallel to the $x$-axis and their shared $y$ coordinates is denoted by $y_0$. Without loss of generality, we suppose $y_0 \leq 0$, with the $a_2$ phase part of $\partial B_{R_0-1}$ positioned above $\{y=y_0\}$, while the $a_1$ phase part is positioned below $\{y=y_0\}$ (see Figure 1).

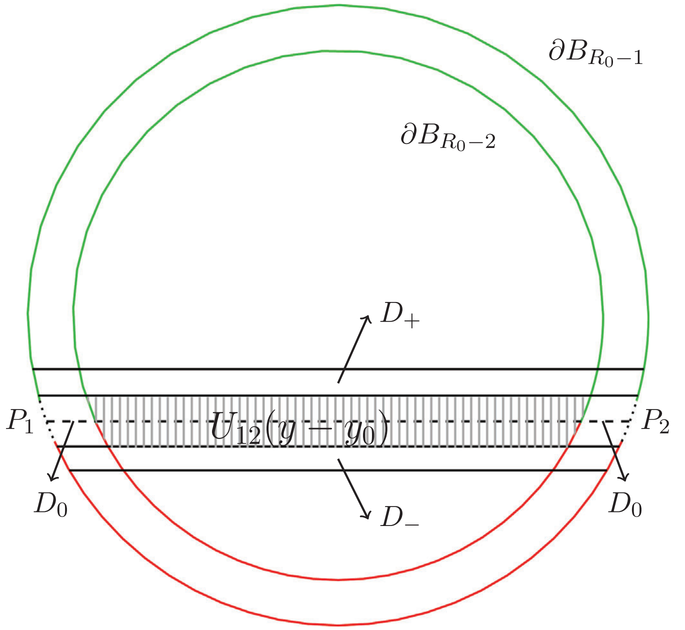


**FIGURE 1** Energy upper bound in $B_{R_0-1}$ for Case 2. Red: $a_1$; Green: $a_2$.

We pick an 1D minimizing connection $U_{12}$ defined in the Hypothesis (H2). Moreover, it is well-known that $U_{ij}$ will converge to $a_1$ or $a_2$ exponentially (see e.g., [2, Proposition 2.4]),

$$|U_{12}(\eta)-a_2|\le Ke^{-k\eta},\quad |U_{12}(\eta)-a_1|\le Ke^{k\eta},$$

for some constants $K$, $k>0$. Let $h=R_0^{\frac{1}{3}}$. If $y_0\le -R_0+3h$, elementary geometry implies that $\mathcal{H}^1(\widehat{z_A^1 z_B^1})\sim O(R_0^{\frac{2}{3}})$, which means the $a_1$ phase on $\partial B_{R_0-1}$ has much smaller measure compared to that of the $a_2$ phase. Hence we can invoke similar analysis as in Case 1 to show that $E(\tilde{u},B_{R_0-1})\le CR_0^{\frac{2}{3}}\ll O(R_0)$. This, together with (4.24), yields a contradiction with (3.13). Therefore, we only consider the case $y_0>-R_0+3h$. Set

$$v(x,y)=\begin{cases}U_{12}(y-y_0), & (x,y)\in \overline{B}_{R_0-2}\cap\{y_0-h\le y\le y_0+h\},\\ a_2, & (x,y)\in B_{R_0-1}\cap\{y\ge y_0+2h\},\\ a_1, & (x,y)\in B_{R_0-1}\cap\{y\le y_0-2h\}.\end{cases}$$

On $B_{R_0-2}\cap\{y=y_0-h\}$ (or $B_{R_0-2}\cap\{y=y_0+h\}$), by the exponential decay of $U_{12}$ we have

$$|v(x,y)-a_1\,(\text{ or }a_2)|\le Ke^{-kR_0^{\frac{1}{3}}}.$$

And for $(x,y)\in D_0:=A_{R_0-1,R_0-2}\cap\{y_0-h\le y\le y_0+h\}$, we take $v(x,y)$ to linearly interpolate in $x$ between $v|_{\partial B_{R_0-1}}$ and $v|_{\partial B_{R_0-2}}$ for each $y$. Note that $D_0$ consists of two small regions with measure controlled by $O(R_0^{\frac{1}{3}})$. For $(x,y)\in D_+:=B_{R_0-1}\cap\{y_0+h\le y\le y_0+2h\}$ and $(x,y)\in D_-:=B_{R_0-1}\cap\{y_0-2h\le y\le y_0-h\}$, let $v(x,y)$ linearly interpolate in $y$ between $v(x,y_0\pm 2h)$ and

$v(x, y_0 \pm h)$. See Figure 1 for an illustration for all these subdomains. We estimate energy of $v$:

$$E(v, \overline{B}_{R_0-2} \cap \{y_0 - h \le y \le y_0 + h\}) \le \sigma * \mathcal{H}^1(P_1P_2) \le 2R_0\sigma,$$

$$E(v, B_{R_0-1} \cap \{y \ge y_0 + 2h\} \cap \{y \le y_0 - 2h\}) = 0,$$

$$E(v, D_0) \le C\mathcal{H}^2(D_0) \le CR_0^{\frac{1}{3}},$$

$$E(v, D_\pm) \le Ch \cdot \mathcal{H}^1(P_1P_2) \cdot K^2 e^{-2kR_0^{\frac{1}{3}}} = o(R_0),$$

given $R_0$ chosen to be sufficiently large. Altogether,

$$E(v, B_{R_0-1}) \le 2R_0\sigma + o(R_0).$$

Utilizing the minimality of $\tilde{u}$ in $B_{R_0-1}$ and (4.24), we obtain

$$\int_{B_{R_0}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz \le 2R_0\sigma + o(R_0),$$

which yields a contradiction with (3.12) when we select $R_0$ to be large enough. Therefore Case 2 is also eliminated and we get

$$Y_i \neq \emptyset, \quad \forall i \in \{1, 2, 3\}.$$

The proof of Lemma 4.1 is complete. □

From Lemma 4.1, there are $z_1$, $z_2$, $z_3 \in \partial B_{R_0}$ such that

$$|u(z_i) - a_i| \le \delta, \quad \forall i \in \{1, 2, 3\}.$$

Next we apply the same analysis for the two-phase scenario to obtain the existence of three arcs $\overset{\frown}{z_l^1 z_r^1}$, $\overset{\frown}{z_l^2 z_r^2}$, $\overset{\frown}{z_l^3 z_r^3}$ that satisfy the following properties,

(1) $z_i \in \overset{\frown}{z_l^i z_r^i}$, for $i = 1, 2, 3$.
(2) $|u(z_l^i) - a_i| \le \delta$, $|u(z_r^i) - a_i| \le \delta$, for $i = 1, 2, 3$.
(3) $\overset{\frown}{z_l^1 z_r^1}$, $\overset{\frown}{z_l^2 z_r^2}$, $\overset{\frown}{z_l^3 z_r^3}$ are disjoint. For each $i \in \{1, 2, 3\}$, we have

$$z \in \overset{\frown}{z_l^i z_r^i} \Rightarrow |u(z) - a_j| > \delta, \ \forall j \neq i.$$

(4) Suppose $\overset{\frown}{z_r^1 z_l^2}$, $\overset{\frown}{z_r^2 z_l^3}$, $\overset{\frown}{z_r^3 z_l^1}$ are the three small arcs that separate the arcs $\overset{\frown}{z_l^1 z_r^1}$, $\overset{\frown}{z_l^2 z_r^2}$ and $\overset{\frown}{z_l^3 z_r^3}$. It holds that

$$|u(z) - a_i| > \delta, \quad \forall i \in \{1, 2, 3\}, \ \forall z \in \overset{\frown}{z_r^1 z_l^2} \cup \overset{\frown}{z_r^2 z_l^3} \cup \overset{\frown}{z_r^3 z_l^1},$$

And consequently,

$$C_1 \le \mathcal{H}^1(\overset{\frown}{z_r^1 z_l^2} \cup \overset{\frown}{z_r^2 z_l^3} \cup \overset{\frown}{z_r^3 z_l^1}) \le \frac{C_2}{\delta^2} \tag{4.25}$$

We introduce the following notations for convenience.

$$
\begin{aligned}
&I_i := \widehat{z_l^i z_r^i} \text{ denotes the set of phase } a_i \text{ on } \partial B_{R_0}, i = 1,2,3;\\
&I_{12} = \widehat{z_r^1 z_l^2},\ I_{23} = \widehat{z_r^2 z_l^3},\ I_{31} = \widehat{z_r^3 z_l^1} \text{ denote transitional arcs between phases,}\\
&d_1 := \min\{\mathcal{H}^1(I_{ij}),\ i \neq j \in \{1,2,3\}\}, \quad d_2 := \max\{\mathcal{H}^1(I_{ij}),\ i \neq j \in \{1,2,3\}\}.
\end{aligned}
$$

Then (4.25) implies

$$
\frac{C_1}{3} \le d_1 \le d_2 \le \frac{C_2}{\delta^2}. \tag{4.26}
$$

By Property (2), there will be an approximate "phase transition" inside each of the separating arcs $I_{12}, I_{23}, I_{31}$, and the energy on these arcs is estimated by

$$
\int_{I_{12}\cup I_{23}\cup I_{31}} \left(\frac{1}{2}|\partial_T u|^2 + W(u)\right) d\mathcal{H}^1 \ge 3\sigma - C\delta^2, \tag{4.27}
$$

for some constant $C = C(W)$. Next we show that on each arc $I_i$, $u(z)$ will be uniformly close to $a_i$, with the distance controlled by $C\delta$.

