actually subsets of the plane which are not single points (but on which the vector field vanishes identically, as the theorem says).

Labyrinths in the Plane

Theorem 8*.5.6. is hard to understand if you cannot imagine what a counterexample might look like. The reader should run the Lorenz equation, Example 6.1.6, on a computer program such as *DiffEq*, *3DViews* in the *MacMath* package, and consider carefully the resulting Lorenz attractor, pictured for instance in Figure 6.1.8.

It might seem "obvious" that nothing like the Lorenz attractor can exist in the plane, but this is just a failure of the imagination. We will show in Example 8*.5.7 that such things, called *labyrinths*, do exist. Nevertheless, the Poincaré–Bendixson Theorem asserts that in $\mathbb{R}^2$ a solution cannot wind around such a labyrinth, accumulating on something like the Lorenz attractor.

Example 8*.5.7 (Labyrinth). Consider the region $U \subset \mathbb{R}^2$ bounded by three semi-circles as represented in Figure 8*.5.4. Fill each semi-circle by concentric semi-circles, and imagine the "differential equation" whose flow curves are precisely these arcs. These curves can obviously be continued forever unless they run into one of the centers of the circles or the point θ.

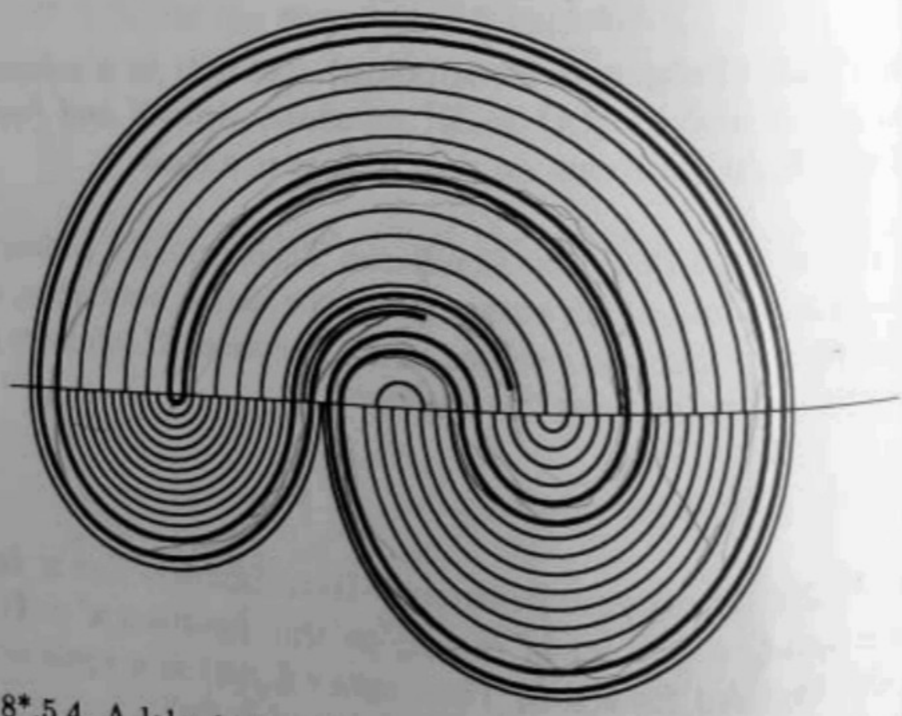

FIGURE 8*.5.4. A labyrinth with an unending orbit, unless you land on a singularity. Part of a typical orbit is drawn as a thicker path, starting and ending near the center. ▲

Example 8*.5.7 becomes especially interesting if θ, the meeting point of the two semi-circles on one side, is irrational. Consider the following two

Lemmas and their implications

Lemma 8*.5.8. *If θ is irrational, then for all but countably many $x \in [-1,1]$, the trajectory through x has limit set equal to U.*

Proof. Consider the two mappings $u_1, u_2 : [-1,1] \to [-1,1]$, which give, for any $x \in [-1,1]$, the other endpoint of the top or bottom semi-circle, one endpoint of which is x. The bottom map is simply $u_1 : x \mapsto -x$, and the top map is

$$u_2(x) = \begin{cases} -x - 1 + \theta & \text{if } x < \theta \\ -x + 1 + \theta & \text{if } x > \theta \,, \end{cases}$$

so we see that $u_2 \circ u_1(x) = x + 1 + \theta \mod 2$. Exercise 8*.5.2 asks you to prove the result from here. □

Remark. You can use the *MacMath* program *Analyzer* to iterate this function, even though it is discontinuous. Enter the function

$$0.5 * ((\mathrm{sgn}(x - \theta) + 1) * (-x + \theta + 1) + (\mathrm{sgn}(\theta - x) + 1) * (-x + \theta - 1)),$$

say with $\theta = \pi/11$, and then with $\theta = 3/11$, and see the difference, as shown in Figure 8*.5.5.