**Lemma 4.2.** *There exists a constant $C$ which only depends on $W$, such that for any $i \in \{1,2,3\}$ and $z \in I_i$, it holds that*

$$
|u(z) - a_i| < C\delta.
$$

*Proof.* Let $C$ be a large constant to be determined later. Without loss of generality, we suppose by contradiction there exists $z^1 \in I_1$ such that $|u(z^1) - a_1| = C\delta$ and

$$
|u(z^1) - a_1| = \max\{|u(z) - a_1|,\ z \in I_1\}.
$$

We can choose suitable constants $\delta$ and $C$ satisfying

$$
C\delta < \delta_W, \tag{4.28}
$$

where $\delta_W$ is the constant in Lemma 2.1. As a result, for any $z \in \widehat{z_l^1 z^1}$, it holds that

$$
W(u(z)) \ge \frac{1}{2} c_W |u(z) - a_1|^2.
$$

We compute

$$\begin{aligned}
&\int_{\widehat{z_l^1 z^1}} \left(\frac{1}{2}|\partial_T u|^2 + W(u)\right) d\mathcal{H}^1 \\
\geq &\int_{\widehat{z_l^1 z^1}} \left(\frac{1}{2}|\partial_T |u(z)-a_1||^2 + \frac{1}{2}c_W|u(z)-a_1|^2\right) d\mathcal{H}^1 \\
\geq &\int_{\widehat{z_l^1 z^1}} \left(\sqrt{c_W}\cdot\frac{1}{2}\partial_T(|u(z)-a_1|^2)\right) d\mathcal{H}^1 \\
\geq &\frac{\sqrt{c_W}}{2}[(C\delta)^2-\delta^2] \\
= &\frac{\sqrt{c_W}(C^2-1)}{2}\delta^2.
\end{aligned}$$

Combining this with (4.27), we can select a sufficiently large $C$ (which does not affect (4.28) as we still have the freedom to choose $\delta$) to obtain

$$\begin{aligned}
&\int_{\partial B_{R_0}} \left(\frac{1}{2}|\partial_T u|^2 + W(u)\right) d\mathcal{H}^1 \\
\geq &\int_{I_{12}\cup I_{23}\cup I_{31}\cup \widehat{z_l^1 z^1}} \left(\frac{1}{2}|\partial_T u|^2 + W(u)\right) d\mathcal{H}^1 \\
\geq &3\sigma + C(W)\delta^2,
\end{aligned}$$

which yields a contradiction with (3.12), as $\varepsilon$ can be chosen arbitrarily small in the beginning. This completes the proof of Lemma 4.2. □

## 5 | REFINED ENERGY UPPER/LOWER BOUND IN $B_{R_0}$

We take $A, B, C$ to be the midpoints of the arcs $I_{12}$, $I_{13}$ and $I_{23}$, respectively. There exists a point $D \in \bar{B}_{R_0}$ such that $|DA| + |DB| + |DC|$ is minimized. It is well-known that if the triangle $ABC$ possesses internal angles that are all less than $\frac{2\pi}{3}$, then $D$ is the point in the interior of $ABC$ such that

$$\angle ADB = \angle BDC = \angle ADC = \frac{2\pi}{3}.$$

However, if one angle, say $\angle BAC$ is greater than or equal to $\frac{2\pi}{3}$, then $D$ coincide with the vertex $A$ of this triangle.

We define the triod

$$T_{ABC} = DA \cup DB \cup DC.$$

Then we invoke the energy upper bound established in [11, formula (3.19)], which is written in the following proposition within our specific context.

**Proposition 5.1** (Energy upper bound). *There exist $C > 0$ and $\alpha \in (0,1)$, both of which are independent of $\varepsilon$ and $R_0$, such that when $R_0$ is sufficiently large, the following energy upper bound holds:*

$$\int_{B_{R_0}} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dz \leq \sigma(|DA| + |DB| + |DC|) + CR_0^{\alpha}. \tag{5.29}$$

*Proof.* The proof follows essentially the proof of [11, Proposition 3.3] (Upper bound construction part), and therefore, it is omitted herein. We only briefly outline the core idea.

To construct an energy competitor $\tilde{u}$, we first define a well-behaved boundary data on $\partial B_{R1}$ such that $\tilde{u}$ equals to $a_i$ on the rescaled arc $\frac{R_1}{R_0} I_i$ for any $i = 1, 2, 3$, where $R_1 = R_0 - R_0^{\alpha}$. On the annulus $A_{R_0,R_1}$, let $\tilde{u}$ linearly interpolate between $\tilde{u}|_{\partial B_{R_0}}$ and $\tilde{u}|_{\partial B_{R_1}}$ with the energy $E(\tilde{u}, A_{R_0,R_1})$ controlled by $C(R_0^{\alpha} + R_0^{1-\alpha})$. For the construction inside $B_{R_1}$, near the interface, say $DA$, we set $\tilde{u}(z) = U_{12}(d(z, DA))$ within a thin rectangle with a width of $R_0^{\alpha}$ and the long side parallel to $DA$. Here $d(\cdot)$ is the signed distance function. The energy within these rectangles will be approximately $\sigma(|DA| + |DB| + |DC|)$. These three thin rectangles partition $B_{R-1}$ into three subdomains, within which we simply take $\tilde{u}$ equal to phase $a_i$ corresponding to the boundary data. Again, interpolations are required in a $R^{\alpha}$-outer layer of the rectangles, with the energy being proved to be negligible. This construction parallels that of $v(x, y)$ in $B_{R_0-1}$ for the upper bound estimate of Case 2 in the proof of Lemma 4.1.

We also mention that a similar approach to construct the energy competitor also appears in [3, Appendix A], for the special case when $D$ coincides with the origin. □

**Corollary 5.2.** *By selecting sufficiently small $\varepsilon$ and sufficiently large $R_0$ in Section 3, we can guarantee that the point $D$ is located in a small neighborhood of the origin, that is, $D \in B_{\tau R_0}$, where $\tau = \min\{\frac{1}{100}, \frac{C_1}{20}\}$ for the constant $C_1$ in (4.25). Furthermore, it follows that $D$ locates in the interior of the triangle $ABC$ and $\angle ADB = \angle BDC = \angle ADC = \frac{2\pi}{3}$.*

*Remark* 5.1. We will explain the choice of $\tau$ in the proof of Proposition 5.3.

*Proof.* Assume by contradiction $D \notin B_{\tau R_0}$, then an elementary calculation implies there exists a positive constant $\mu \sim O(\tau^2)$ such that

$$|DA| + |DB| + |DC| < (3 - \mu)R_0.$$

This estimate, together with (5.29). yields a contradiction with the lower bound in (3.13) provided $R_0$ is sufficiently large. □

Next we establish the lower bound for $E(u, B_{R_0})$.

**Proposition 5.3.** *There exists a constant $C(W)$ such that*

$$\int_{B_{R_0}} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dz \geq \sigma(|DA| + |DB| + |DC|) - CR_0^{\frac{2}{3}}. \tag{5.30}$$

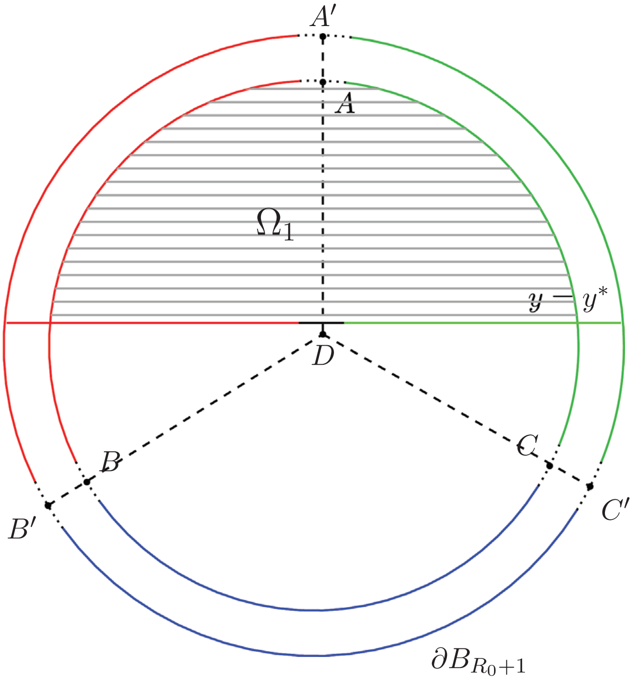


**FIGURE 2** Triod $T_{ABC}$ with the center $D$ close to origin, and its extension to $T_{A'B'C'} \in B_{R_0+1}$.

*Proof.* Up to a possible rotation, we assume $\overrightarrow{DA}$ is parallel to the positive $y$–axis, $\overrightarrow{DB}$ is parallel to the vector $(-\sqrt{3},-1)$ and $\overrightarrow{DC}$ is parallel to the vector $(\sqrt{3},-1)$, as shown in Figure 2. We set

$$A=(x_A,y_A),\ B=(x_B,y_B),\ C=(x_C,y_C),\ D=(x_D,y_D).$$

By Corollary 5.2,

$$|OD|=\sqrt{x_D^2+y_D^2}<\tau R_0,\quad \text{for } \tau=\min\{\frac{1}{100},\frac{C_1}{20}\}.$$

Without loss of generality we assume that

$$x_D=x_A\geq 0,$$

and therefore by simple geometry it holds that

$$y_C\geq y_B.$$

Now we define an extension $\tilde{u}$ of $u|_{B_{R_0}}$ to a larger ball $B_{R_0+1}$, which satisfies a simpler boundary condition on $\partial B_{R_0+1}$. We set $\tilde{I}_i,\tilde{I}_{ij}$ as the image of arcs $I_i,I_{ij}$ under the homothetic transformation: $z\to\frac{R_0+1}{R_0}z$, that is,

$$\tilde{I}_i:=\{z\in\partial B_{R_0+1}:\ \frac{R_0}{R_0+1}z\in I_i\},\quad \forall i\in\{1,2,3\},$$

$$\tilde{I}_{ij}:=\{z\in\partial B_{R_0+1}:\ \frac{R_0}{R_0+1}z\in I_{ij}\},\quad \forall i\neq j\in\{1,2,3\}.$$

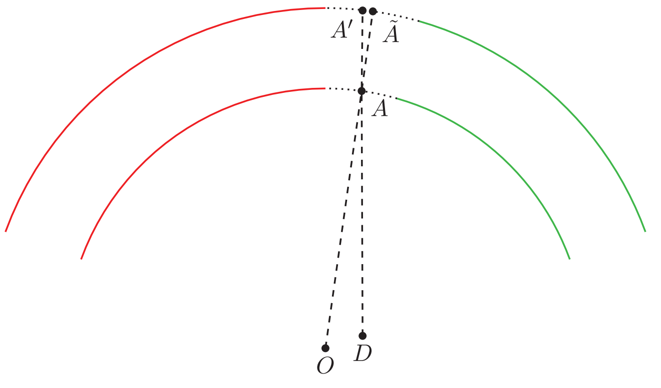


**FIGURE 3** $A'$ locates within $\tilde{I}_{12}$.

The boundary value for $\tilde{u}$ on $\partial B_{R_0+1}$ is given by

$$\tilde{u}(z) = \begin{cases} a_i, & z \in \tilde{I}_i, \\ \text{smooth connection of } a_i,\ a_j, & z \in \tilde{I}_{ij}. \end{cases} \tag{5.31}$$

We define $\tilde{u}$ in the annulus $A_{R_0+1,R_0}$ by linearly interpolating between $\tilde{u}|_{\partial B_{R_0+1}}$ and $u|_{\partial B_{R_0}}$,

$$\tilde{u}(r,\theta) = (r - R_0)\tilde{u}(R_0+1,\theta) + (R_0+1-r)u(R_0,\theta), \quad r \in (R_0, R_0+1).$$

Following the same computation as in the previous estimate of energy for the two-phase boundary data (see Case 2 in the proof of Lemma 4.1), we can obtain the following upper bound for the energy in $A_{R_0+1,R_0}$:

$$\int_{A_{R_0+1,R_0}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \le C(\delta, W).$$

Consequently, in order to get (5.30), it suffices to prove

$$\int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \ge \sigma(|DA| + |DB| + |DC|) - CR_0^{\frac{2}{3}}, \tag{5.32}$$

with the boundary value (5.31). It follows from Corollary 5.2 that $\tilde{u}$ equals to $a_1, a_2, a_3$, respectively on three arcs of $\partial B_{R_0+1}$ with the arc angle close to $\frac{2\pi}{3}$. Moreover, these arcs are separated by small arcs of size $\frac{C}{\delta^2}$ due to the estimate (4.25).

Next we prove the lower bound (5.32). The argument essentially follows the proof of [3, Proposition 3.1], with necessary adjustments tailored to accommodate the scenario where $D$ is not the origin. Firstly, we extend $DA$ to intersect with $\partial B_{R_0+1}$ at $A'$. We claim that $A' \in \tilde{I}_{12}$. Take $\tilde{A}$ to be the midpoint of $\tilde{I}_{12}$ ($\tilde{A}$ is the intersection of the ray $\overrightarrow{OA}$ and $\partial B_{R_0+1}$). As illustrated in Figure 3, it follows from Corollary 5.2 and (4.26) that

$$|A'\tilde{A}| \le |OD| \cdot \frac{2}{R_0} \le 2\tau \le \frac{C_1}{10} \le \frac{3d_1}{10}.$$

which implies that $A' \in \tilde{I}_{12}$ since $\mathcal{H}^1(\tilde{I}_{12}) \geq d_1$. Similarly, we extend $DB$, $DC$ to intersect with $\partial B_{R_0+1}$ at $B' \in \tilde{I}_{13}$ and $C' \in \tilde{I}_{23}$, respectively.

The coordinates for $A', B', C'$ are denoted as

$$A' = (x'_A, y'_A), \quad B' = (x'_B, y'_B), \quad C' = (x'_C, y'_C),$$

and satisfy

$$x'_A = x_D = x_A, \quad y'_C \geq y'_B,$$
$$\frac{y'_B - y_D}{x'_B - x_D} = \frac{1}{\sqrt{3}}, \quad \frac{y'_C - y_D}{x'_C - x_D} = -\frac{1}{\sqrt{3}}.$$

For any $y \in (y'_C, R_0 + 1)$, we define the horizontal line segment $\gamma_y$ and functions $\lambda_1(y)$, $\lambda_2(y)$ and $\lambda_3(y)$ as follows.

$$\gamma_y := \{(x, y) : x \in \mathbb{R}\} \cap B_{R_0+1},$$
$$\lambda_i(y) := \mathcal{H}^1(\gamma_y \cap \{|\tilde{u}(x, y) - a_i| \leq R_0^{-\frac{1}{6}}\}), \quad i \in \{1, 2, 3\}.$$

By the boundary condition on $\partial B_{R_0+1}$ we know for any $y \in [y'_C + d_2, y'_A - d_2]$, it holds that $\lambda_1(y) > 0$ and $\lambda_2(y) > 0$.

We set

$$y^* := \min\{y \in [y'_C + d_2, R_0 + 1] : \lambda_1(y) + \lambda_2(y) \geq \mathcal{H}^1(\gamma_y) - R_0^{\frac{2}{3}}\}, \tag{5.33}$$
$$\zeta(x) := \min\{y^*, \sqrt{(R_0 + 1)^2 - x^2}\},$$
$$K := \{x \in [x'_B + d_2, x'_C - d_2] : |\tilde{u}(x, \zeta(x)) - a_i| < R_0^{-\frac{1}{6}}, \ i = 1 \text{ or } 2\},$$
$$L := \{y \in [y'_C + d_2, y^*] : \lambda_3(y) > 0\},$$
$$\Omega_1 := \{z = (x, y) \in B_{R_0+1} : y \geq y^*\},$$
$$\Omega_2 := \{z = (x, y) \in B_{R_0+1} : y < y^*\}.$$

Note that when $(R_0 + 1)^2 - y^2 < \frac{R_0^{\frac{4}{3}}}{4}$, $\mathcal{H}^1(\gamma_y) - R_0^{\frac{2}{3}} < 0$, and therefore the set on which we minimize to get $y^*$ is not empty. We will examine two scenarios based on the value of $y^*$. Additionally, we can bound the measure of $K$ from below by the definition of $y^*$ and the boundary constraint $\tilde{u}|_{\partial B_{R_0+1}}$:

$$\mathcal{H}^1(K) \geq x'_C - x'_B - 2d_2 - R_0^{\frac{2}{3}}. \tag{5.34}$$

*Case* 1. $y * < y'_A - d_2$.

For any $y \in [y^*, y'_A - d_2)$, the horizontal line $\gamma_y$ intersects $\partial B_{R_0+1}$ at two points, where $\tilde{u}$ takes the value $a_1$ and $a_2$, respectively. The one dimensional energy estimate Lemma 2.2 yields

$$\int_{\gamma_y} \left( \frac{1}{2} |\partial_x \tilde{u}|^2 + W(\tilde{u}) \right) dx \geq \sigma.$$

Integrating along the vertical direction implies

$$\int_{\Omega_1}\left(\frac{1}{2}|\partial_x\tilde{u}|^2+W(\tilde{u})\right)dz\geq\sigma(y'_A-y^*-d_2)\geq\sigma(y'_A-y^*)-C(\delta). \tag{5.35}$$

On the domain $\Omega_2$, we claim that there exists a constant $C$ such that

$$\mathcal{H}^1([y'_C+d_2,y^*]\setminus L)<CR_0^{\frac{2}{3}}.$$

Indeed, we set

$$S:=\{y\in[y'_C+d_2,y^*]:\lambda_3(y)=0\}=[y'_C+d_2,y^*]\setminus L.$$

For $y\in S$, the definitions of $y^*$ and $S$ imply that $\lambda_1(y)+\lambda_2(y)+\lambda_3(y)<\mathcal{H}^1(\gamma_y)-R_0^{\frac{2}{3}}$, or equivalently

$$\mathcal{H}^1(\{x\in[-\sqrt{(R_0+1)^2-y^2},\sqrt{(R_0+1)^2-y^2}]:|\tilde{u}(x,y)-a_i|>R_0^{-\frac{1}{6}},\ \forall i\})>R_0^{\frac{2}{3}}.$$

From the energy upper bound (5.29) and Lemma 2.1 we have that

$$\begin{aligned}4R_0&\geq\int_S\int_{\gamma_y}W(\tilde{u})\,dxdy\geq\mathcal{H}^1(S)\cdot(\frac{1}{2}c_WR_0^{-\frac{1}{3}})\cdot R_0^{\frac{2}{3}}\\&\Rightarrow\mathcal{H}^1(S)\leq CR_0^{\frac{2}{3}},\end{aligned}$$

for some constant $C$ depending on $W$. The claim is established.