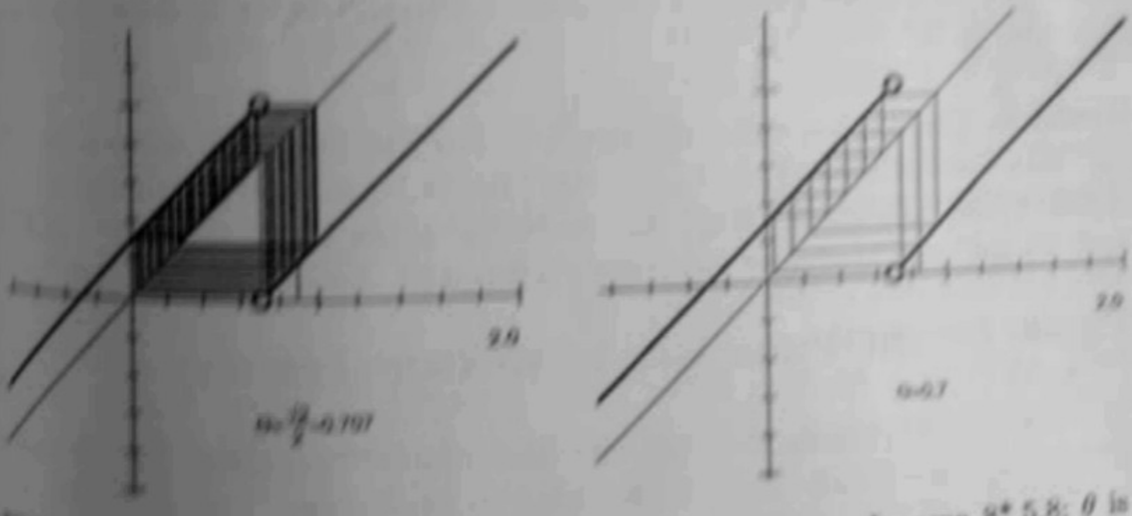

FIGURE 8*.5.5. Different iteration behaviors, to illustrate Lemma 8*.5.8; θ is *irrational* on the left, *rational* on the right.

The labyrinth of Example 8*.5.7 is not quite a counterexample to the Poincaré–Bendixson Theorem, because U contains singularities. In order to remove the singularities, try Example 8*.5.9.

Example 8*.5.9. Start with the idea of Example 8*.5.7, but open up the exceptional trajectories to a union of tadpole-shaped regions V, with narrowing infinitely long tails, as indicated in Figure 8*.5.6, and fill in the opened region, also as indicated. Now a trajectory in $U - V$ has limit set

$\overline{U-V}$ and, in particular, contains no singularities. See Exercise 8*.5.3 for further exploration of this example. ▲

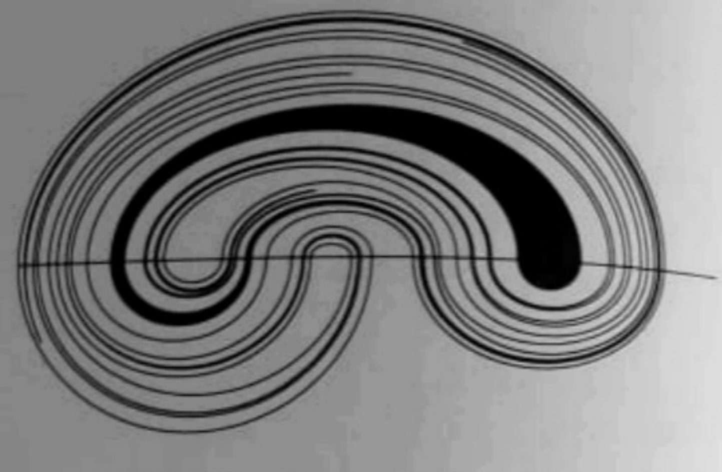

FIGURE 8*.5.6. A labyrinth without singularities, to illustrate Example 8*.5.9.

Why is the labyrinth of Example 8*.5.9 still not a counterexample to the Poincaré–Bendixson Theorem 8*.5.6? Because there is *no direction* to these flow curves, you cannot put arrows on the curves in a coherent way! So the Poincaré–Bendixson Theorem has met the labyrinth challenge in $\mathbb{R}^2$. Another labyrinth example is given in Exercise 8*.5#4 which especially shows how wrong the Poincaré–Bendixson Theorem must be for differential equations in $\mathbb{R}^3$.

Remark. There is a great similarity between this labyrinth "counterexample" and the cardboard model for the Lorenz equations of Example 8.7.1. Nothing like Lemma 8*.5.10, which excludes the labyrinth, holds in $\mathbb{R}^3$, and "chaos" can therefore rear its ugly head in the Lorenz attractor.

Proof of the Poincaré–Bendixson Theorem

The first step in proving the Poincaré–Bendixson Theorem 8*.5.6 is Lemma 8*.5.10, the key result that avoids chaos in the plane.

Lemma 8*.5.10 (Monotonicity). *If I is a segment in $\mathbb{R}^2$ transverse to* $\mathbf{f}$*, which a solution* $\mathbf{u}$ *of* $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ *crosses at three points* $A_i = \mathbf{u}(t_i)$ *with* $t_1 < t_2 < t_3$*, then* A_2 *is between* A_1 *and* A_3 *on* I*.*

Proof of Lemma. The construction is shown in Figure 8*.5.7. The set $\mathbf{u}[t_1, t_2] \cup [A_1, A_2]$ forms a simple closed curve, which bounds a region U. The vector field either enters or leaves U. Changing the sign of the vector field if necessary, we can suppose it enters.

Then $\mathbf{u}(t_2, \infty)$ is contained in the interior of U, but the part of $I - A_2$ containing A_1 is in the boundary of U or outside. Hence, A_3 is in the