Next we want to estimate the energy in $\Omega_2$ in both vertical and horizontal directions. We split the potential $W(\tilde{u})$ as

$$W(\tilde{u})=\sin^2\theta\,W(\tilde{u})+\cos^2\theta\,W(\tilde{u}),$$

for some $\theta\in[0,\frac{\pi}{2}]$ to be determined.

For $x\in K$, $A(x)=(x,\zeta(x))$ and $B(x)=(x,-\sqrt{(R_0+1)^2-x^2})$ are the two intersection points of the vertical line $\{(x,y):y\in\mathbb{R}\}$ with $\partial\Omega_2$. We have $\tilde{u}(B(x))=a_3$ by the boundary condition and $|\tilde{u}(A(x))-a_i|\leq R_0^{-\frac{1}{6}}$ for $i=1$ or $2$ by the definition of $K$. Applying Lemma 2.2 and then integrating with respect to $x$ yields

$$\begin{aligned}&\int_{\Omega_2}\left(\frac{1}{2}|\partial_y\tilde{u}|^2+\sin^2\theta\,W(\tilde{u})\right)dz\\&\geq\sin\theta\int_{x\in K}\int_{-\sqrt{(R_0+1)^2-x^2}}^{\zeta(x)}\left(\frac{1}{2\sin\theta}|\partial_y\tilde{u}|^2+\sin\theta\,W(\tilde{u})\right)dydx\\&\geq\sin\theta(x'_C-x'_B-2d_2-R_0^{\frac{2}{3}})(\sigma-CR_0^{-\frac{1}{3}})\\&\geq\sin\theta\left[(x'_C-x'_B)\sigma-CR_0^{\frac{2}{3}}\right].\end{aligned} \tag{5.36}$$

Here we utilize $x'_C - x'_B \sim \sqrt{3}R_0$ and $d_2 \le \frac{C}{\delta^2} \ll R_0^{\frac{2}{3}}$ to derive the last inequality.

By definition, for any $y \in L$,

$$\tilde{u}(-\sqrt{(R_0+1)^2 - y^2}, y) = a_1, \quad \tilde{u}(\sqrt{(R_0+1)^2 - y^2}, y) = a_2,$$

$$\exists\,(x_0, y) \in \gamma_y, \ \text{such that } |\tilde{u}(x_0, y) - a_3| \le R_0^{-\frac{1}{6}},$$

which implies that there are approximately two phase transitions along $\gamma_y$. Thus we can estimate

$$\begin{aligned}
&\int_{\Omega_2\cap\{y\in L\}} \left(\frac{1}{2}|\partial_x \tilde{u}|^2 + \cos^2\theta\, W(\tilde{u})\right) dz \\
\ge &\cos\theta \int_{y\in L}\int_{\gamma_y} \left(\frac{1}{2\cos\theta}|\partial_x \tilde{u}|^2 + \cos\theta\, W(\tilde{u})\right) dxdy \\
= &\cos\theta \int_{y\in L} \left\{\int_{-\sqrt{(R_0+1)^2-y^2}}^{x_0} + \int_{x_0}^{\sqrt{(R_0+1)^2-y^2}}\right\} \left(\frac{1}{2\cos\theta_2}|\partial_x \tilde{u}|^2 + \cos\theta\, W(u)\right) dx \\
\ge &\cos\theta \cdot (y^* - y'_C - d_2 - CR_0^{\frac{2}{3}}) \cdot (2\sigma - CR_0^{-\frac{1}{3}}) \\
\ge &2\cos\theta (y^* - y'_C)\sigma - CR_0^{\frac{2}{3}},
\end{aligned} \tag{5.37}$$

where the last inequality follows from $y^* - y'_C < 2R_0$ and $d_2 \ll R_0^{\frac{2}{3}}$.

In case $y'_B + d_2 \ge y'_C - d_2$, we directly proceed with the above estimate (5.37) to derive (5.39). Otherwise, if $y'_B + d_2 < y'_C - d_2$, for any $y \in (y'_B + d_2, y'_C - d_2)$, $\gamma_y$ will intersect with $\partial B_{R_0+1}$ at two points where $\tilde{u}$ equals to $a_1$, $a_3$, respectively. We have that

$$\begin{aligned}
&\int_{\Omega_2\setminus\{y\in L\}} \left(\frac{1}{2}|\partial_x \tilde{u}|^2 + \cos^2\theta\, W(\tilde{u})\right) dxdy \\
\ge &\cos\theta \int_{y'_B+d_2}^{y'_C-d_2}\int_{\gamma_y} \left(\frac{1}{2\cos\theta}|\partial_x \tilde{u}|^2 + \cos\theta\, W(\tilde{u})\right) dxdy \\
= &\cos\theta\, (y'_C - y'_B - 2d_2)\sigma \\
\ge &\cos\theta\, (y'_C - y'_B)\sigma - C.
\end{aligned} \tag{5.38}$$

Adding (5.37) and (5.38) gives

$$\int_{\Omega_2} \left(\frac{1}{2}|\partial_x \tilde{u}|^2 + \cos^2\theta\, W(\tilde{u})\right) dz \ge \cos\theta\,(2y^* - y'_B - y'_C) - CR_0^{\frac{2}{3}}. \tag{5.39}$$

This, together with (5.36), implies that

$$\int_{\Omega_2} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \ge \left[\sin\theta\,(x'_C - x'_B) + \cos\theta(2y^* - y'_B - y'_C)\right]\sigma - CR_0^{\frac{2}{3}},$$

holds for any $\theta \in [0, \frac{\pi}{2}]$. Taking $\theta = \arctan \frac{x'_C - x'_B}{2y^* - y'_B - y'_C}$ to maximize the right-hand side, we obtain

$$\int_{\Omega_2} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \geq \sigma \cdot \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2} - CR_0^{\frac{2}{3}}. \tag{5.40}$$

Combining (5.35) and (5.40) gives that

$$\begin{aligned}
&\int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \\
&\geq \sigma\left[(y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}\right] - CR_0^{\frac{2}{3}}.
\end{aligned} \tag{5.41}$$

We are left with solving the following minimization problem

$$\begin{aligned}
\min \quad & (y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}, \\
\text{subject to} \quad & y^* \in (y'_C + d_2, y'_A - d_2).
\end{aligned} \tag{5.42}$$

Direct calculation shows that

$$\min \left\{(y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}\right\} = |DA'| + |DB'| + |DC'|,$$

and the minimum is reached at

$$y^* = y_D.$$

The calculation is elementary and provided in Appendix A. Therefore (5.30) holds for the case $y^* < y'_A - d_2$.

*Case* 2. $y^* \geq y'_A - d_2$. We show that in this case the energy is strictly larger than $\sigma(|DA'| + |DB'| + |DC'|)$. Split

$$W(\tilde{u}) = \sin^2\theta\, W(\tilde{u}) + \cos^2\theta\, W(\tilde{u}), \text{ for some } \theta \in (0, \frac{\pi}{2}).$$

By the boundary data we have

$$\begin{aligned}
&\tilde{u}(x, \sqrt{(R_0+1)^2 - x^2}) = a_1,\ \tilde{u}(x, -\sqrt{(R_0+1)^2 - x^2}) = a_3,\ \forall x \in [x_{B'} + d_2, x'_A - d_2], \\
&\tilde{u}(x, \sqrt{(R_0+1)^2 - x^2}) = a_2,\ \tilde{u}(x, -\sqrt{(R_0+1)^2 - x^2}) = a_3,\ \forall x \in [x'_A + d_2, x'_C - d_2],
\end{aligned}$$

which allows us to estimate the energy in the vertical direction,

$$\begin{aligned} &\int_{B_{R_0+1}} \left(\frac{1}{2}|\partial_y \tilde{u}|^2 + \sin^2\theta\, W(\tilde{u})\right) dz \\ \geq &\int_{x_{B'}+d_2}^{x'_A-d_2} \sigma \sin\theta\, dx + \int_{x_{A'}+d_2}^{x'_C-d_2} \sigma\sin\theta\, dx \\ \geq &\sigma \sin\theta\,(x'_C - x'_B) - C. \end{aligned} \tag{5.43}$$

Applying the same argument as in the claim of Case 1, we have that most of $y \in [y'_C + d_2, y'_A - d_2]$ belong to $L$, more precisely,

$$\mathcal{H}^1([y'_C + d_2, y'_A - d_2] \setminus L) < CR_0^{\frac{2}{3}}.$$

For $y \in L \cap [y'_C + d_2, y'_A - d_2]$, by definition there are two phase transitions: from $a_1$ to $a_3$ and from $a_3$ to $a_2$. We perform a similar computation as in (5.37) to obtain

$$\begin{aligned} &\int_{B_{R_0+1}\cap\{y\in L\}} \left(\frac{1}{2}|\partial_x\tilde{u}|^2 + \cos^2\theta\, W(\tilde{u})\right) dxdy \\ \geq &\cos\theta \int_{y\in L\cap[y'_C+d_2,y'_A-d_2]} \int_{\gamma_y} \left(\frac{1}{2\cos\theta}|\partial_x\tilde{u}|^2 + \cos\theta\, W(\tilde{u})\right) dxdy \\ \geq &2\cos\theta(y'_A - y'_C)\sigma - CR_0^{\frac{2}{3}}. \end{aligned} \tag{5.44}$$

Adding (5.43) and (5.44) and maximizing with respect to $\theta$ yield

$$\begin{aligned} \int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz &\geq \sigma\sqrt{(x'_C - x'_B)^2 + 4(y'_A - y'_C)^2} - CR_0^{\frac{2}{3}} \\ &\geq \frac{49}{50} 2\sqrt{3}R_0\sigma. \end{aligned} \tag{5.45}$$

where the last line follows from Corollary 5.2 which implies that $x'_C - x'_B \geq (1-2\tau)\sqrt{3}R_0$ and $y'_A - y'_C \geq (1-2\tau)\frac{3R_0}{2}$. Note that

$$|DA'| + |DB'| + |DC'| \leq 3R_0 + 3,$$

which together with (5.45) leads to

$$\int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz > \sigma(|DA'| + |DB'| + |DC'|).$$

This completes the proof of Proposition 5.3. □

At the end of this section, we will derive that $y^*$ is not far from $y_D$. Set

$$\beta := 1 - \min\left\{\frac{1}{6}, \frac{1-\alpha}{2}\right\} \in (0,1), \tag{5.46}$$

where $\alpha \in (0,1)$ is the constant in Proposition 5.1. Then we have the following lemma:

**Lemma 5.4.** *There is a constant $C$ depending on $W$, such that*

$$|y^* - y_D| \le CR_0^{\beta},$$

*where $y^*$ is defined in (5.33).*

*Proof.* By examining the proof of Proposition 5.3, we have

$$\int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz \le \int_{B_{R_0}} \left(\frac{1}{2}|\nabla \tilde{u}|^2 + W(\tilde{u})\right) dz + C(\delta, W).$$

This together with the upper bound (5.29) and (5.41) implies

$$\begin{aligned} &\sigma\left[(y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}\right] \\ &\qquad \le \sigma(|DA| + |DB| + |DC|) + CR_0^{\alpha} + CR_0^{\frac{2}{3}}. \end{aligned} \tag{5.47}$$

From Appendix A, the left-hand side attains its minimum value $\sigma(|DA'| + |DB'| + |DC'|)$ when $y^* = y_D$. When the functional is perturbed by a small quantity $R_0^{\alpha} + R_0^{\frac{2}{3}}$, it is straightforward to show that $y^*$ can be perturbed away from $y_D$ by an amount no greater than $CR_0^{\beta}$, where $\beta$ is defined by (5.46). □

## 6 | LOCALIZATION OF THE DIFFUSE INTERFACE

We insist on the choice of $A, B, C, D$ at the beginning of Section 5, and assume without loss of generality that $\overrightarrow{DA}$ is parallel to the positive $y$-axis. Then the triod defined by

$$T_{ABC} = DA \cup DB \cup DC$$

divides $B_{R_0}$ into three regions:

$$\begin{aligned} \mathcal{D}_1 &:= \text{the region enclosed by } DA,\ DB,\ \partial B_{R_0}; \\ \mathcal{D}_2 &:= \text{the region enclosed by } DA,\ DC,\ \partial B_{R_0}; \\ \mathcal{D}_3 &:= \text{the region enclosed by } DB,\ DC,\ \partial B_{R_0}. \end{aligned}$$

For any $\gamma > 0$, we define the diffuse interface $\mathcal{I}_\gamma$ by

$$\mathcal{I}_\gamma := \{z : |u(z) - a_i| \ge \gamma,\ \forall\, i = 1, 2, 3\}.$$

From now on we fix $\gamma = 2C\delta$, where $C = C(W)$ is the constant in Lemma 4.2 which measures the closeness of $u$ to $a_i$ on arc $I_i$. Note that $C$ is independent of $\delta$, which allows $\gamma$ to be arbitrarily small by choosing suitably small $\delta$. The main result of this section is the following proposition which states that $\mathcal{I}_\gamma$ is contained in a $O(R_0^{\beta})$ neighborhood of $T_{ABC}$.

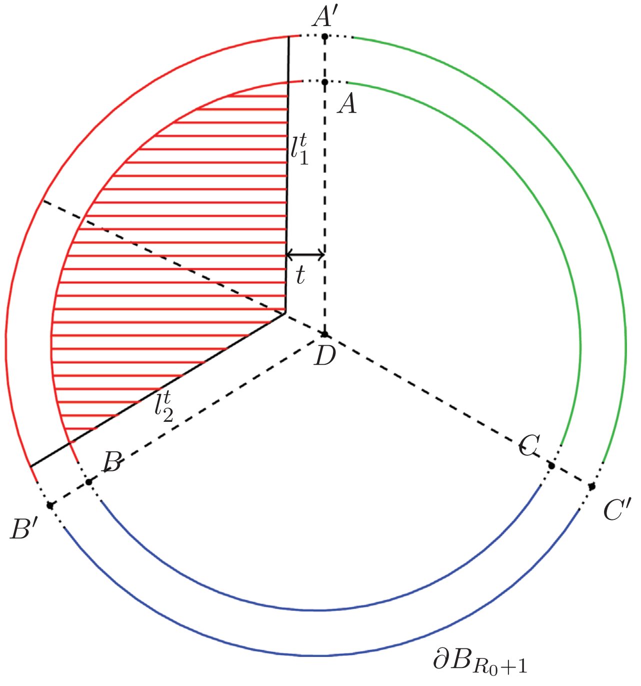


**FIGURE 4** Definition of $l_1^t, l_2^t$. In the red shaded region, $|u(z)-a_1| \le \gamma$.

**Proposition 6.1.** *There exists a constant $C = C(\gamma, W)$, such that for sufficiently small $\varepsilon$ and associated $R_0$, it holds that*

$$\mathcal{I}_\gamma \cap B_{R_0} \subset N_{CR_0^\beta}(T_{ABC}) := \{z \in B_{R_0} : \operatorname{dist}(z, T_{ABC}) \le CR_0^\beta\}. \tag{6.48}$$

*Moreover, there are positive constants $K = K(M,W)$ and $k = k(W)$ such that*

$$|u(z) - a_i| \le K e^{-k(\operatorname{dist}(z,\partial \mathcal{D}_i) - CR_0^\beta)^+}, \quad z \in \mathcal{D}_i,\ i = 1,2,3, \tag{6.49}$$

*where $(a)^+ = \max\{a, 0\}$.*

*Proof.* Let $C_0$ be the constant in Lemma 5.4 and $\tilde{u}$ be defined as the extension of $u|_{B_{R_0}}$ onto $B_{R_0+1}$ in the proof of Proposition 5.3.

For any $t \in [0, \frac{1}{2}R_0]$, we define the line segments (see Figure 4)

$$\begin{aligned} l_1^t &:= \{(x_D - t, y) : \frac{y - y_D}{t} \ge \frac{\sqrt{3}}{3}\} \cap \overline{B}_{R_0+1}, \\ l_2^t &:= \{(x,y) : \frac{y - (y_D + \frac{\sqrt{3}}{3}t)}{x - (x_D - t)} = \frac{\sqrt{3}}{3},\ x \le x_D - t\} \cap \overline{B}_{R_0+1}. \end{aligned} \tag{6.50}$$

From Lemma (5.4), when $t \ge \sqrt{3}C_0R_0^\beta$, we have that $l_1^t \subset \overline{\Omega}_1$, for $\Omega_1$ defined by $B_{R_0+1} \cap \{y \ge y^*\}$.

We set

$$\mathcal{A} := \{t : t \in [\sqrt{3}C_0R_0^{\beta}, \frac{1}{2}R_0],\ \max_{z\in l_1^t} |\tilde{u}(z) - a_1| > \gamma\}.$$

If $\mathcal{A} = \emptyset$ then we trivially get (6.53). Now suppose $\mathcal{A} \neq \emptyset$, we show the measure of $\mathcal{A}$ is $O(R_0^{\beta})$. The proof relies on the part of the lower bound over $\Omega_1$ derived in (5.35) only considers the horizontal gradient, thereby allowing for the addition of vertical displacement to provide improvement. See [1, Lemma 4.3] for a similar idea.

For any $t \in \mathcal{A}$, there exists a point $z_t \in l_1^t$ such that

$$|\tilde{u}(z_t) - a_1| > \gamma.$$

By the boundary data on $\partial B_{R_0+1}$,

$$\tilde{u}(x_D - t, \sqrt{(R_0+1)^2 - (x_D - t)^2}) = a_1.$$

Thus there exists a constant $C_1 := C_1(\gamma, W)$ such that

$$\int_{y_D+\frac{\sqrt{3}t}{3}}^{\sqrt{(R_0+1)^2-(x_D-t)^2}} \left(\frac{\lambda}{2}|\partial_y \tilde{u}(x_D - t, y)|^2 + \frac{1}{\lambda}W(\tilde{u}(x_D - t, y))\right) dy \geq C_1, \quad \forall \lambda > 0. \tag{6.51}$$

Set

$$\kappa := \frac{\mathcal{H}^1(\mathcal{A})C_1}{\sigma(y_A' - y^*)}.$$

We compute the energy on $\Omega_1$:

$$\begin{aligned}
&\int_{\Omega_1} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dxdy \\
&= \int_{\Omega_1} \left(\frac{1}{2}|\partial_x\tilde{u}|^2 + \frac{1}{1+\kappa^2}W(\tilde{u})\right) dxdy \\
&\quad + \int_{\Omega_1} \left(\frac{1}{2}|\partial_y\tilde{u}|^2 + \frac{\kappa^2}{1+\kappa^2}W(\tilde{u})\right) dydx \\
&\geq \frac{1}{\sqrt{1+\kappa^2}} \int_{\Omega_1} \left(\frac{\sqrt{1+\kappa^2}}{2}|\partial_x\tilde{u}|^2 + \frac{1}{\sqrt{1+\kappa^2}}W(\tilde{u})\right) dxdy \\
&\quad + \frac{\kappa}{\sqrt{1+\kappa^2}} \int_{\Omega_1} \left(\frac{\sqrt{1+\kappa^2}}{2\kappa}|\partial_y\tilde{u}|^2 + \frac{\kappa}{\sqrt{1+\kappa^2}}W(\tilde{u})\right) dydx \\
&\geq \frac{1}{\sqrt{1+\kappa^2}}\sigma(y_A' - y^* - d_2) \\
&\quad + \frac{\kappa}{\sqrt{1+\kappa^2}} \int_{\mathcal{A}} \int_{l_1^t} \left(\frac{\sqrt{1+\kappa^2}}{2\kappa}|\partial_y\tilde{u}|^2 + \frac{\kappa}{\sqrt{1+\kappa^2}}W(\tilde{u})\right) dydx
\end{aligned}$$

$$\geq \frac{1}{\sqrt{1+\kappa^2}}\sigma(y'_A - y^*) + \frac{\kappa}{\sqrt{1+\kappa^2}}\mathcal{H}^1(\mathcal{A})C_1 - C(\delta)$$
$$= \sqrt{\sigma^2(y'_A - y^*)^2 + (C_1\mathcal{H}^1(\mathcal{A}))^2} - C(\delta). \tag{6.52}$$

This, together with (5.40), updates the total energy in $B_{R_0+1}$:

$$\int_{B_{R_0+1}} \left(\frac{1}{2}|\nabla\tilde{u}|^2 + W(\tilde{u})\right) dz$$
$$\geq \sqrt{\sigma^2(y'_A - y^*)^2 + (C_1\mathcal{H}^1(\mathcal{A}))^2} + \sigma\sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2} - CR_0^{\frac{2}{3}}$$
$$\geq \sigma\left[(y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}\right] - CR_0^{\frac{2}{3}} + \frac{(C_1\mathcal{H}^1(\mathcal{A}))^2}{2\sigma(y'_A - y^*)}$$
$$\geq \sigma(|DA'| + |DB'| + |DC'|) - CR_0^{\frac{2}{3}} + \frac{(C_1\mathcal{H}^1(\mathcal{A}))^2}{2\sigma(y'_A - y^*)}.$$

Here $-C(\delta)$ in (6.52) is absorbed by $-CR_0^{\frac{2}{3}}$ as $R_0$ can be chosen arbitrarily large and $\delta$ is a constant only depending on $W$. Combining this with the upper bound (5.29) implies

$$\frac{(C_1\mathcal{H}^1(\mathcal{A}))^2}{2\sigma(y'_A - y^*)} \leq C(R_0^{\frac{2}{3}} + R_0^{\alpha}) \Rightarrow \mathcal{H}^1(\mathcal{A}) \leq C_2(\gamma, W)R_0^{\beta}. \tag{6.53}$$

We define

$$\mathcal{B} := \{t : t \in [\sqrt{3}C_0R_0^{\beta}, \frac{1}{2}R_0],\ \max_{z\in l_2^t} |\tilde{u}(z) - a_1| > \gamma\}. \tag{6.54}$$

An analogous computation implies

$$\mathcal{H}^1(\mathcal{B}) \leq C_2(\gamma, W)R_0^{\beta}. \tag{6.55}$$

Set

$$C_3(\gamma, W) := \sqrt{3}C_0 + 3C_2.$$

From (6.53) and (6.55) it follows that there exists $t_0 \in [\sqrt{3}C_0R_0^{\beta}, C_3R_0^{\beta}]$ (note that $R_0$ is chosen to be large enough so that $C_3R_0^{\beta} < \frac{1}{2}R_0$) such that

$$|\tilde{u}(z) - a_1| \leq \gamma, \quad \forall z \in l_1^{t_0} \cup l_2^{t_0}.$$

Let $\mathcal{D}_{1,\gamma} \subset \mathcal{D}_1$ denote the region enclosed by $l_1^{t_0}$, $l_2^{t_0}$ and $\partial B_{R_0}$. By Lemma 4.2, we have

$$|u(z) - a_1| \leq \gamma, \quad \forall z \in \partial\mathcal{D}_{1,\gamma}.$$

Here we write $u$ instead of $\tilde{u}$ because $u = \tilde{u}$ on $\overline{B}_{R_0}$.

By the variational maximum principle (see Lemma 2.3), given $\gamma$ (or equivalently, $\delta$) small enough, we conclude that

$$|u(z) - a_1| \leq \gamma \ \text{ on } \mathcal{D}_{1,\gamma}. \tag{6.56}$$

This further implies that the diffuse interface $\mathcal{I}_\gamma \cap \mathcal{D}_1$ is contained in a $C_3 R_0^\beta$ neighborhood of $T$. The same argument works for $\mathcal{I}_\gamma \cap \mathcal{D}_j$ for $j = 2, 3$ and we conclude the proof of the first part (6.48).

Utilizing (6.56), hypothesis (H1) and the comparison principle for elliptic equations, the exponential decay estimate (6.49) follows (cf. [2, Proposition 6.4]), which completes the proof. □

## 7 | RESCALING TO THE UNIT DISK $B_1$

In the following two sections, we initially choose a small $\delta$, ensuring that the specified smallness criteria for $\delta$ in the proofs of Lemma 4.2 and Proposition 6.1 are met. The choice of $\delta$ depends only on the potential $W$. Subsequently we determine a possibly even smaller $\varepsilon$ and the associated $R_0(\varepsilon) \in (R(\varepsilon), 2R(\varepsilon))$ (see Section 3 for the selection of $R_0(\varepsilon)$), ensuring all energy bounds established in Sections 3, 5, and 6 are satisfied, and also fulfilling the requirements on the largeness of $R_0$ in the proofs of Lemma 4.1 and Lemma 4.2. Also it is noteworthy that for any $R > R(\varepsilon)$, there exists $R_0 \in (R, 2R)$ such that all the results remain valid. Then we consider the rescalings

$$\begin{aligned}
u_{R_0}(z) &:= u(R_0 z), \ \text{for } z \in \overline{B_1}, \\
\tilde{A} &:= \left(\frac{x_A}{R_0}, \frac{y_A}{R_0}\right), \\
\tilde{B} &:= \left(\frac{x_B}{R_0}, \frac{y_B}{R_0}\right), \\
\tilde{C} &:= \left(\frac{x_C}{R_0}, \frac{y_C}{R_0}\right), \\
\tilde{D} &:= \left(\frac{x_D}{R_0}, \frac{y_D}{R_0}\right), \\
I^i_{R_0} &:= \{z \in \partial B_1 \, : \, R_0 z \in I_i\}, \ \ i = 1,2,3, \\
\mathcal{D}^i_{R_0} &:= \{z \in B_1 \, : \, R_0 z \in \mathcal{D}_i\}, \ \ i = 1,2,3, \\
T_{R_0} &:= \tilde{D}\tilde{A} \cap \tilde{D}\tilde{B} \cap \tilde{D}\tilde{C}.
\end{aligned}$$

From Section 4 and Proposition 6.1 we have the following properties:

(1) On $\partial B_1$,

$$|u_{R_0}(z) - a_1| \leq C\delta, \ \forall z \in I^i_{R_0}, \ i = 1,2,3,$$

where the constant $C$ only depends on $W$.

(2) The diffuse interface is contained in an $O(R_0^{\beta-1})$ neighborhood of $T_{R_0}$, that is,

$$\{z \in B_1 \, : \, |u_{R_0} - a_i| \geq 2C\delta, \ \forall i = 1,2,3\} \subset \{z \in B_1 \, : \, \operatorname{dist}(z, T_{R_0}) \leq C(\delta, W) R_0^{\beta-1}\}.$$

(3) For $i = 1, 2, 3$, within $\mathcal{D}^i_{R_0}$,

$$|u_{R_0}(z) - a_i| \leq Ke^{-kR_0(\mathrm{dist}(z,\partial\mathcal{D}^i_{R_0}) - CR_0^{\beta-1})^+}.$$

Denote the approximated triple junction map associated with $T_{R_0}$ by

$$U_{R_0}(z) := a_1\chi_{\mathcal{D}^1_{R_0}}(z) + a_2\chi_{\mathcal{D}^2_{R_0}}(z) + a_3\chi_{\mathcal{D}^3_{R_0}}(z), \quad z \in B_1, \tag{7.57}$$

The aforementioned properties imply the following estimate on the $L^1$-closeness of $u_{R_0}$ and $U_{R_0}$.

**Proposition 7.1.** *There is a constant $C_1 = C_1(W)$ that satisfies*

$$\|u_{R_0} - U_{R_0}\|_{L^1(B_1)} \leq C_1 R_0^{\beta-1}. \tag{7.58}$$

*Proof.* Let $C = C(W)$ be the constant in Proposition 6.1. Note that this constant initially depends on $\delta$ and $W$. However, since we have fixed a $\delta$ that depends on $W$, it now becomes a constant solely dependent on $W$. We have

$$\begin{aligned}\int_{\mathcal{D}^1_{R_0}} |u_{R_0}(z) - U_{R_0}(z)|\,dz &= \int_{\mathcal{D}^1_{R_0}} |u_{R_0}(z) - a_1|\,dz \\ &= \int_{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0})<CR_0^{\beta-1}} |u_{R_0}(z) - a_1|\,dz + \int_{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0})\geq CR_0^{\beta-1}} |u_{R_0}(z) - a_1|\,dz \\ &=: \Lambda_1 + \Lambda_2.\end{aligned}$$

Using $\mathcal{H}^2(\mathcal{D}^1_{R_0} \cap \{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0}) < CR_0^{\beta-1}\}) \leq CR_0^{\beta-1}$, and $|u| \leq M$ by (3.9), we get that

$$\Lambda_1 \leq C(M)R_0^{\beta-1}. \tag{7.59}$$

We observe that $\partial D^1_{R_0}$ consists of a circular arc $I^1_{R_0}$ and two line segments $\tilde{D}\tilde{A}$, $\tilde{D}\tilde{B}$. When $\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0}) \geq CR_0^{\beta-1}$,

$$\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0}) = \min\left\{\mathrm{dist}(z,\tilde{D}\tilde{A}), \mathrm{dist}(z,\tilde{D}\tilde{B}), \mathrm{dist}(z,I^1_{R_0})\right\}.$$

We compute

$$\int_{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0})\geq CR_0^{\beta-1}} e^{-kR_0(\mathrm{dist}(z,\tilde{D}\tilde{A})-CR_0^{\beta-1})}\,dz \leq 2\int_0^1 e^{-kR_0 r}\,dr \leq \frac{2}{kR_0}.$$

Similarly, we have

$$\int_{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0})\geq CR_0^{\beta-1}} e^{-kR_0(\mathrm{dist}(z,\tilde{D}\tilde{B})-CR_0^{\beta-1})}\,dz \leq 2\int_0^1 e^{-kR_0 r}\,dr \leq \frac{2}{kR_0}.$$

$$\int_{\mathrm{dist}(z,\partial\mathcal{D}^1_{R_0})\geq CR_0^{\beta-1}} e^{-kR_0(\mathrm{dist}(z,\partial B_1)-CR_0^{\beta-1})}\,dz \leq 2\pi\int_0^1 (1-r)e^{-kR_0 r}\,dr \leq \frac{2\pi}{kR_0}.$$

Summing the inequalities above, we get

$$\Lambda_2 \leq C(k)R_0^{-1}. \tag{7.60}$$

Combining (7.59) and (7.60) yields

$$\int_{\mathcal{D}^1_{R_0}} |u_{R_0}(z) - U_{R_0}(z)|\, dz \leq CR_0^{\beta-1}. \tag{7.61}$$

The same estimate also applies for $\mathcal{D}^2_{R_0}$ and $\mathcal{D}^3_{R_0}$ and (7.58) immediately follows. □

## 8 | PROOF OF THEOREM 1.2

In this final section we will conclude the proof of the main theorem. It suffices to show that the sequential limit $u_0$ in (3.10) is unique, or equivalently, independent of the sequence $\{r_k\}$.

We argue by contradiction. Suppose there are two sequences of radii $\{r_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ that converge to two distinct triple junction maps, that is

$$u_{r_k}(z) := u(r_k z) \to u_1(z) \text{ in } L^1_{loc}(\mathbb{R}^2),\ r_k \to \infty \text{ as } k \to \infty, \tag{8.62}$$

$$u_{s_k}(x) := u(s_k z) \to u_2(z) \text{ in } L^1_{loc}(\mathbb{R}^2),\ s_k \to \infty \text{ as } k \to \infty, \tag{8.63}$$

where $u_1 = \sum_{j=1}^{3} a_j \chi_{\mathcal{D}^j_I}$, for $\mathcal{P}_I := \{\mathcal{D}^1_I, \mathcal{D}^2_I, \mathcal{D}^3_I\}$ is a minimal 3-partition of $\mathbb{R}^2$, with $\partial\mathcal{P} = \bigcup_{j=1}^{3} \partial\mathcal{D}^j_I$ consists of three rays emanating from the origin and form 120-degree angles pairwise. Similarly, $u_2$ can be represented as $u_2 = \sum_{j=1}^{3} a_k \chi_{\mathcal{D}^j_{II}}$ where $\mathcal{P}_{II} := \{\mathcal{D}^1_{II}, \mathcal{D}^2_{II}, \mathcal{D}^3_{II}\}$ is another minimal 3-partition, which is centered at the origin.

For $\varepsilon \ll 1$, by Sections 3 and 7 we can find a $R(\varepsilon)$ such that for any $R \geq R(\varepsilon)$, there exists $R_0 \in (R, 2R)$ such that the rescaled function $u_{R_0}(x) = u(R_0 x)$ is close to the approximate triple junction map $U_{R_0}$ which is defined through Sections 4, 6, 7. The $L^1$ closeness is controlled by $O(R_0^{\beta-1})$ for some $\beta \in (0,1)$. Now we fix the choice of $R_0 \in (R(\varepsilon), 2R(\varepsilon))$, and then find a sequence of radii $\{R_i\}_{i=1}^{\infty}$ such that $R_i \in (2^i(R(\varepsilon)), 2^{i+1}R(\varepsilon))$ and $u_{R_i}$ satisfies

$$\|u_{R_i} - U_{R_i}\|_{L^1(B_1)} \leq CR_i^{\beta-1}$$

for some $U_{R_i} := \sum_{j=1}^{3} a_j \chi_{\mathcal{D}^j_i}$ with $\{\mathcal{D}^j_i\}$ is a partition of $B_1$ by three rays forming 120-degree angles pairwise (not necessarily centered at the origin). For each $i \in \mathbb{N}^+$, we denote

$$\tilde{D}_i := \bigcap_{j=1}^{3} \partial\mathcal{D}^j_i \text{ is the junction point for three phases of } U_{R_i},$$

$$\tilde{A}_i := \partial B_1 \cap \partial\mathcal{D}^1_i \cap \partial\mathcal{D}^2_i,$$

$$\tilde{B}_i := \partial B_1 \cap \partial\mathcal{D}^1_i \cap \partial\mathcal{D}^3_i,$$

$$\tilde{C}_i := \partial B_1 \cap \partial \mathcal{D}_i^2 \cap \partial \mathcal{D}_i^3,$$

$$T_i := \tilde{D}_i\tilde{A}_i \cap \tilde{D}_i\tilde{B}_i \cap \tilde{D}_i\tilde{C}_i \text{ is the triod at the scale } R_i.$$

The construction of this $U_{R_i}$ proceeds in the same manner as in the beginning of Section 5. We first choose $A_i, B_i, C_i$ as the midpoints of the three transitional arcs between different phases on $\partial B_{R_i}$, followed by the selection of $D_i$ such that $|D_iA_i| + |D_iB_i| + |D_iC_i|$ is minimized. Then we divide the coordinates by $R_i$ to rescale all the points $(A_i, B_i, C_i, D_i)$ to $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i, \tilde{D}_i) \in \overline{B}_1$. All the earlier results presented in Sections 4–7 remain valid upon substituting $(A, B, C, D)$ with $(A_i, B_i, C_i, D_i)$, respectively, and replacing $R_0$ by $R_i$.

For each $i$, we denote by $\theta_i \in (0, 2\pi]$ the angle of the vector $\overrightarrow{D_iA_i}$, which represents the direction of the $a_1$-$a_2$ interface for the approximate triple junction map at scale $R_i$. Given that the angles between each pair of interfaces are all 120 degrees, $U_{R_i}$ is completely determined by $\theta_i$ and the coordinates of $D_i$. The following lemma says that in two consecutive scales, $\theta_i$ will not change too much.

**Lemma 8.1.** *There is a constant $C = C(W)$ such that for any $i \in \mathbb{N}^+$,*

$$|\theta_i - \theta_{i+1}| \le CR_i^{\beta-1}. \tag{8.64}$$

*Proof.* Without loss of generality, we assume $\tilde{D}_{i+1} = (0,0)$ and $\tilde{A}_{i+1} = (0,1)$, which means the approximated $a_1$-$a_2$ interface at scale $R_{i+1}$ aligns with $y$-axis. Set the coordinates

$$D_i = (x_D^i, y_D^i), \quad A_i = (x_A^i, y_A^i).$$

According to Corollary 5.2 we have

$$\sqrt{(x_D^i)^2 + (y_D^i)^2} \le \tau R_i, \qquad \text{for some } \tau \le \frac{1}{100}.$$

We consider the original minimizer $u$ on $B_{R_{i+1}}$. By Proposition 6.1,

$$\begin{aligned} |u(x,y) - a_1| &\le 2C\delta, \quad \text{if } x \le -C_0R_{i+1}^{\beta},\ y \ge C_0R_{i+1}^{\beta},\ (x,y) \in B_{R_{i+1}}, \\ |u(x,y) - a_2| &\le 2C\delta, \quad \text{if } x \ge C_0R_{i+1}^{\beta},\ y \ge C_0R_{i+1}^{\beta},\ (x,y) \in B_{R_{i+1}}, \end{aligned} \tag{8.65}$$

for some constant $C_0(W)$.

Since $R_i \in (2^iR(\varepsilon), 2^{i+1}R(\varepsilon))$, we have

$$\frac{1}{4} < \frac{R_i}{R_{i+1}} < 1.$$

Note that rescaling does not affect the direction of straight lines. Thus in order to show (8.64), it suffices to prove that three exists a constant $C$ such that

$$|x_A^i - x_D^i| \le CR_i^{\beta}. \tag{8.66}$$

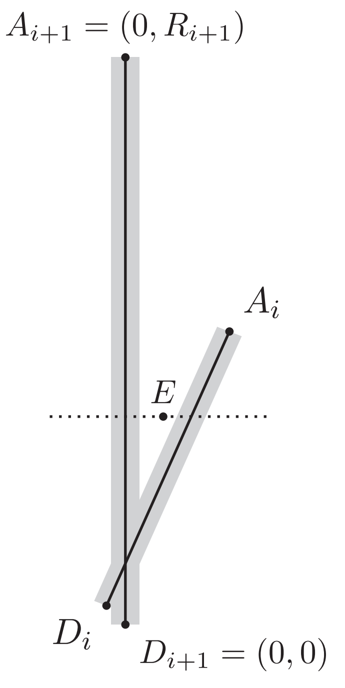


**FIGURE 5** Grey region: $O(R_i^\beta)$ transition layer between $a_1$ and $a_2$. A significant difference between $\theta_i$ and $\theta_{i+1}$ leads to a contradiction that $E$ belongs to both the $a_1$ phase and the $a_2$ phase.

Now we take $C = 48C_0$ and show that violation of (8.66) will yield a contradiction. Suppose $|x_A^i - x_D^i| > 48C_0R_i^\beta$. Then it holds that

$$|\frac{2x_A^i + x_D^i}{3} - \frac{x_A^i + 2x_D^i}{3}| > 16C_0R_i^\beta,$$

which further implies that the distance of either $(\frac{2x_A^i+x_D^i}{3}, \frac{2y_A^i+y_D^i}{3})$ or $(\frac{x_A^i+2x_D^i}{3}, \frac{y_A^i+2y_D^i}{3})$ (which are two trisection points of the line segment $D_iA_i$) from the $y$-axis exceeds $8C_0R_i^\beta$.

Suppose $\frac{2x_A^i+x_D^i}{3} > 8C_0R_i^\beta$ (the case $\frac{2x_A^i+x_D^i}{3} < -8C_0R_i^\beta$ or $|\frac{x_A^i+2x_D^i}{3}| > 8C_0R_i^\beta$ can be analyzed similarly). We set

$$E := (5C_0R_i^\beta, \frac{2y_A^i + y_D^i}{3}).$$

According to (8.65), we have $|u(E) - a_2| \le 2C(W)\delta$ because $5C_0R_i^\beta > C_0R_{i+1}^\beta$. On the other hand, we also have

$$5C_0R_i^\beta - \frac{2x_A^i + x_D^i}{3} \le -3C_0R_i^\beta,$$

which implies that $|u(E) - a_1| \le 2C(W)\delta$ through a similar estimate as in (8.65) at scale $R_i$. Consequently, we arrive at a contradiction (as illustrated in Figure 5), thereby concluding the proof. □

Since $\mathcal{P}_I$ and $\mathcal{P}_{II}$ are two distinct triple junctions centered at the origin, their directions of $a_1$-$a_2$ interface should be different, that is,

$$\varphi_1 - \varphi_2 = \alpha, \quad \text{for some angle } \alpha \neq 0.$$

Here $\varphi_1$ denotes the angle of $\partial\mathcal{D}_I^1 \cap \partial\mathcal{D}_I^2$ and $\varphi_2$ denotes the angle of $\partial\mathcal{D}_{II}^1 \cap \partial\mathcal{D}_{II}^2$.

For each sufficiently large $k$, there exist $i(k), j(k) \in \mathbb{N}^+$ such that

$$r_k \in [R_{i(k)}, R_{i(k)+1}], \quad s_k \in [R_{j(k)}, R_{j(k)+1}],$$

with $i(k)$ and $j(k)$ tend to infinity as $k$ increases.

On $B_1$, $u_{r_k}$ is converging to a triple junction map $u_1$ thanks to (8.62); while for the comparable scaling $u_{R_{i(k)}}$ is close to another triple junction map $U_{R_{i(k)}}$. By the same argument as in the proof of Lemma 8.1 we can conclude that

$$\lim_{k\to\infty} \|U_{R_{i(k)}} - u_1\|_{L^1(B_1)} = 0,$$

$$\lim_{k\to\infty} |\theta_{i(k)} - \varphi_1| = 0.$$

Similarly, we have

$$\lim_{k\to\infty} \|U_{R_{j(k)}} - u_2\|_{L^1(B_1)} = 0,$$

$$\lim_{k\to\infty} |\theta_{j(k)} - \varphi_2| = 0.$$

Therefore we obtain two subsequences of $\theta_i$ that converge to distinct angles. However, by Lemma 8.1, we have for any $i < j$,

$$\begin{aligned} |\theta_i - \theta_j| &\leq \sum_{l=i}^{j-1} C R_l^{\beta-1} \\ &\leq \sum_{l=i}^{j-1} C\left(2^l R(\varepsilon)\right)^{\beta-1} \\ &= C 2^{i(\beta-1)} R(\varepsilon)^{\beta-1} \sum_{l=0}^{j-i-1} 2^{l(\beta-1)} \\ &\leq \frac{C 2^{i(\beta-1)} R(\varepsilon)^{\beta-1}}{1-2^{\beta-1}} \to 0 \ \text{as } i \to \infty, \end{aligned}$$

indicating that $\{\theta_i\}$ is a Cauchy sequence, which yields a contradiction. The proof of Theorem 1.2 is complete.

## REFERENCES

1. N. D. Alikakos and G. Fusco, *Sharp lower bounds for vector Allen-Cahn energy and qualitative properties of minimizes under no symmetry hypotheses*, Bull. Hellenic Math. Soc. **67** (2023), 12–58.
2. N. D. Alikakos, G. Fusco, and P. Smyrnelis, *Elliptic systems of phase transition type*, Birkhäuser, Cham, 2018.
3. N. D. Alikakos and Z. Geng, *On the triple junction problem without symmetry hypotheses*, Arch. Ration. Mech. Anal. **248** (2024), no. 2, 24.

4. W. K. Allard and F. J. Almgren Jr, *On the radial behavior of minimal surfaces and the uniqueness of their tangent cones*, Ann. Math. **113** (1981), 215–265.
5. S. Baldo, *Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids*, Ann. Inst. Henri Poincare (C) Anal. Non Lineaire **7** (1990), no. 2, 67–90.
6. L. Bronsard, C. Gui, and M. Schatzman, *A three-layered minimizer in* $\mathbb{R}^2$ *for a variational problem with a symmetric three-well potential*, Commun. Pure Appl. Math. **49** (1996), no. 7, 677–715.
7. G. Flores, P. Padilla, and Y. Tonegawa, *Higher energy solutions in the theory of phase transitions: a variational approach*, J. Differ. Equ. **169** (2001), no. 1, 190–207.
8. G. Fusco, *Minimizing under relaxed symmetry constraints: triple and N-junctions*, Ann. Sc. Norm. Super. Pisa Cl. sci. **XXV** (2022), 887–940.
9. D. Gazoulis, *On the* $\gamma$*-convergence of the Allen–Cahn functional with boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A (2024), 1–23.
10. C. Gui and M. Schatzman, *Symmetric quadruple phase transitions*, Indiana Univ. Math. J. **57** (2008), no. 2, 781–836.
11. E. Sandier and P. Sternberg, *Allen-cahn solutions with triple junction structure at infinity*, Comm. Pure Appl. Math. **77** (2024), no. 11, 4163–4211.
12. L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. Math. **118** (1983), 525–571.
13. P. Sternberg and W. P. Zeimer, *Local minimisers of a three-phase partition problem with triple junctions*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 6, 1059–1073.


## APPENDIX A: MINIMIZATION PROBLEM ABOUT $y^*$

Recall the minimization problem (5.42):

$$\min \quad f(y^*) = (y'_A - y^*) + \sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2},$$

subject to $y^* \in (y'_C + d_2, y'_A - d_2)$. Taking derivatives with respect to $y^*$,

$$\frac{\partial f}{\partial y^*} = -1 + \frac{2(2y^* - y'_B - y'_C)}{\sqrt{(x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2}},$$

$$\frac{\partial^2 f}{\partial {y^*}^2} = \frac{4(x'_C - x'_B)^2}{((x'_C - x'_B)^2 + (2y^* - y'_B - y'_C)^2)^{\frac{3}{2}}} > 0.$$

From the expression above we have that when $y \sim y'_C$, $\frac{\partial f}{\partial y^*} < 0$, whereas when $y \sim y'_A$, $\frac{\partial f}{\partial y^*} > 0$. Therefore there is only one critical point for $f(y^*)$ on the interval $(y'_C + d_2, y'_A - d_2)$, which is a minimum point. Moreover, the minimum $y^*_{min}$ satisfies

$$3(2y^*_{min} - y'_B - y'_C)^2 = (x'_C - x'_B)^2$$

Since $\overrightarrow{DB'}$ parallels $(-\sqrt{3}, -1)$ and $\overrightarrow{DC'}$ parallels $(\sqrt{3}, -1)$,

$$\frac{2y_D - y'_B - y'_C}{x'_C - x'_B} = \frac{1}{\sqrt{3}},$$

which immediately implies $y^*_{min} = y_D$.

Finally we compute

$$|DA'| = y'_A - y_D,$$

$$|DB'| + |DC'| = 2|y_D - y'_B| + 2|y_D - y'_C| = \sqrt{(x'_C - x'_B)^2 + (2y_D - y'_B - y'_C)^2}.$$

Hence,

$$\min\ f(y^*) = f(y_D) = |DA'| + |DB'| + |DC'|.$$